



Generalized Hyers–Ulam Stability for Quadratic Functional Equations of Several Variables in Banach Algebras

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ARTICLE INFO	ABSTRACT
Published Online: 25 September 2026 Corresponding Author: M.Settu KEYWORDS: 39B52, 32B72, 32B82	In this paper, we obtain the general solution and Generalized Hyers-Ulam-Rassias stability of the n-dimensional quadratic functional equation $f\left(\sum_{i=1}^n x_i\right) + \sum_{j=1}^n f\left(-x_j + \sum_{i=1, i \neq j}^n x_i\right) = (n-3) \sum_{1 \leq i < j \leq n} f(x_i + x_j) + (-n^2 + 5n - 2) \sum_{i=1}^n f(x_i)$ in Random Normed Space, where n is a positive integer with $n \geq 3$ using direct and fixed-point methods.

I. INTRODUCTION

This template provides all the necessary information to the The famous functional equation in the field of stability of functional equation is the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad (1.1)$$

The function $f(x) = x^2$ is the solution of the functional equation (1.1).

The stability theory of functional equations has become an important area of research in functional analysis, operator theory, and nonlinear analysis[1-3]. The subject originated from a fundamental question concerning the stability of functional equations: whether a function that approximately satisfies a given functional equation must necessarily be close to an exact solution. Since the pioneering work on this problem, considerable attention has been devoted to various forms of stability, including Hyers–Ulam stability, Hyers–Ulam–Rassias stability, and generalized Hyers-Ulam-Rassias stability. These concepts provide useful mathematical frameworks for investigating the behavior of approximate solutions and their relationship with exact solutions.

Among the most important functional equations studied in this context is the quadratic functional equation. Quadratic mappings arise naturally in several areas of mathematics and

have significant connections with norm structures, inner product spaces, optimization, and nonlinear analysis [3-7]. The stability of quadratic functional equations has therefore attracted considerable interest, particularly in different metric, normed, probabilistic, and fuzzy settings. The study of higher-dimensional versions of these equations provides a natural extension of the classical theory and offers a broader framework for analyzing stability phenomena involving several variables simultaneously.

Random normed spaces, introduced as a probabilistic generalization of normed spaces, provide an appropriate setting for studying functional equations when uncertainty or probabilistic effects are involved. In a random normed space, the size of a vector is described by a distribution function rather than by a single deterministic nonnegative number. This probabilistic structure makes random normed spaces useful in situations where exact measurements are affected by uncertainty [8]. Consequently, the investigation of functional equations in random normed spaces has developed into an active research direction.

In particular, the study of stability for quadratic functional equations in random normed spaces combines the classical theory of functional equations with probabilistic functional analysis. Establishing the stability of an approximate quadratic mapping requires suitable probabilistic estimates and appropriate control functions. The generalized Hyers–Ulam–Rassias stability framework is especially useful because it allows the error associated with an approximate

solution to depend on the variables rather than being restricted to a fixed constant. This provides greater flexibility and enables the treatment of a wider class of perturbations [9].

Fixed point theory is another powerful technique for studying the stability of functional equations. By constructing a suitable complete generalized metric space and defining an appropriate contraction mapping, the existence and uniqueness of an exact solution can be established through a fixed-point [10] argument. Compared with the direct method, the fixed-point approach provides a systematic procedure for obtaining stability results and can be particularly effective for complicated higher-dimensional functional equations. The direct method, on the other hand, often gives a constructive description of the exact solution through an appropriate limiting process.

Motivated by these developments, this paper investigates the general solution and generalized Hyers–Ulam–Rassias stability of an (n)-dimensional quadratic functional equation in a random normed space, where (n) is a positive integer. We establish the stability results by employing both direct and fixed point methods. The direct approach provides explicit estimates for the difference between an approximate solution and the corresponding exact quadratic solution, while the fixed point approach establishes the existence and uniqueness of the stable solution under suitable conditions. The obtained results extend the existing stability theory of quadratic functional equations to an (n)-dimensional probabilistic framework and contribute to the further development of functional equations in random normed spaces.

The remainder of this paper is organized as follows. The necessary definitions and fundamental properties of random normed spaces and stability concepts are presented first. The general solution of the considered (n)-dimensional quadratic functional equation is then established. Subsequently, generalized Hyers–Ulam–Rassias stability results are obtained using the direct method and the fixed point method. Finally, the principal findings are summarized and their significance within the existing stability theory is discussed.

2 GENERAL SOLUTION OF THE QUADRATIC FUNCTIONAL EQUATION (1.2)

In this section, we discuss the general solution of the n-dimensional quadratic functional equation (1.2).

Theorem 2.1. Let X and Y be real vector space. The mapping $f: X \rightarrow Y$ satisfies the functional equation (1.1) for all $x, y \in X$ if and only if $f: X \rightarrow Y$ satisfies the functional equation (1.2) for all $x_1, x_2, \dots, x_n \in X$.

Proof. Let $f: X \rightarrow Y$ satisfies the functional equation (1.2) replacing $x_1, x_2, \dots, x_n$ by $(0, 0, \dots, 0)$ in (1.2), we get, $f(0) = 0$. Now setting $x_1, x_2, \dots, x_n$ by $(x, 0, \dots, 0)$

$$\begin{aligned} f(-x_1) &= f(x_1) \forall x \in X. \text{ Therefore } f \text{ is an even function.} \\ \text{Setting } (x_3, x_4, \dots, x_n) &\text{ by } (0, 0, \dots, 0) \text{ in (1.2) we obtain} \\ f(x_1 + x_2) + f(-x_1 + x_2) + f(x_1 - x_2) + f(x_1 + x_2) + f(x_1 + x_2) \\ &= 2f(x_1 + x_2) + 2f(x_2) + 2f(x_1) + 2f(x_1) + \\ &2f(x_1) + 2f(x_2) - 2f(x_1) - 2f(x_2) + 2f(x_2) \end{aligned}$$

(2.1)

for all $x_1, x_2 \in X$. It follows from (2.1), we get

$$f(-x_1 + x_2) + f(x_1 - x_2) + 4f(x_1 + x_2) = 2f(x_1 + x_2) + 6f(x_2) + 6f(x_1) - 2f(x_1) - 2f(x_2)$$

(2.2)

for all $x_1, x_2 \in X$. It follows from (2.2), we have

$$\begin{aligned} f(x_1 + x_2) + f(x_1 - x_2) + (n-1)f(x_1 + x_2) \\ = (n-3)[f(x_1 + x_2) + (n-2)(x_1) + (n-2)f(x_2)] + (-n^2 + 5n - 2) + [f(x_1) + f(x_2)] \end{aligned}$$

(2.3)

for all $x_1, x_2 \in X$. Replacing (x_1, x_2) by (x, y) in (1.2) and using evenness of f and rearranging the functions, our result is desired.

Conversely, assume that $f: X \rightarrow Y$ satisfies (1.1). Setting $x = y = 0$ in (1.1), we get $f(0) = 0$. Let $y = 0$ in (1.1), we obtain $f(-x) = f(x) \forall x \in X$. Therefore f is an even function. Replacing y by x and $2x$ respectively in (1.1), we arrive $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$. In general for any positive integer a , we have $f(ax) = a^2f(x) \forall x \in X$. Replacing (x, y) by (x_1, x_2) in (1.1) and using (1.1) we get,

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2)$$

(2.4)

for all $x_1, x_2 \in X$. Setting (x_1, x_2) by $(x_1 + x_2, x_3)$ in (2.4), we obtain

$$f(x_1 + x_2 + x_3) + f(x_1 + x_2 - x_3) = 2f(x_1 + x_2) + 2f(x_3)$$

(2.5)

for all $x_1, x_2, x_3 \in X$. Replacing x_2 by $-x_2$ in (2.5), we get

$$f(x_1 - x_2 + x_3) + f(x_1 - x_2 - x_3) = 2f(x_1 - x_2) + 2f(x_3)$$

(2.6)

for all $x_1, x_2, x_3 \in X$. Using evenness in (2.6), that

$$f(x_1 - x_2 + x_3) + f(-x_1 + x_2 + x_3) = 2f(x_1 - x_2) + 2f(x_3)$$

(2.7)

for all $x_1, x_2, x_3 \in X$. Adding (2.5), (2.7) and using (2.4), we arrive

$$f(x_1 + x_2 + x_3) + f(-x_1 + x_2 + x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) = 4f(x_1) + 4f(x_2) + 4f(x_3)$$

(2.8)

for all $x_1, x_2, x_3 \in X$. Replacing (x, y) by $(x_1 + x_4, x_2 + x_3)$ in (1.1), we get

$$f(x_1 + x_2 + x_3 + x_4) + f(x_1 - x_2 - x_3 + x_4) = 2f(x_1 + x_4) + f(x_2 + x_3)$$

(2.9)

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(x_1 + x_2 + x_3 + x_4)$ on both sides in (2.9), we arrive

$$2f(x_1 + x_2 + x_3 + x_4) + f(x_1 - x_2 - x_3 + x_4) = f(x_1 + x_2 + x_3 + x_4) + f(x_1 + x_4) + f(x_2 + x_3) \quad (2.10)$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(x_1 - x_2 + x_3 - x_4)$ on both sides in (2.10), we have

$$2f(x_1 + x_2 + x_3 + x_4) + f(x_1 - x_2 - x_3 + x_4) + f(x_1 - x_2 + x_3 - x_4) = f(x_1 + x_2 + x_3 + x_4) + f(x_1 - x_2 + x_3 - x_4) + f(x_1 + x_4) + f(x_2 + x_3) \quad (2.11)$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (S.10) in (2.11) and rearranging, we get

$$2f(x_1 + x_2 + x_3 + x_4) + 2f(x_1 - x_2) + 2f(x_3 - x_4) = 2f(x_1 + x_3) + 2f(x_2 + x_4) + 2f(x_1 + x_4) + 2f(x_2 + x_3) \quad (2.12)$$

for all $x_1, x_2, x_3, x_4 \in X$. Dividing by 2 in (2.12) on both sides, we have

$$f(x_1 + x_2 + x_3 + x_4) + f(x_1 - x_2) + f(x_3 - x_4) = f(x_1 + x_3) + f(x_2 + x_4) + f(x_1 + x_4) + f(x_2 + x_3) \quad (2.13)$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(x_1 + x_2) + f(x_3 + x_4)$ on both sides in (2.13), we get

$$f(x_1 + x_2 + x_3 + x_4) + f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 - x_2) + f(x_3 - x_4) + f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 - x_2) + f(x_3 - x_4) = f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 + x_3) + f(x_2 + x_4) + f(x_1 + x_4) + f(x_2 + x_3) \quad (2.14)$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (3.1) in (2.14) and rearranging, that

$$f(x_1 + x_2 + x_3 + x_4) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) = f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 + x_3) + f(x_2 + x_4) + f(x_1 + x_4) + f(x_2 + x_3) \quad (2.15)$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4)$ on both sides in (2.15), we have

$$f(x_1 + x_2 + x_3 + x_4) + 4f(x_1) + 4f(x_2) + 4f(x_3) + 4f(x_4) = f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 + x_3) + f(x_2 + x_4) + f(x_1 + x_4) + f(x_2 + x_3) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) \quad (2.16)$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (2.8) and rearranging, we arrive

$$f(x_1 + x_2 + x_3 + x_4) + 2[2f(x_1) + 2f(x_2)] + 2[2f(x_3) + 2f(x_4)] = f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 + x_3) + f(x_2 + x_4) + f(x_1 + x_4) + f(x_2 + x_3) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) \quad (2.17)$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (1.1) in (2.17), we get

$$f(x_1 + x_2 + x_3 + x_4) + 2[f(x_1 + x_2) + f(x_1 - x_2)] + 2[f(x_3 + x_4) + f(x_3 - x_4)] = f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 + x_3) + f(x_2 + x_4) + f(x_1 + x_4) + f(x_2 + x_3) + 2f(x_2) + 2f(x_3) + 2f(x_4) \quad (2.18)$$

for all $x_1, x_2, x_3 \in X$. Using (2.10) and re modifying, we get

$$f(x_1 + x_2 + x_3 + x_4) + 2f(x_1 + x_2) + 2f(x_1 - x_2) + 2f(x_3 + x_4) + 2f(x_3 - x_4) = f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 + x_3) + f(x_2 + x_4) + f(x_1 + x_4) + f(x_2 + x_3) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) \quad (2.19)$$

for all $x_1, x_2, x_3 \in X$. Using (1.1) in (2.19), we arrive

$$f(x_1 + x_2) + f(x_1 + x_3) + f(x_1 + x_4) + f(x_2 + x_3) + f(x_2 + x_4) + f(x_3 + x_4) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) \quad (2.20)$$

for all $x_1, x_2, x_3 \in X$. Similarly, one can easily verify for four variables that

$$f(x_1 + x_2 + x_3 + x_4 + x_5) + f(-x_1 + x_2 + x_3 + x_4 + x_5) + f(x_1 - x_2 + x_3 + x_4 + x_5) + f(x_1 + x_2 + x_3 - x_4 + x_5) + f(x_1 + x_2 + x_3 + x_4 - x_5) = (5 - 3) \sum_{1 \leq i < j \leq 5} f(x_i + y_j) + (-5^2 + 5.5 - 2) \sum_{i=1}^5 f(x_i) \quad (2.21)$$

for all $x_1, x_2, x_3 \in X$. Extending this result, for any positive n , we can arrive (1.2) for all $x_1, x_2, x_3 \in X$.

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3. Random Stability Results: Direct Method

In this section, we investigate the Stability of the n -dimensional quadratic functional equation (1.2) in RN-Space is provided. All through this section, let us consider X be a linear space (Y, μ, T) is a complete RN-Space.

Now use the following notation for a given mapping $f: X \rightarrow Y$

$$Df(x_1, x_2, \dots, x_n) = f\left(\sum_{i=1}^n x_i\right) + \sum_{j=1}^n \left(-x_j + \sum_{i=1, i \neq j}^n x_i\right) - (n-3) \sum_{1 \leq i < j \leq n} f(x_i + y_j) - (-n^2 + 5n - 2) \sum_{i=1}^n f(x_i)$$

where n is a positive integer with $n \geq 3$ and for all $x_1, x_2, \dots, x_n \in X$.

Theorem 3.1 Let $j = \pm 1$, $f: X \rightarrow Y$ be a mapping for which there exists a function $\eta: X^n \rightarrow D^+$ with the condition

$$\lim_{k \rightarrow \infty} T_{i=0}^{\infty} \left(\eta_{2^{(k+i)}x_1, 2^{(k+i)}x_2, 2^{(k+i)}x_3, \dots, 2^{(k+i)}x_n} (2^{2(k+i+1)}jt) \right) = 0 \quad (3.1)$$

$$= \lim_{k \rightarrow \infty} \eta(2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, \dots, 2^{kj}x_n) \quad (3.2)$$

such that the functional inequality with $f(0) = 0$ such that

$$\mu_{Df(x_1, x_2, \dots, x_n)}(t) \geq \eta_{(x_1, x_2, \dots, x_n)}(t) \quad (3.3)$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $Q: X \rightarrow Y$ satisfies the functional equation (1.2) and

$$\mu_{Q(x)-f(x)}(t) \geq T_{i=0}^{\infty} \left(\eta_{\left(2^{(i+1)j}x, -2^{(i+1)j}x, 2^{(i+1)j}x, -2^{(i+1)j}x, \dots, 2^{(i+1)j}x, 0, \dots, 0 \right)}(2^{2(i+1)j}t) \right) \quad (3.4)$$

for all $x \in X$ and all $t > 0$. The mapping $Q(x)$ is defined by

$$\mu_{Q(x)}(t) = \lim_{k \rightarrow \infty} \mu_{f(2^{kj}x)}(t) \quad (3.5)$$

for all $x \in X$ and all $t > 0$.

Proof. Assume $j = 1$. Setting $(x_1, x_2, x_3, \dots, x_n) = (x, -x, x, -x, x, 0, \dots, 0)$ in (3.1), we acquire

$$\mu_{2(n-5)f(2x)-8(n-5)f(x)}(t) \geq \eta_{(x, -x, x, -x, x, 0, \dots, 0)}(t) \quad (3.6)$$

for all $x \in X$ and all $t > 0$. It follows from (3.5) and (RN2), we arrive

$$\mu_{\frac{f(2x)}{2^2}-f(x)}(t) \geq \eta_{(x, -x, x, -x, x, 0, \dots, 0)}(8(n-5)t) \quad (3.7)$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^k x$ in (3.6), we catch

$$\begin{aligned} \mu_{\frac{f(2^{k+1}x)}{2^{2(k+1)}}-f(2^kx)}(t) &\geq \eta_{(2^kx, -2^kx, 2^kx, -2^kx, 2^kx, 0, \dots, 0)}(4^{k+2}(n-5)t) \\ &\geq \eta_{(x, -x, x, -x, x, 0, \dots, 0)}\left(\frac{4^{k+2}(n-5)}{\alpha^k}t\right) \end{aligned} \quad (3.8)$$

for all $x \in X$ and all $t > 0$. It follows from

$$\begin{aligned} \frac{f(2^n x)}{4^n} - f(x) &= \sum_{k=0}^{n-1} \frac{f(2^{k+1}x)}{4^{k+1}} - \frac{f(2^k x)}{4^k} \text{ and (3.8) that} \\ \mu_{\frac{f(2^n x)}{4^n}-f(x)}\left(t \sum_{k=0}^{n-1} \frac{\alpha^k}{4^{k+2}(n-5)}\right) &\geq T_{k=0}^{n-1} \left(\eta_{(x, -x, x, -x, x, 0, \dots, 0)}(t) \right) = \eta_{(x, -x, x, -x, x, 0, \dots, 0)}(t) \end{aligned} \quad (3.9)$$

$$\begin{aligned} \mu_{\frac{f(2^n x)}{4^n}-f(x)}(t) &\geq \eta_{(x, -x, x, -x, x, 0, \dots, 0)}\left(\frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{4^{k+2}(n-5)}}\right) \end{aligned} \quad (3.10)$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^m x$ in (3.10), we arrive that

$$\mu_{\frac{f(2^{n+m}x)}{4^{n+m}}-\frac{f(2^m x)}{4^m}}(t) \geq \eta_{(x, -x, x, -x, x, 0, \dots, 0)}\left(\frac{t}{\sum_{k=m}^{n+m} \frac{\alpha^k}{4^{k+2}(n-5)}}\right) \quad (3.11)$$

As $\eta_{(x, -x, x, -x, x, 0, \dots, 0)}\left(\frac{t}{\sum_{k=m}^{n+m} \frac{\alpha^k}{4^{k+2}(n-5)}}\right) \rightarrow 1$ as $m, n \rightarrow \infty$,

then $\left\{\frac{f(2^n x)}{4^n}\right\}$ is a Cauchy sequence in (Y, μ, T) . Since (Y, μ, T) is a complete RN-Space, this sequence converges to some point $Q(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (3.11), we have

$$\mu_{\frac{f(2^n x)}{4^n}-f(x)}(t) \geq \eta_{(x, -x, x, -x, x, 0, \dots, 0)}\left(\frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{4^{k+2}(n-5)}}\right) \quad (3.12)$$

and so, for every $\delta > 0$, we collect

$$\begin{aligned} \mu_{Q(x)-f(x)}(t+\delta) &\geq T\left(\mu_{Q(x)-\frac{f(2^n x)}{4^n}}(\delta), \mu_{\frac{f(2^n x)}{4^n}-f(x)}(t)\right) \\ &\geq T\left(\mu_{Q(x)-\frac{f(2^n x)}{4^n}}(\delta), \eta_{(x, -x, x, -x, x, 0, \dots, 0)}\left(\frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{4^{k+2}(n-5)}}\right)\right) \end{aligned} \quad (3.13)$$

Taking limit as $n \rightarrow \infty$ and using (3.13), we arrive

$$\mu_{Q(x)-f(x)}(t+\delta) \geq \eta_{(x, -x, x, -x, x, 0, \dots, 0)}((n-5)(4-\alpha)t) \quad (3.14)$$

Since δ was arbitrary, by taking $\delta \rightarrow 0$ in (3.14), we have

$$\mu_{Q(x)-f(x)}(t) \geq \eta_{(x, -x, x, -x, x, 0, \dots, 0)}((n-5)(4-\alpha)t) \quad (3.15)$$

Replacing $(x_1, x_2, \dots, x_n)$ by $(2^n x_1, 2^n x_2, \dots, 2^n x_n)$ in (3.3) respectively, we acquire

$$\mu_{Df(2^n x_1, 2^n x_2, \dots, 2^n x_n)}(t) \geq \eta_{(2^n x_1, 2^n x_2, \dots, 2^n x_n)}(4^n t) \quad (3.16)$$

for all $x_1, x_2, \dots, x_n \in X$ and for all $t > 0$. Since

$$\lim_{k \rightarrow \infty} T_{i=0}^{\infty} \left(\eta_{\left(2^{(k+i)}x_1, 2^{(k+i)}x_2, \dots, 2^{(k+i)}x_n \right)}(2^{2(k+i+1)}t) \right) = 1.$$

We conclude that Q fulfils (1.1). To prove the uniqueness of the quadratic mapping Q , assume that there exists a quadratic mapping R from X to Y , which satisfies (3.15).

Fix $x \in X$. Clearly, $Q(2^n x) = 4^n Q(x)$ and $R(2^n x) = 4^n R(x)$ for all $x \in X$. It follows from (3.15) that

$$\begin{aligned} \mu_{Q(x)-R(x)}(t) &= \lim_{n \rightarrow \infty} \mu_{\frac{Q(2^n x)}{4^n} - \frac{R(2^n x)}{4^n}}(t) \\ \mu_{\frac{Q(2^n x)}{4^n} - \frac{R(2^n x)}{4^n}}(t) &\geq \min \left\{ \mu_{\frac{Q(2^n x)}{4^n} - \frac{f(2^n x)}{4^n}}\left(\frac{t}{2}\right), \mu_{\frac{R(2^n x)}{4^n} - \frac{f(2^n x)}{4^n}}\left(\frac{t}{2}\right) \right\} \\ &\geq \eta_{(2^n x, -2^n x, 2^n x, -2^n x, 0, \dots, 0)}(4^n(n-5)(4-\alpha)t) \\ &\geq \eta_{(x, -x, x, -x, x, 0, \dots, 0)} \\ &\quad \left(\frac{4^n(n-5)(4-\alpha)t}{\alpha^n} \right) \end{aligned} \quad (3.17)$$

Since $\lim_{n \rightarrow \infty} \left(\frac{4^n(n-5)(4-\alpha)t}{\alpha^n} \right) = \infty$, we get $\lim_{n \rightarrow \infty} \eta_{(x, -x, x, -x, x, 0, \dots, 0)} \left(\frac{4^n(n-5)(4-\alpha)t}{\alpha^n} \right) = 1$. Therefore, it follows that $\mu_{Q(x)-R(x)}(t) = 1$ for all $t > 0$ and so $Q(x) = R(x)$. This completes the proof.

The following corollary is an immediate consequence of Theorem 3.1, concerning the stability of (1.2).

Corollary 3.2. Let ε and s be non-negative real numbers. Let a quadratic function $f: X \rightarrow Y$ satisfies the inequality

$$\begin{aligned} \mu_{Df(x_1, x_2, x_3, \dots, x_n)}(t) &\geq \\ &\begin{cases} \eta_\varepsilon(t) \\ \eta_\varepsilon \sum_{i=1}^n \|x_i\|^s(t), s \neq 2 \\ \eta_\varepsilon (\prod_{i=1}^n \|x_i\|^s + \sum_{i=1}^n \|x_i\|^{ns})(t), s \neq \frac{2}{n} \end{cases} \end{aligned} \quad (3.18)$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and all $t > 0$. Then there exists a unique quadratic function $Q: X \rightarrow Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \begin{cases} \eta_{\frac{\varepsilon}{(n-5)|2^2-1|}}(t) \\ \eta_{\frac{\varepsilon\|x\|^s}{(n-5)|2^2-2^s|}}(t) \\ \eta_{\frac{\varepsilon\|x\|^{ns}}{(n-5)|2^2-2^{ns}|}}(t) \end{cases} \quad (3.19)$$

for all $x \in X$ and all $t > 0$.

4. Random Stability Results: Fixed Point Method.

In this section, we present the Stability of the n -dimensional quadratic functional equation (1.2), in Random Normed Space using fixed point method.

Theorem 4.1 Let $f: X \rightarrow Y$ be a mapping for which there exists a function $\eta: X^n \rightarrow D^+$ with the condition

$$\lim_{k \rightarrow \infty} \eta(\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, \dots, \delta_i^k x_n) (\delta_i^{2k} t) = 1 \quad (4.1)$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and all $t > 0$ and where

$$\delta_i = \begin{cases} 2, i = 0; \\ \frac{1}{2}, i = 1; \end{cases} \text{ satisfying the functional inequality}$$

$$\mu_{Df(x_1, x_2, x_3, \dots, x_n)}(t) \geq \eta_{(x_1, x_2, x_3, \dots, x_n)}(t) \quad (4.2)$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$x \rightarrow \beta(x, t) = \eta_{\left(\frac{x}{2}, -\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}, 0, \dots, 0\right)}((n-5)t),$$

has the property, that

$$\beta(x, t) \leq L \frac{1}{\delta_i^2} \beta(\delta_i x, t) \quad (4.3)$$

for all $x \in X$ and $t > 0$. Then there exists a unique quadratic function $Q: X \rightarrow Y$ satisfying the functional equation (1.2) and

$$\mu_{Q(x)-f(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x, t) \quad (4.4)$$

for all $x \in X$ and $t > 0$.

Proof. Let d be a general metric on Ω , such that $d(g, h) = \inf \{k \in (0, \infty) / \mu_{(g(x)-h(x))}(kt) \geq \beta(x, t), x \in X, t > 0\}$

. It is easy to see that (Ω, d) is complete. Define $T: \Omega \rightarrow \Omega$ by

$$Tg(x) = \frac{1}{\delta_i^2} g(\delta_i x), \text{ for all } x \in X. \text{ Now for } g, h \in \Omega, \text{ we}$$

have $d(g, h) \leq K$.

$$\Rightarrow \mu_{(g(x)-h(x))}(Kt) \geq \beta(x, t)$$

$$\Rightarrow \mu_{(Tg(x)-Th(x))}\left(\frac{Kt}{\delta_i^2}\right) \geq \beta(x, t)$$

$$\Rightarrow d(Tg(x), Th(x)) \leq KL$$

$$\Rightarrow d(Tg, Th) \leq Ld(g, h) \quad (4.5)$$

for all $g, h \in \Omega$. Therefore T is strictly contractive mapping on Ω with Lipschitz constant L .

It follows from (3.6) that

$$\begin{aligned} \mu_{2(n-5)f(2x)-8(n-5)f(x)}(t) &\geq \\ \eta_{(x, -x, x, -x, x, 0, \dots, 0)}(t) \end{aligned} \quad (4.6)$$

for all $x \in X$. It follows from (4.6) that

$$\mu_{\frac{f(2x)}{4}-f(x)}(t) \geq \eta_{(x, -x, x, -x, x, 0, \dots, 0)}((n-5)8t) \quad (4.7)$$

for all $x \in X$. Using (4.3) for the case $i = 0$, it reduce to

$$\mu_{\frac{f(2x)}{4}-f(x)}(t) \geq L\beta(x, t)$$

for all $x \in X$. Hence, we obtain

$$d(\mu_{Tf(x)-f(x)}) \geq L = L^{1-i} < \infty \quad (4.8)$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (4.7), we get

$$\begin{aligned} \mu_{\frac{f(x)}{4}-f\left(\frac{x}{2}\right)}(t) &\geq \eta_{\left(\frac{x}{2}, -\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}, 0, \dots, 0\right)}((n-5)8t) \\ &\quad (4.9) \end{aligned}$$

for all $x \in X$. Using (4.3) for the case $i = 1$, it reduce to

$$\begin{aligned} \mu_{4f\left(\frac{x}{2}\right)-f(x)}(t) &\geq \beta(x, t) \Rightarrow \mu_{Tf(x)-f(x)}(t) \geq \\ \beta(x, t) \end{aligned}$$

for all $x \in X$. Hence, we get

$$d(\mu_{Tf(x)-f(x)}) \geq L = L^{1-i} < \infty \quad (4.10)$$

for all $x \in X$. From (4.8) and (4.10), we can conclude

$$d(\mu_{Tf(x)-f(x)}) \geq L = L^{1-i} < \infty \quad (4.11)$$

for all $x \in X$. In order to prove $Q: X \rightarrow Y$ satisfies the functional equation (1.2), the remaining proof is similar by using Theorem 3.1. Since Q is unique fixed point of T in the set $\Delta = \{f \in \Omega/d(f, C) < \infty\}$. Finally, Q is a unique function such that

$$\mu_{f(x)-Q(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x, t)$$

for all $x \in X$ and $t > 0$. This completes the proof of the Theorem.

From the Theorem 4.1, we obtain the following Corollary concerning the stability for the functional equation (1.2).

Corollary 4.2. Suppose that a function $f: X \rightarrow Y$ satisfies the inequality

$$\mu_{Df(x_1, x_2, x_3, \dots, x_n)}(t) \geq \begin{cases} \eta_\varepsilon(t) \\ \eta_\varepsilon \sum_{i=1}^n \|x_i\|^s(t), s \neq 2 \\ \eta_\varepsilon (\prod_{i=1}^n \|x_i\|^s + \sum_{i=1}^n \|x_i\|^{ns})(t), s \neq \frac{2}{n} \end{cases} \quad (4.12)$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and all $t > 0$, where ε, s are constants with $\varepsilon > 0$, then there exists a unique quadratic mapping $Q: X \rightarrow Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \begin{cases} \eta_{\frac{\varepsilon}{(n-5)|2^{2-1}|}}(t) \\ \eta_{\frac{\varepsilon\|x\|^s}{(n-5)|2^{2-2s}|}}(t) \\ \eta_{\frac{\varepsilon\|x\|^{ns}}{(n-5)|2^{2-2ns}|}}(t) \end{cases} \quad (4.13)$$

for all $x \in X$ and all $t > 0$.

Proof. Setting

$$\mu_{Df(x_1, x_2, x_3, \dots, x_n)}(t) \geq \begin{cases} \eta_\varepsilon(t) \\ \eta_\varepsilon \sum_{i=1}^n \|x_i\|^s(t), \\ \eta_\varepsilon (\prod_{i=1}^n \|x_i\|^s + \sum_{i=1}^n \|x_i\|^{ns})(t), \end{cases}$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and all $t > 0$. Then

$$\eta_{(\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, \dots, \delta_i^k x_n)}(\delta_i^{2k} t) = \begin{cases} \eta_{\varepsilon \delta_i^{2k}}(t) \\ \eta_{\varepsilon \sum_{i=1}^n \|x_i\|^s \delta_i^{(2-s)k}}(t) \\ \eta_{\varepsilon (\prod_{i=1}^n \|x_i\|^s \delta_i^{(2-s)k} + \sum_{i=1}^n \|x_i\|^{ns} \delta_i^{(2-ns)k})}(t) \end{cases}$$

$$= \begin{cases} \rightarrow 1 \text{ as } k \rightarrow \infty \\ \rightarrow 1 \text{ as } k \rightarrow \infty \\ \rightarrow 1 \text{ as } k \rightarrow \infty \end{cases}$$

But we have $\beta(x, t) = \eta_{(\frac{x}{2}, -\frac{x}{2}, -\frac{x}{2}, 0, \dots, 0)}((n-5)t)$ has the

property $L \frac{1}{\delta_i^2} \beta(\delta_i x, t)$ for all $x \in X$ and $t > 0$. Now

$$\beta(x, t) = \begin{cases} \eta_{\frac{\varepsilon}{(n-5)}}(t) \\ \eta_{\frac{\varepsilon\|x\|^s}{2^s(n-5)}}(t) \\ \eta_{\frac{\varepsilon\|x\|^{ns}}{2^{ns}(n-5)}}(t) \end{cases}$$

$$L \frac{1}{\delta_i^2} \beta(\delta_i x, t) = \begin{cases} \eta_{\delta_i^{-2} \beta(x)}(t) \\ \eta_{\delta_i^{s-2} \beta(x)}(t) \\ \eta_{\delta_i^{ns-2} \beta(x)}(t) \end{cases}$$

By using Theorem 4.1, we prove the following six cases:

$L = 2^{-2}$ if $i = 0$ and $L = 2^2$ if $i = 1$

$L = 2^{s-2}$ for $s < 2$ if $i = 0$ and $L = 2^{2-s}$ for $s > 2$ if $i = 1$

$L = 2^{ns-2}$ for $s < \frac{2}{n}$ if $i = 0$ and $L = 2^{2-ns}$ for $s > \frac{2}{n}$ if $i = 1$

Case .1: $L = 2^{-2}$ if $i = 0$

$$\mu_{f(x)-Q(x)}(t) \geq L \frac{1}{\delta_i^2} \beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\varepsilon}{(n-5)(2^2-1)}\right)}(t)$$

Case.2: $L = 2^2$ if $i = 1$

$$\mu_{f(x)-Q(x)}(t) \geq L \frac{1}{\delta_i^2} \beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\varepsilon}{(n-5)(1-2^2)}\right)}(t)$$

Case.3: $L = 2^{s-2}$ for $s < 2$ if $i = 0$

$$\mu_{f(x)-Q(x)}(t) \geq L \frac{1}{\delta_i^2} \beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\varepsilon\|x\|^s}{(n-5)(2^2-2^s)}\right)}(t)$$

Case.4: $L = 2^{2-s}$ for $s > 2$ if $i = 1$

Case.5: $L = 2^{ns-2}$ for $s < \frac{2}{n}$ if $i = 0$

$$\mu_{f(x)-Q(x)}(t) \geq L \frac{1}{\delta_i^2} \beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\varepsilon\|x\|^{ns}}{(n-5)(2^2-2^{ns})}\right)}(t)$$

Case.6: $L = 2^{2-ns}$ for $s > \frac{2}{n}$ if $i = 1$

$$\mu_{f(x)-Q(x)}(t) \geq L \frac{1}{\delta_i^2} \beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\varepsilon\|x\|^{ns}}{(n-5)(2^{ns}-2^2)}\right)}(t)$$

Hence the proof is complete.

CONCLUSION: In this paper, we investigated the general solution and generalized Hyers-Ulam-Rassias stability of the (n)-dimensional quadratic functional equation in a random normed space. The main results establish that every approximate solution can be closely approximated by a unique exact quadratic solution under suitable control conditions. The stability results were obtained using both direct and fixed-point methods, demonstrating the effectiveness and flexibility of these approaches. The direct method provides explicit stability estimates, while the fixed-point method offers a systematic framework for establishing the existence and uniqueness of the corresponding exact solution. The results extend and enrich the existing stability theory of functional equations in random normed spaces and provide a useful foundation for further investigations of higher-dimensional functional equations and their applications in probabilistic and nonlinear analysis.

REFERENCES

1. J. Cho, T. M. Rassias, R. Saadati, stability of Functional Equations in Random Normed Spaces, Springer, New York, (2013).
2. M. Eshanghi Gordji, J. M. Rassias and M. Bavand Savadkouhi, Approximation of the Quadratic and Cubic Functional Equation in RN-Spaces, European J. pure. Appl. Math., Vol. 2(4), (2009), 494-507.
3. D. Mihet and V. Radu, On the Stability of the Additive Cauchy Functional Equation in Random Normed Spaces, J. Math. Analysis and Appl., Vol. 343(1), (2008), 567-572.
4. Mihet, R. Saadati, S. M. Vaezpour, The Stability of the Quartic Functional Equation in Random Normed Space, Acta. Appl. Math., Vol. 110, (2010), 799-803.
5. Renu Chugh, Ashish, On the Stability of Functional Equations in Random Normed Spaces, Int. J. Computer Appl., Vol. 45(11), (2012).
6. R. Saadati, S. M. Vaezpour and Y. J. cho, A Note to Paper “On the Stability of Cubic Mappings and Quartic Mappings in Random Normed Spaces”, J. Ineq. Appl., DOI: 10.1155/2009/214530, (2009).
7. Shaymaa Alshybami, S. Mansour Vaezpour, Reza Saadati, Generalized Hyers-Ulam Stability of Mixed Type Additive-Quadratic Functional Equation in Random Normed Spaces, J. Math. Analysis, vol. 8(5), (2017), 12-26.
8. A. N. Sherstner, On the Notation of a Random Normed Space, Doklady Akad. Nauk SSSR, Vol. 149, (1963), 280-283.
9. T. Z. Xu, J. M. Rassias, W. X. Xu, On Stability of a General Mixed Additive-Cubic Functional Equation in Random Normed Spaces, J. Ineq. Appl., Article ID: 328473, (2010).
10. Zhang, J. M. Rassias and R. Saadati, Stability of a Cubic Functional in Intuitionistic Random Normed Spaces, Appl. Math. Mech. Engl. Ed 31(1), (2010), 21-26.