



On Contra hc-Continuous Functions

Amer Khrjia Abed

Mathematics Department, College of Education for Pure Sciences, University of Al- Muthanna, Samawa 66001, Iraq

ARTICLE INFO	ABSTRACT
Published Online: 01 February 2025	In this paper, by means of hc – open sets, we introduce and investigate certain ramifications of contra – continuous and allied functions, namely, contra hc – continuous , perfectly contra hc – functions, contra hc – open functions and contra hc – closed functions . along with their several properties, characterizations and mutual relationships. Further, we
Corresponding Author: Amer Khrjia Abed	introduce new type of graphs, called contra hc – closed graphs via hc – open sets. The relationship between these graphs and contra hc – continuous functions are studied .
KEYWORDS: hc – open(closed) ; contra hc – continuous functions ; pre – hc – open(closed) functions ; contra hc – open(closed) functions ; perfectly hc – continuous function	

1. INTRODUCTION

In 2021[1], Fadhil H. A. obtained a new class of sets in a topological space, known as h – open sets and in 2022 Elsaid L. , Amjad A. Al-Rehili [2] introduced and studied the notion of hc – open . In the year 1996 , Dontchev J. [3] introduced a concept known as contra – continuous . In 1999 [4], Jafari S. , Noiri T. introduced new generalization of contra continuity called contra super continuity. In 2009 [5], Jawad J. K. , Mustafa , H. J. introduced and studied Certain Types of contra – continuous fuction. Also in 2023 [6], Fadhil H. A introduced new type of continuity called contra h – continuous function .

The aim of this research is to study the properties of one types of contra – continuous functions , namely, contra hc – continuous, this type introduced by uses concept of hc – open sets . We define this class of functions by the requirement that the inverse image of each open set in the codomain is hc – closed set in the domain.

2. PRELIMINARIES

In this work X, Y, Z means topological space, If (X, τ) is a topological space and ω is subset of X then $\text{int}(\omega), \text{Cl}(\omega)$ means the interior of ω , closure of ω respectively . A space X is said to be extremelly disconnected if the closure of any open set is open [3] . A subset ω of X is called regular open if $\omega = \text{int Cl}(\omega)$, respectively regular Closed if $\omega = \text{Cl}(\text{int}(\omega))$ [7] . ω is called δ – open if $\omega = \bigcup_{i \in I} \mathcal{M}_i$ where $\mathcal{M}_i$ is regular open for each $i \in I$ [8] . A subset ω_0 of the topological space X is called h – open set if for each non-empty set ω_1 in X , $\omega_1 \neq X$ and $\omega_1 \in O(X)$, where , $\omega_0 \subseteq \text{int}(\omega_0 \cup \omega_1)$.

The complement of the h – open set is said to be h – closed , every open [closed] in a topological space X is h – open[h – closed] [1] . For a topological space X and $\omega \subseteq X$, Intersection of all open sets of X containing ω is called kernel of ω and is denoted by $\ker(\omega)$ [3]. By a topological function $f : X \rightarrow Y$, it is meant a function from a topological space X to another topological space Y and denoted by $(TO.f)$ $O(X) [C(X)]$ is the set of all open [closed] sets in X , $O_h(X) [C_h(X)]$ is the set of all h – open [h – closed] sets in X . We will use the symbol ■ to indicate end of the proof.

Definition 2.1[2] Let ω be a subset and h – open of a space X , ω is said to be hc – open if for every $x \in \omega$, there exists a closed set $\mathcal{M}$ and $x \in \mathcal{M} \subseteq \omega$. The set of all hc – open in a topological space X is defend by notation $O_{hc}(X)$, and $X - \omega$ in a topological space X is called hc – closed .

The denote $C_{hc}(X)$ means of the set of all hc – closed sets in a topological spaces X , ω is called hc – clopen if $\omega \in O_{hc}(X)$ and $\omega \in C_{hc}(X)$

Remark 2.2, Let X be a topological space , clearly

- 1) If $\omega \in O(X)$, then $\omega \in O_h(X)$ [1]
- 2) If $\omega \in O_{hc}(X)$, then $\omega \in O_h(X)$ [2] .
- 3) If ω is clopen set in X , then $\omega \in O_{hc}(X)$ [2] .
- 4) If X is finite space and $\omega \in O_{hc}(X)$, then $\omega \in C(X)$ [2].

But The converse of this statements , need not be true in general as explain in the following example.

Let $X = \{\ell_1, \ell_2, \ell_3\}$ with topology $T = \{X, \emptyset, \{\ell_1\}, \{\ell_2\}, \{\ell_1, \ell_2\}\}$, then

- 1) $\{\ell_3\} \in O_h(X)$ but $\{\ell_3\} \notin O(X)$
- 2) $\{\ell_2\} \in O_h(X)$ but $\{\ell_2\} \notin O_{hc}(X)$

- 3) $\{\ell_2, \ell_3\} \in O_{hc}(X)$ but not clopen .
- 4) If $T = \{X, \emptyset, \{\ell_1\}\}$, $\{\ell_2, \ell_3\} \in C(X)$, but $\{\ell_2, \ell_3\} \notin O_{hc}(X)$.

Remark 2.3 [2] If X is a topological space , then

- 1) If X is a regular space and ω is open , then $\omega \in O_{hc}(X)$
- 2) If X is an extremelly disconnected and ω is δ – open , then $\omega \in O_{hc}(X)$

Remark 2.4 [2] Let X be a topological space and $\omega_1, \omega_2 \in O_{hc}(X)$ then

- 1) $\omega_1 \cap \omega_2 \in O_{hc}(X)$.
- 2) $\omega_1 \cup \omega_2 \in O_{hc}(X)$.

Definition 2.5 Let X be a topological spacelet $\omega \subseteq X, p \in X$ we say that p is:

- 1) hc – adherent point of ω , if every $G \in O_{hc}(X)$ and containing p meets ω . The set of all hc – adherent point of ω is called the hc – closure of ω and denoted by $hc - Cl(\omega)$ [2]
- 2) hc – interior point of ω , if there exists an $G \in O_{hc}(X)$ and containing p contained in ω . The set of all hc – interior points of ω is called hc – interior of ω , and denoted by $hc - int(\omega)$. [2]

Remark 2.6 [2]

- i) $\omega \in C_{hc}(X)$ if and only if $\omega = hc - Cl(\omega)$.
- ii) $\omega \in O_{hc}(X)$ if and only if $\omega = hc - int(A)$.

Definition 2.7 A subset ω of a topological space X is said to be hc – dense in X if $hc - Cl(\omega) = X$. [2]

For example in remark (2.2) , $\omega = \{\ell_1, \ell_3\}$ is hc – dense , since $hc - Cl(\{\ell_1, \ell_3\}) = X$.

Definition 2.8 A topological space X is said to be

- 1) Urysohn space, if for each pair of distinct points $x_1 \& x_2$ in X , there exists ω_1 and $\omega_2 \in O(X)$ such that $x_1 \in \omega_1, x_2 \in \omega_2$ & $Cl(\omega_1) \cap Cl(\omega_2) = \emptyset$. [7]
- 2) ultra – normal, if each pair of non–empty disjoint closed sets can be separated by disjoint closed sets. [9]
- 3) hc – Hausdorff space, if for each two points $x_1 \neq x_2$ in X , there exists two ω_1 & $\omega_2 \in O_{hc}(X)$, such that $x_1 \in \omega_1, x_2 \in \omega_2$ & $\omega_1 \cap \omega_2 = \emptyset$.
- 4) hc – normal space, if for each disjoint $H_1, H_2 \in C(X)$, there exists two ω_1 & $\omega_2 \in O_{hc}(X)$ such that $H_1 \subseteq \omega_1, H_2 \subseteq \omega_2$ & $\omega_1 \cap \omega_2 = \emptyset$.

Definition 2.9 Let X be a topological space, X is said to be hc – disconnected if it is the union of two nonempty hc – open disjoint subsets , otherwise X called is hc – connected For example : let $X = \{\ell_1, \ell_2, \ell_3\}$ with topology $T = \{X, \emptyset, \{\ell_1\}\}$ is hc – connected .

Definition 2.10. A $TO.f : X \rightarrow Y$ is called:

- 1) perfectly continuous, if every $\omega \in O(Y)$, then $f^{-1}(\omega)$ is clopen subset of X . [5]
- 2) h – continuous , if $\omega \in O(Y)$, then $f^{-1}(\omega) \in O_h(X)$. [1]

- 3) hc – continuous , if $\omega \in O(Y)$, then $f^{-1}(\omega) \in O_{hc}(X)$. [2]
- 4) contra – continuous , if every $\omega \in O(Y)$, then $f^{-1}(\omega) \in C(X)$. [3]
- 5) contra h – continuous , if every $\omega \in O(Y)$, then $f^{-1}(\omega) \in C_h(X)$. [6]
- 6) pre – hc – open, if $\omega \in O_{hc}(X)$, then $f(\omega) \in O_{hc}(Y)$.
- 7) pre – hc – closed, if $\omega \in C_{hc}(X)$, then $f(\omega) \in C_{hc}(Y)$.
- 8) contra hc – open, if $\omega \in O_{hc}(X)$, then $f(\omega) \in C_{hc}(Y)$
- 9) contra hc – closed , if $\omega \in C_{hc}(X)$, then , $f(\omega) \in O_{hc}(Y)$.

3. CONTRA hc – CONTINUOUS FUNCTION

In this section, we introduce contra hc – continuous functions and perfectly hc – continuous functions , We studied their properties and the relationship between them.

Definition 3.1 Let $f : X \rightarrow Y$ be a $TO.f$ f is said to be contra hc – continuous if for each $\omega \in O(Y)$, $f^{-1}(\omega) \in C_{hc}(X)$.

For example:

Let $f : (\mathfrak{R}, \tau_d) \rightarrow (\mathfrak{R}, \tau_d)$ be a $(TO.f)$, where τ_d is discrete topology on $\mathfrak{R}$ define, f by $f(x) = x$, then f is contra hc – continuous

Theorem 3.2 For a $TO.f : X \rightarrow Y$, the following statements are equivalent:

- i) f is contra hc – continuous .
- ii) For every $H \in C(Y)$, $f^{-1}(H) \in O_{hc}(X)$.
- iii) For each $x \in X$ and each $H \in C(Y)$ with $f(x) \in H$, there exists $\omega^* \in O_{hc}(X)$ such that $x \in \omega^* \subseteq \omega, f(\omega) \subseteq H$

Proof: i) $\rightarrow$ ii) Obvious by definition (3.1).

ii) $\rightarrow$ iii) Let $H \in C(Y)$ and let $f(x) \in H$ where $x \in X$. Then by (ii), $f^{-1}(H) \in O_{hc}(X)$, Also $x \in f^{-1}(H)$ Take $\omega = f^{-1}(H)$. Then $\omega^* \in O_{hc}(X)$ and containing $x, \omega^* \subseteq \omega$ & $f(\omega) \subseteq H$.

iii) $\rightarrow$ i) Let $x \in X$ & $H \in O(Y)$, $f(x) \in H$ & $(Y - H) \in C(Y)$, then $f^{-1}(Y - H) = X - f^{-1}(H)$, & $f(x) \in H$. Hence by iii) , there exists $\omega^* \in O_{hc}(X)$ of X with $x \in \omega^*$ such that $f(\omega^*) \subseteq H$. Then $f^{-1}(H) \in O_{hc}(X)$, therefore, f is contra hc – continuous . ■

Lemma 3.3 [4] The following properties hold for subsets ω_1, ω_2 of a space X :

- i) $x \in \ker(\omega_1)$ if and only if $\omega_1 \cap H \neq \emptyset$ for any $H \in C(X)$
- ii) $\omega_1 \subseteq \ker(\omega_1)$ and $\omega_1 = \ker(\omega_1)$ if $\omega_1 \in O(X)$
- iii) If $\omega_1 \subseteq \omega_2$, then $\ker(\omega_1) \subseteq \ker(\omega_2)$.

Theorem 3.4 Let $f : X \rightarrow Y$ be a bijective $TO.f$. Then the following statements are equivalent:

- i) f is contra hc – continuous
- ii) $f(hc - Cl(\omega_1)) \subseteq \ker(f(\omega_1))$ for every subset ω_1 of X

iii) $hc - Cl(f^{-1}(\omega_2)) \subseteq f^{-1}(\ker(\omega_2))$ for every subset ω_2 of Y

Proof: (i) $\rightarrow$ (ii) Let ω_1 be any subset of X . Suppose that $y \in \ker(f(\omega_1))$. By Lemma (3.3 (i)), there exists $\mathcal{F} \in C(Y)$ & containing $f(x)$ such that $(\omega_1) \cap \mathcal{F} = \emptyset$. Then $\omega_1 \cap f^{-1}(\mathcal{F}) = \emptyset$. Since $f^{-1}(H) \in O_{hc}(X)$ by (i), $hc - Cl(\omega_1) \cap f^{-1}(\mathcal{F}) = \emptyset$. That implies $f(hc - Cl(\omega_1) \cap \mathcal{F}) = \emptyset$ and so $y \notin f(hc - Cl(\omega_1))$. This shows that $f(hc - Cl(\omega_1)) \subseteq \ker(f(\omega_1))$

(ii) $\rightarrow$ (iii) Let ω_2 be any subset of Y . Then by (ii), $f(hc - Cl(f^{-1}(\omega_2))) \subseteq \ker(f)$ & $(f^{-1}(\omega_2)) = \ker(\omega_2)$. Therefore, $hc - Cl(f^{-1}(\omega_2)) \subseteq f^{-1}(\ker(\omega_2))$.

(iii) $\rightarrow$ (i) Let $\omega_2 \in O(Y)$, then $hc - Cl(f^{-1}(\omega_2)) \subseteq f^{-1}(\ker(\omega_2)) = f^{-1}(\omega_2)$ by (iii) and Lemma (3.3(ii)). But $f^{-1}(\omega_2) \subseteq hc - Cl(f^{-1}(\omega_2))$. So $f^{-1}(\omega_2) = hc - Cl(f^{-1}(\omega_2))$. This means that $f^{-1}(\omega_2) \in C_{hc}(X)$ set in X so that f is contra hc - continuous. ■

Remark 3.5 :

1) If $f: X \rightarrow Y$ is a TO. f and contra hc - continuous function, then f contra h - continuous, since every hc - closed set is h - closed. But the converse need not be true. While the converses is not true in general as the following example:

Let $X = Y = \{\ell_1, \ell_2, \ell_3\}$, $\tau_1 = \{\emptyset, X, \{\ell_1\}, \{\ell_1, \ell_2\}\}$, $\tau_2 = \{\emptyset, Y, \{\ell_1\}, \{\ell_2, \ell_3\}\}$. Clearly, the identity function $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ is a contra h - continuous, but is not contra hc - continuous, since $f^{-1}(\{\ell_2, \ell_3\}) = \{\ell_2, \ell_3\} \in C_h(X)$ but $\{\ell_2, \ell_3\} \notin C_{hc}(X)$.

2) There is no relation between the contra - continuous function and contra hc - continuous. Consider the

following examples : Let $X = Y = \{\ell_1, \ell_2, \ell_3\}$, $\tau_1 = \{\emptyset, X, \{\ell_1\}, \{\ell_2\}, \{\ell_1, \ell_2\}\}$, $\tau_2 = \{\emptyset, Y, \{\ell_3\}\}$. Clearly, the identity TO. $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ is a contra - continuous, but is not contra hc - continuous, and a TO. $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ such that $f(\ell_1) = f(\ell_2) = \ell_3$, $f(\ell_3) = \ell_2$ is contra hc - continuous, but is not contra - continuous.

Theorem 3.6 Let $f: X \rightarrow Y$ be a TO. f ,

- 1) if X is T_1 - space and f contra h - continuous, then f is contra hc - continuous,
- 2) if X is finite space and f contra hc - continuous, then f is continuous.
- 3) if X is regular space and f contra - continuous, then f is contra hc - continuous,
- 4) if X is extremally disconnected and f contra δ - continuous, then f is contra hc - continuous

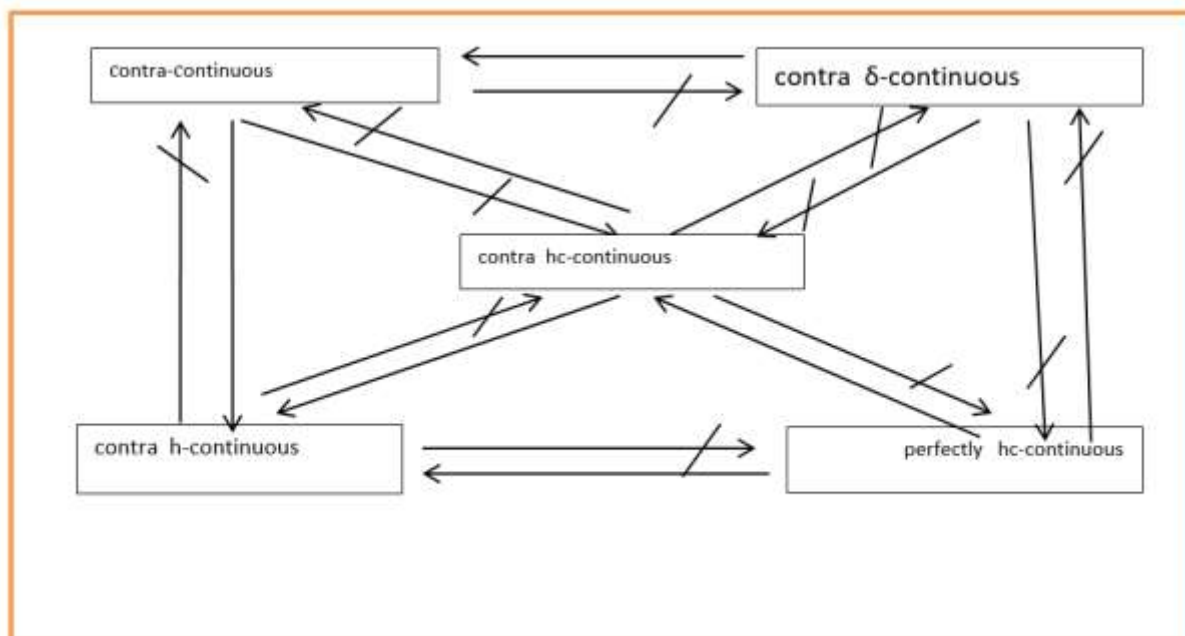
Proof: It directly follows from the definitions (3.1), (2.10) and remark (2.2), (2.3). ■

Definition 3.10 A TO. $f: X \rightarrow Y$ is called perfectly hc - continuous, if $f^{-1}(\omega)$ is hc - clopen in X for each open set ω in Y .

Theorem 3.11 Every perfectly hc - continuous function is hc - continuous and contra hc - continuous.

Proof: It directly follows from the definitions (3.1), (3.10), (2.10). ■

The next diagram explains the relations of these types of contra - continuous functions



Theorem 3.12 If $f_1 : X \rightarrow Y$ is contra hc – continuous TO. f , $f_2 : X \rightarrow Y$ is contra hc – continuous TO. f and Y is Urysohn, then $\pi = \{x \in X : f_1(x) = f_2(x)\}$ is hc – closed in X .

Proof: Let $x \in X \setminus \pi$. Then $f_1(x) \neq f_2(x)$. Since Y is Urysohn, there exists $\omega_1 \& \omega_2 \in C(Y)$ such that $f_1(x) \in \omega_1, f_2(x) \in \omega_2$ & $Cl(\omega_1) \cap Cl(\omega_2) = \emptyset$. Since f is contra hc – continuous, $f_1^{-1}(Cl(\omega_1)) \in O_{hc}(X)$. Since f_2 is contra hc – continuous, $f_2^{-1}(Cl(\omega_2)) \in O_{hc}(X)$. Let $O_1 = f_1^{-1}(Cl(\omega_1))$ & $O_2 = f_2^{-1}(Cl(\omega_2))$ & set $\rho = O_1 \cap O_2$. Then ρ is a hc – open set containing x in X .

Now

$f_1(\rho) \cap f_2(\rho) \subseteq f_1(O_1) \cap f_2(O_2) \subseteq Cl(\omega_1) \cap Cl(\omega_2) = \emptyset$. This implies that $\rho \cap \pi = \emptyset$ where ρ is hc – open. So x is not a hc – cluster point of ρ . Hence $x \notin hc - Cl(\pi)$ & this completes the proof. ■

Theorem 3.13 Let $f_1 : X \rightarrow Y$ be a contra hc – continuous TO. f and $f_2 : X \rightarrow Y$ be a contra hc – continuous TO. f . If Y is Urysohn and $f_1 = f_2$ on hc – dense set $\omega \subseteq X$, then $f_1 = f_2$ on X .

Proof: Let $\pi = \{x \in X : f_1(x) = f_2(x)\}$. Since f_1 is contra hc – continuous, f_2 is contra hc – continuous & Y is Urysohn, by Theorem (3.12), π is hc – closed in X . By assumption, we have $f_1 = f_2$ on ω where ω is hc – dense in X . Since $\omega \subseteq \pi$, ω is hc – dense & π hc – closed, we have $X = hc - Cl(\omega) \subseteq hc - Cl(\pi) = \pi$. Hence $f_1 = f_2$ on X . ■

Theorem 3.14 If $f : X \rightarrow Y$ is closed TO. f injective and contra hc – continuous and Y is ultra– normal, then X is hc – normal.

Proof: Let $H_1, H_2 \in C(X)$ and disjoint. Since f is closed and injective, $f(H_1), f(H_2) \in C(Y)$ and are disjoint. Since Y is ultra normal, there exists two clopen sets $\omega_1 \& \omega_2$ in Y , such that $f(H_1) \subseteq \omega_1, f(H_2) \subseteq \omega_2$ & $\omega_1 \cap \omega_2 = \emptyset$. Since f is contra hc – continuous, $f^{-1}(\omega_1) \& f^{-1}(\omega_2) \in O_{hc}(X)$. Also $H_1 \subseteq f^{-1}(\omega_1), H_2 \subseteq f^{-1}(\omega_2)$ & $f^{-1}(\omega_1) \cap f^{-1}(\omega_2) = \emptyset$. This shows that X is hc – normal. ■

Theorem 3.15 If a TO. $f : X \rightarrow Y$ is injective, contra hc – continuous and Y is a Urysohn space, then X is hc – Hausdorff.

Proof: Let $x, y \in X$ with $x \neq y$. Since f is injective, $f(x) \neq f(y)$. Since Y is a Urysohn space, there exists two open sets $O_1 \& O_2$ in Y such that $f(x) \in O_1, f(y) \in O_2$ & $Cl(O_1) \cap Cl(O_2) = \emptyset$. Since f is contra hc – continuous, by theorem (3.2) there exists $\omega_1 \& \omega_2 \in O_{hc}(X)$, such that $x \in \omega_1, y \in \omega_2$ & $f(\omega_1) \subseteq Cl(O_1), f(\omega_2) \subseteq Cl(O_2)$. Then

$f(\omega_1) \cap f(\omega_2) = \emptyset$ & so $f(\omega_1 \cap \omega_2) = \emptyset$. This implies that $\omega_1 \cap \omega_2 = \emptyset$ & hence X is hc – Hausdorff. ■

Lemma 3.16 For a topological space X , X is hc – connected if and only if the subsets of X which are both hc – open and hc – closed are the sets X and $\emptyset$.

Proof: $\Rightarrow$ Let $\omega \in O_{hc}(X) \& \in C_{hc}(X)$. Then $X \setminus \omega \in O_{hc}(X) \& \in C_{hc}(X)$. Since X is hc – connected & X is the disjoint union of hc – open sets ω and $X \setminus \omega$, one of these must be empty. Hence either $\omega = \emptyset$ or $\omega = X$.

$\Leftarrow$ Suppose that X is not hc – connected. Then $X = \omega_1 \cup \omega_2$ where ω_1, ω_2 are nonempty & $\omega_1, \omega_2 \in O_{hc}(X)$, such that $\omega_1 \cap \omega_2 = \emptyset$. Since $\omega_2 = X \setminus \omega_1 \in O_{hc}(X), \omega_1 \in O_{hc}(X) \& \in C_{hc}(X)$. By assumption, $\omega_1 = \emptyset$ or X . That is, either $\omega_1 = \emptyset$ or $\omega_2 = \emptyset$, which is a contradiction. Therefore X is hc – connected. ■

Theorem 3.17 For a topological space X , The only subsets of X which are both hc – open and hc – closed are the sets X and $\emptyset$ If and only if each contra hc – continuous TO. f of X into a discrete space Y with at least two points is a constant function.

Proof: $\Rightarrow$ Let $f : X \rightarrow Y$ be a contra hc – continuous function from a topological space X into a discrete topological space Y . Then for each $y \in Y, \{y\}$ is both open & closed in Y . Since f is Contra hc – continuous, $f^{-1}(y) \in O_{hc}(X) \& \in C_{hc}(X)$. Hence X is covered by hc – open and hc – closed covering $\{f^{-1}(y) : y \in Y\}$. $f^{-1}(y) = \emptyset$ or X for each $y \in Y$. If $f^{-1}(y) = \emptyset$ for each $y \in Y$, then f is fiasco function. Hence there exists only one point $y \in Y$ such that $f^{-1}(y) = X$, which shows that f is a constant function.

$\Leftarrow$ Let $\omega_0 \in O_{hc}(X) \& \in C_{hc}(X)$. Suppose $\omega_0 \neq \emptyset$. Let $f : X \rightarrow Y$ be a contra hc – continuous function from a topological space X into a discrete topological space Y defined by $f(\omega_0) = \{\gamma\}$ and $f(X \setminus \omega_0) = \{\delta\}$, where $\gamma, \delta \in Y$ and $\gamma \neq \delta$. Since f is constant so that $\omega_0 = X$. ■

Theorem 3.18 Let X be a hc – connected space and Y be any topological space. If $f : X \rightarrow Y$ is surjective and contra hc – continuous TO. f , then Y is not a discrete space.

Proof: Let Y be a discrete space & ω be any proper non-empty subset of Y . Then ω is both open & closed in Y . Since f is contra hc – continuous, $f^{-1}(\omega) \in O_{hc}(X) \& \in C_{hc}(X)$. Since X is hc – connected, by lemma (3.16), the only subsets of X which are both hc – open & hc – closed are the sets X and $\emptyset$. Hence $f^{-1}(\omega)$ is either X or $\emptyset$. If $f^{-1}(\omega) = \emptyset$, then it contradicts to the fact that $\omega \neq \emptyset$ and f is surjective. If $f^{-1}(\omega) = X$, then f is fails function. Hence Y is not a discrete space. ■

Theorem 3.19 If $TO.f f: X \rightarrow Y$ is surjective, contra hc – continuous and X is hc – connected, then Y is connected.

Proof: Assume that Y is not connected. Then $Y = \omega_1 \cup \omega_2$ where ω_1 & ω_2 are nonempty open subsets in Y such that $\omega_1 \cap \omega_2 = \emptyset$. Set $H_1 = Y \setminus \omega_1$ & $H_2 = Y \setminus \omega_1$. Then H_1 and H_2 are nonempty closed subsets in Y . Since f is surjective & contra hc – continuous, then $f^{-1}(H_1) \& f^{-1}(H_2) \in O_{hc}(X)$.
Now, $f^{-1}(H_1) \cap f^{-1}(H_2) = \emptyset$ & $f^{-1}(H_1) \cup f^{-1}(H_2) = X$. This contradicts to the fact that X is hc – connected & so Y is connected. ■

Theorem 3.20 If X is hc – connected space and $f: X \rightarrow Y$ is contra hc – continuous $TO.f$, Y is T_1 – space, then f is constant function.

Proof: Let X be hc – connected. Now, since Y is a T_1 – space, $\rho = \{f^{-1}(y) : y \in Y\}$ is disjoint hc – open partition of X . If $|\rho| \geq 2$ (where $|\rho|$ denotes the cardinality ρ), then X is the union of two nonempty disjoint hc – open sets. Since X is hc – connected, we get $|\rho| = 1$. Hence, f is constant. ■

Theorem 3.21 A space X is hc – connected if every contra hc – continuous $TO.f$ from a space X into any T_0 – space Y is constant.

Proof: suppose that X isn't hc – connected, Let $Y = \{\alpha^*, \alpha^{**}\}$ & $\sigma = \{Y, \emptyset, \{\alpha^*\}, \{\alpha^{**}\}\}$ be a topology for Y . Let $f: X \rightarrow Y$ be a function such that $f(\omega) = \{\alpha^*\}$ & $f(X \setminus \omega) = \{\alpha^{**}\}$. Then f is non constant & contra hc – continuous such that Y is T_0 , this is a contradiction, also implies that a space X have to be hc – connected. ■

Theorem 3.22 Let $f_1: X \rightarrow Y$ is and $f_2: X \rightarrow Z$ are a $TO.f$, if

- i) f_1 is contra hc – continuous and f_2 is continuous then $f_2 \circ f_1$ is contra hc – continuous
- ii) f_1 is contra hc – continuous and f_2 is contra – continuous then $f_2 \circ f_1$ is hc – continuous function.

Proof: Clearly, It can be derived from the definitions.
Type equation here.

Let $f_1: X \rightarrow Y$ & $f_2: Y \rightarrow Z$ be two $TO.f$. The case when the composition $f_2 \circ f_1$ is contra hc – continuous has been studied in the following theorem:

Theorem 3.23 Let X, Y and Z be three topological spaces, $f_1: X \rightarrow Y$ and $f_2: Y \rightarrow Z$ be two $TO.f$ if,

- i) $f_2 \circ f_1$ is contra hc – continuous and f_2 is open injection, then f_1 is contra n-continuous.
- ii) $f_2 \circ f_1$ is contra hc – continuous and f_2 is closed injection, then f_1 is contra hc – continuous.

- iii) $f_2 \circ f_1$ is contra hc – continuous and f_1 is pre – hc – closed surjection, then f_2 is contra hc – continuous.
- iv) $f_2 \circ f_1$ is contra hc – continuous and f_1 is pre – hc – open surjection, then f_2 is contra hc – continuous.

proof:

- i) Let f_2 be open & injection, let $\omega \in O(Y)$, then $f_2(\omega) \in O(Z)$. Since $f_2 \circ f_1$ is contra hc – continuous, then $(f_2 \circ f_1)^{-1}(f_2(\omega)) = f_1^{-1}(f_2^{-1}(f_2(\omega))) = f_1^{-1}(\omega) \in C_{hc}(X)$, therefore f_1 is contra hc – continuous.
- ii) Let $f_2 \circ f_1$ be contra hc – continuous & f_2 be closed and injection, let $\mathcal{M} \in C(Y)$, then $f_2(\mathcal{M}) \in C(Z)$. Since $f_2 \circ f_1$ is contra hc – continuous, then $(f_2 \circ f_1)^{-1}(f_2(\mathcal{M})) = f_1^{-1}(f_2^{-1}(f_2(\mathcal{M}))) = f_1^{-1}(\mathcal{M}) \in O_{hc}(X)$, therefore f_1 is contra hc – continuous.
- iii) Let $f_2 \circ f_1$ be contra hc – continuous & f_1 be pre – hc – closed surjection, let $\omega \in O(Z)$, then $(f_2 \circ f_1)^{-1}(\omega) \in C_{hc}(X)$. Since f_1 is pre – hc – closed & surjection which implies $(f_1(f_2 \circ f_1)^{-1}(\omega)) = f_1(f_1^{-1}(f_2^{-1}(\omega))) = f_2^{-1}(\omega) \in C_{hc}(Y)$, therefore f_2 is contra hc – continuous.
- iv) Let $f_2 \circ f_1$ be contra hc – continuous & f_1 be pre – hc – open surjection, let $\mathcal{M} \in C(Z)$, then $(f_2 \circ f_1)^{-1}(\mathcal{M}) \in O_{hc}(X)$. Since f_1 is pre – hc – open & surjection which implies $f_1((f_2 \circ f_1)^{-1}(\mathcal{M})) = f_1(f_1^{-1}(f_2^{-1}(\mathcal{M}))) = f_2^{-1}(\mathcal{M}) \in O_{hc}(Y)$, therefore f_2 is contra hc – continuous. ■

4. CONTRA hc – CLOSED GRAPH

In this section, we have presented hc – closed and contra hc – closed graph, and we studied relationship with contra hc – continuous functions.

Definition 4.1 The graph $G(f)$ of a $TO.f f: X \rightarrow Y$ is called:

- i) hc – closed in $X \times Y$, if and only if for each $(x, y) \in \{(X \times Y) \setminus G(f)\}$, there exist $\omega_1 \in O_{hc}(X)$, ω_1 containing the element x and $\omega_2 \in O(Y)$, ω_2 containing the element y such that $f(\omega_1) \cap \omega_2 = \emptyset$.
- ii) contra hc – closed in $X \times Y$ if and only if for each $(x, y) \in \{(X \times Y) \setminus G(f)\}$ there exist $\omega_1 \in O_{hc}(X)$, ω_1 containing the element x and a $\omega_2 \in C(Y)$, ω_2 containing the element y such that $f(\omega_1) \cap \omega_2 = \emptyset$.

For example: Let $X = \{\ell_1, \ell_2\}$, $\tau_1 = \{\emptyset, X, \{\ell_1\}, \{\ell_2\}\}$, $Y = \{\alpha_1, \alpha_2, \alpha_3\}$ $\tau_2 = \{\emptyset, Y, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_2, \alpha_3\}\}$, such that $f(\ell_1) = f(\ell_2) = \alpha_1$, then $G(f) = \{(\ell_1, \alpha_1), (\ell_2, \alpha_1)\}$ is hc – closed in $X \times Y$

But not contra hc – closed in $X \times Y$.

If $(\ell_1) = f(\ell_2) = \alpha_3$, then $G(f) = \{(\ell_1, \alpha_3), (\ell_2, \alpha_3)\}$ is contra hc – closed in $X \times Y$, while not hc – closed in $X \times Y$.

Theorem 4.2 If $f: X \rightarrow Y$ is hc – continuous TO. f and Y is T_1 , then $G(f)$ is contra hc – closed in $X \times Y$.

Proof: Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ & since Y is T_1 there exist open set O^* of Y , such that $f(x) \in O^*$, $y \notin O^*$. Since f is hc – continuous, there exist hc – open set O^{**} of X containing x such that $f(O^{**}) \subseteq O^*$. Therefore $f(O^{**}) \cap (Y \setminus O^*) = \emptyset$ & $\{Y \setminus O^*\}$ is a closed set in Y containing y . Hence by above definition, $G(f)$ is contra hc – closed in $X \times Y$. ■

Theorem 4.3 If Y is a Urysohn and $f: X \rightarrow Y$ is contra hc – continuous TO. f , then $G(f)$ is contra hc – closed in $X \times Y$.

Proof: assume that $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ & since Y is Urysohn, there exist open sets ω_1, ω_2 in Y such that $f(x) \in \omega_1$, $y \in \omega_2$ & $Cl(\omega_1) \cap Cl(\omega_2) = \emptyset$. Now, since f is contra hc – continuous, there exist O^* hc – open set such that $f(O^*) \subseteq Cl(\omega_1)$ which implies that $f(O^*) \cap Cl(\omega_2) = \emptyset$. Hence by definition (4.1), $G(f)$ is contra hc – closed in $X \times Y$. ■

Theorem 4.4 Let $f_1: X \rightarrow Y$ be a TO. f and $f_2: X \rightarrow X \times Y$ be a graph function of f_1 , defined by $f_2(x) = (x, f_1(x))$ for every $x \in X$. If f_2 is contra hc – continuous, then f_1 is contra hc – continuous.

Proof: Let ω be an open set in Y . Then $X \times \omega$ is open in $X \times Y$. Since f_2 is contra hc – continuous, $f_2^{-1}(X \times \omega) = f_1^{-1}(\omega)$ is hc – closed in X . This shows that f is contra hc – continuous.

5. CONCLUSIONS

In this work, we have presented the idea of hc – open and hc – closed sets and learned about its master properties. Then, we have used this idea to show a kind of contra – continuity. We discussed the master properties of this continuity and we have revealed the relationship between this type of continuity and other types. In addition, we have introduced graph functions using hc – open and hc – closed sets and checked their main properties. Our next works will concentrate on studying further topological concepts associated with the contra hc – continuous.

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