



The Application of Group Theory on Planetary Motion

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ARTICLE INFO	ABSTRACT
Published Online: 08 September 2026 Corresponding Author: Ifan Setiawan Yuntoro KEYWORDS: Group; Cyclic Group; Planet; Rotation; Revolution	Group theory studies the algebraic structure known as groups. A group is a non-empty set with a binary operation that satisfies some axioms. These axioms include associativity, the existence of an identity element, and the existence of an inverse element. An Abelian group is a group that satisfies the commutative law. One of the most well-known examples of a group is the cyclic group $(\mathbb{Z}_n, +_n)$, whose elements repeat periodically after n steps under modular addition. On the other hand, planetary rotation and revolution are continuous dynamical processes that repeat periodically and are governed by gravitational mechanics. In this research, we construct a discrete mathematical model of planetary periodic motion. Since physical planetary motion is continuous, we discretize each planet's rotation and revolution period to the nearest positive integer n in an appropriate time unit, yielding a finite set of positional states $R_n = \{r_0, r_1, \dots, r_{n-1}\}$, where r_k represents the planet's position at time step k. We define the binary operation $r_a * r_b = r_{(a+b) \bmod n}$, which represents the composition of two successive time progressions modulo the period. We then verify that $(R_n, *)$ satisfies the group axioms, forming an Abelian group isomorphic to $(\mathbb{Z}_n, +_n)$. Our purpose here is to show some applications of algebra outside the classroom.

I. INTRODUCTION

A group is a non-empty set with binary operations and satisfies some axioms [1], [2]. So far, the application of mathematics, especially in abstract algebraic structures, has rarely been seen in a collection of things in everyday life. Abstract algebra is often perceived as a purely theoretical discipline, its applications extend broadly across the natural sciences [3]. Some of events that are happening in this world actually applied the principle of algebraic structure, especially group. Group theory is a fundamental tool for analysing symmetry in classical mechanics, quantum mechanics, atomic and molecular physics, and gauge theories [4].

We know that there are 8 planets on our solar system. All of these planets rotate on their axis and revolve around The Sun. This movement is actually a periodic motion [5], [6], [7], [8], [9], [10]. The periodic motion of each planet is shown in the table below [7], [8], [9], [10]

Table 1. The Rotation and Revolution of Planet

Planet	Rotation	Revolution
Mercury	58,6462 days	87,9690 days
Venus	243,018 days	224,701 days
Earth	23,9345 hours	365,257 days
Mars	24,6230 hours	686,971 days
Jupiter	9,92500 hours	11,8630 years
Saturn	10,6562 hours	29,4470 years
Uranus	17,2399 hours	84,0170 years
Neptune	16,1100 hours	164,791 years

Now, in this research, we want to show that a discrete mathematical model of planetary motion can be structured as a group. First, we construct a set from the planet's position at time step k . Then, we define the binary operation $r_a * r_b = r_{(a+b) \bmod n}$, which represents the composition of two successive time progressions modulo the period. Lastly, we will see that this group is isomorphic with the cyclic group $(\mathbb{Z}_n, +_n)$. From this research, we will learn that a group is not just a theory but also has an application on our day-to-day life.

It provides a concrete and accessible motivation of how abstract algebraic structures arise naturally from periodic physical phenomena, which may help students develop intuition for group theory beyond purely formal definitions. We hope this approach encourages further exploration of algebraic modeling in natural sciences.

II. GROUP THEORY

We recall the necessary definitions from group theory.

Definition 2.1. [11], [12] *The relation $' * '$ from $G \times G$ to G is called a binary operation if $' * '$ is a function of $G \times G$ to G .*

Definition 2.2. [2], [12] *Let G is a non-empty set and $' * '$ be a binary operation on G . The set G with binary operation $' * '$ is called a group if it satisfies the following axioms.*

1. *The $' * '$ operation is associative, that is, for all $a, b, c \in G$ satisfies $(a * b) * c = a * (b * c)$.*
2. *There exists an identity element $e \in G$, such that $e * a = a * e = a$ for all $a \in G$.*
3. *For every $a \in G$ there exists an inverse element $a^{-1} \in G$, such that $a * a^{-1} = a^{-1} * a = e$.*

Definition 2.3. [11], [13] *Let G is a group. Then G is called an Abelian group if it satisfies the commutative law, which is for every $a, b \in G$ to satisfy $a * b = b * a$.*

Definition 2.4. [13] *A group G is called cyclic if there is an element $a \in G$ such that for every $x \in G$, $x = a^n$ for an element $n \in \mathbb{Z}$. This element a is called a generator of G and denoted $G = \langle a \rangle$.*

Definition 2.5. [13-15] *Let $(G, \circ)$ and $(H, *)$ be groups, then $f: G \rightarrow H$ is a function. The function f is a group homomorphism if for all $a, b \in G$ satisfy $f(a \circ b) = f(a) * f(b)$.*

Definition 2.6. [13-15] *Let $(G, \circ)$ and $(H, *)$ be groups, then $f: G \rightarrow H$ is a group homomorphism. The group homomorphism $f: G \rightarrow H$ called as*

1. *monomorphism, if f is injective, that is if for any $a, b \in G$ with $f(a) = f(b)$, then $a = b$.*
2. *epimorphism, if f is surjective, that is if for every $y \in H$, there exists $x \in G$ such that $y = f(x)$.*
3. *isomorphism, if f is injective and surjective.*

III. RESULT AND DISCUSSION

In this research, we will see that planetary motion follows the principle of group. Before presenting the main results, we describe how discrete model is constructed from physical data. Planetary rotation and revolution are periodic motions, meaning that after a fixed amount of time, a planet returns to the same position. This natural periodicity is the starting point of our model. Because the motion repeats after full cycle, we can divide one complete rotation into a finite number of equally spaced time steps and label each step as a distinct positional state. The number of these states is determined directly by the rotation period of the planet. If a planet completes one full rotation in n time units, then there are exactly n distinct positional states before the motion repeats.

The physical rotation period T of a planet, as measured astronomically, is generally not an exact integer. For example, Earth's rotation period is approximately 23,93 hours, and Mars rotates in approximately 24,62 hours. To work with a finite set of positional states, we need n to be a positive integer, so we round T to the nearest integer in an appropriate time unit. We use hours for planets whose rotation period is less than seven days, and days for Mercury and Venus whose periods are much longer.

So, we can assume that Earth has 24 positions in a day, as follows, r_0 as position of Earth at 0 o'clock, r_1 as position of Earth at 1 o'clock, and so on until r_{23} as position of Earth at 23 o'clock. Thus, we have the set of 24 positions of Earth. Once the period n is established, we define the set of positional states as $R_n = \{r_0, r_1, \dots, r_{n-1}\}$, where r_k represents the planet's position at the end of time step k , measured from a fixed reference point r_0 . After n steps the planet completes one full rotation and returns to r_0 , so there is no state r_n , which is the same as r_0 .

The operation defines by $r_a * r_b = r_{(a+b) \bmod n}$ which simply means, if the planet is currently at the position reached after a steps, and we advance it by b more steps, the result is the position after $a + b$ steps in total, where we count modulo n because the motion is periodic. It is important to note that r_k is an abstract label for a positional state, not a physical coordinate or angle. The group structure we establish belongs to this discrete model, not to the continuous physical motion of the planet.

Proposition 3.1. *Let a planet have rotation period n time units. Define the set of positional states $R_n = \{r_0, r_1, \dots, r_{n-1}\}$ where r_k denotes the planet's positional state at time step k , for $k \in \{0, 1, \dots, n-1\}$. Define the operation $' * '$ by $r_a * r_b = r_{(a+b) \bmod n}$ for all $r_a, r_b \in R_n$. Then $(R_n, *)$ is an Abelian group.*

Proof. Let $r_a, r_b \in R_n$, where $a, b \in \{0, 1, \dots, n-1\}$. By definition, $r_a * r_b = r_{(a+b) \bmod n}$. The operation $(a + b) \bmod n$ returns the remainder when $a + b$ is divided by n , which by definition of the remainder always satisfies $0 \leq (a + b) \bmod n \leq n - 1$. Therefore, $r_{(a+b) \bmod n} \in R_n$. So, R_n is closed under $' * '$. Moreover, by definition of the remainder the value of $(a + b) \bmod n$ is uniquely determined by a and b , so $r_{(a+b) \bmod n}$ is unique. Hence, $' * '$ is well-defined.

Let $r_a, r_b, r_c \in R_n$, where $a, b, c \in \{0, 1, \dots, n-1\}$. By definition we have

$$\begin{aligned} (r_a * r_b) * r_c &= r_{(a+b) \bmod n} * r_c \\ &= r_{((a+b) \bmod n + c) \bmod n} \\ &= r_{(a+b+c) \bmod n} \\ &= r_{a+(b+c) \bmod n} \\ &= r_{a+((b+c) \bmod n)} \\ &= r_a * r_{(b+c) \bmod n} \\ &= r_a * (r_b * r_c) \end{aligned}$$

So, the operation $' * '$ is associative. We claim that $r_0 \in R_n$ is the identity element. For any $r_a \in R_n$,

$$r_a * r_0 = r_{(a+0) \bmod n} = r_{a \bmod n} = r_a$$

$$r_0 * r_a = r_{(0+a) \bmod n} = r_{a \bmod n} = r_a$$

So, the set R_n under the operation $' * '$ have an identity element.

Let $r_a \in R_n$.

Case 1. For $a = 0$, the inverse of $r_0 \in R_n$ is $r_0 \in R_n$, because

$$r_0 * r_0 = r_{(0+0) \bmod n} = r_{0 \bmod n} = r_0$$

So, $r_0^{-1} = r_0$.

Case 2. For $a \in \{1, 2, \dots, n-1\}$, the inverse of $r_a \in R_n$ is $r_{n-a} \in R_n$, because

$$r_a * r_{n-a} = r_{(a+(n-a)) \bmod n} = r_{n \bmod n} = r_0$$

$$r_{n-a} * r_a = r_{((n-a)+a) \bmod n} = r_{n \bmod n} = r_0$$

So, $r_a^{-1} = r_{n-a}$. Therefore, every element in R_n has an inverse element under the operation $' * '$. Let $r_a, r_b \in R_n$, where $a, b \in \{0, 1, \dots, n-1\}$. By definition we have

$$r_a * r_b = r_{(a+b) \bmod n}$$

$$= r_{(b+a) \bmod n}$$

$$= r_b * r_a$$

Hence, the operation $' * '$ is commutative. This proved $(R_n, *)$ is an Abelian group. ■

By analogy, the orbital period of a planet can also form a group structure.

Proposition 3.2. *Let a planet have revolution period m time units. Define the set of positional states $S_m = \{s_0, s_1, \dots, s_{m-1}\}$ where s_k denotes the planet's orbital positional state at time step k , for $k \in \{0, 1, \dots, m-1\}$. Define the operation $' \circ '$ by $s_a \circ s_b = s_{(a+b) \bmod m}$ for all $s_a, s_b \in S_m$. Then $(S_m, \circ)$ is an Abelian group.*

Proof. Let $s_a, s_b \in S_m$, where $a, b \in \{0, 1, \dots, m-1\}$. By definition, $s_a \circ s_b = s_{(a+b) \bmod m}$. The operation $(a+b) \bmod m$ returns the remainder when $a+b$ is divided by m , which by definition of the remainder always satisfies $0 \leq (a+b) \bmod m \leq m-1$. Therefore, $s_{(a+b) \bmod m} \in S_m$. So, S_m is closed under $' \circ '$. Moreover, by definition of the remainder the value of $(a+b) \bmod m$ is uniquely determined by a and b , so $s_{(a+b) \bmod m}$ is unique. Hence, $' \circ '$ is well-defined.

Let $s_a, s_b, s_c \in S_m$, where $a, b, c \in \{0, 1, \dots, m-1\}$. By definition we have

$$(s_a \circ s_b) \circ s_c = s_{(a+b) \bmod m} \circ s_c$$

$$= s_{((a+b) \bmod m) + c}$$

$$= s_{(a+b+c) \bmod m}$$

$$= s_{a+(b+c)}$$

$$= s_{a+((b+c) \bmod m)}$$

$$= s_a \circ s_{(b+c) \bmod m}$$

$$= s_a \circ (s_b \circ s_c)$$

So, the operation $' \circ '$ is associative. We claim that $s_0 \in S_m$ is the identity element. For any $s_a \in S_m$,

$$s_a \circ s_0 = s_{(a+0) \bmod m} = s_{a \bmod m} = s_a$$

$$s_0 \circ s_a = s_{(0+a) \bmod m} = s_{a \bmod m} = s_a$$

So, the set S_m under the operation $' \circ '$ have an identity element.

Let $s_a \in S_m$.

Case 1. For $a = 0$, the inverse of $s_0 \in S_m$ is $s_0 \in S_m$, because

$$s_0 \circ s_0 = s_{(0+0) \bmod m} = s_{0 \bmod m} = s_0$$

So, $s_0^{-1} = s_0$.

Case 2. For $a \in \{1, 2, \dots, m-1\}$, the inverse of $s_a \in S_m$ is $s_{m-a} \in S_m$, because

$$s_a \circ s_{m-a} = s_{(a+(m-a)) \bmod m} = s_{m \bmod m} = s_0$$

$$s_{m-a} \circ s_a = s_{((m-a)+a) \bmod m} = s_{m \bmod m} = s_0$$

So, $s_a^{-1} = s_{m-a}$. Therefore, every element in S_m has an inverse element under the operation $' \circ '$. Let $s_a, s_b \in S_m$, where $a, b \in \{0, 1, \dots, m-1\}$. By definition we have

$$s_a \circ s_b = s_{(a+b) \bmod m}$$

$$= s_{(b+a) \bmod m}$$

$$= s_b \circ s_a$$

Hence, the operation $' \circ '$ is commutative. This proved $(S_m, \circ)$ is an Abelian group. ■

Furthermore, we can see that the set R_n is isomorphic to the set $\mathbb{Z}_n$.

Proposition 3.3. *The group $(R_n, *)$ is isomorphic to $(\mathbb{Z}_n, +_n)$.*

Proof. Define the assignment $f: \mathbb{Z}_n \rightarrow R_n$ for all $k \in \mathbb{Z}_n$, $f(k) = r_k$. We must show that each element $k \in \mathbb{Z}_n$ determines a unique image $r_k \in R_n$. Since $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ and $R_n = \{r_0, r_1, \dots, r_{n-1}\}$, every $k \in \mathbb{Z}_n$ corresponds to exactly one positional state $r_k \in R_n$. Moreover, if $k = l$ in $\mathbb{Z}_n$, which is, $k \equiv l \bmod n$, then $f(k) = r_k$ and $f(l) = r_l$ shows the same positional state, hence $f(k) = f(l)$. Thus, f is well-defined, then f is a function. Let $a, b \in \mathbb{Z}_n$. By definition of the mapping f and the definition of operation $' * '$ we have

$$f(a +_n b) = f((a+b) \bmod n)$$

$$= r_{(a+b) \bmod n}$$

$$= r_a * r_b$$

$$= f(a) * f(b)$$

Hence, f is a group homomorphism. Let $r_k \in R_n$ be arbitrary. By definition of R_n , every positional state r_k have exactly one $k \in \{0, 1, \dots, n-1\}$. Since $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$, this means $k \in \mathbb{Z}_n$. Therefore, every $r_k \in R_n$ there exists $k \in \mathbb{Z}_n$ such that $f(k) = r_k$. So, f is surjective. Let $a, b \in \mathbb{Z}_n$ and suppose $f(a) = f(b)$. Then $r_a = r_b$. Now recall that the elements of R_n are defined as distinct positional states, which means when r_a and r_b represent the same element of R_n if and only if $a = b$. So, f is injective. This proved, f is an isomorphism. Therefore, $(R_n, *) \cong (\mathbb{Z}_n, +_n)$. ■

By analogy, the set of revolution period of a planet S_m also isomorphic to the set $\mathbb{Z}_m$.

Proposition 3.4. *The group $(S_m, \circ)$ is isomorphic to $(\mathbb{Z}_m, +_m)$.*

Proof. Define the assignment $g: \mathbb{Z}_m \rightarrow S_m$ for all $p \in \mathbb{Z}_m$, $g(p) = s_p$. We must show that each element $p \in \mathbb{Z}_m$ determines a unique image $s_p \in S_m$. Since $\mathbb{Z}_m = \{0, 1, \dots, m-1\}$ and $S_m = \{s_0, s_1, \dots, s_{m-1}\}$, every $p \in \mathbb{Z}_m$

corresponds to exactly one orbital positional state $s_p \in S_m$. Moreover, if $p = q$ in $\mathbb{Z}_m$, which is, $p \equiv q \pmod m$, then $g(p) = s_p$ and $g(q) = s_q$ shows the same orbital positional state, hence $g(p) = g(q)$. Thus, g is well-defined, then g is a function. Let $a, b \in \mathbb{Z}_m$. By definition of the mapping g and the definition of operation $' \circ '$ we have

$$g(a+_mb) = g((a+b) \pmod m)$$

$$\begin{aligned} &= S_{(a+b) \pmod m} \\ &= s_a \circ s_b \\ &= g(a) \circ g(b) \end{aligned}$$

Hence, g is a group homomorphism. Let $s_p \in S_m$ be arbitrary. By definition of S_m , every orbital positional state s_p have exactly one $p \in \{0, 1, \dots, m-1\}$. Since $\mathbb{Z}_m = \{0, 1, \dots, m-1\}$, this means $p \in \mathbb{Z}_m$. Therefore, every $s_p \in S_m$ there exists $p \in \mathbb{Z}_m$ such that $g(p) = s_p$. So, g is surjective. Let $a, b \in \mathbb{Z}_m$ and suppose $g(a) = g(b)$. Then $s_a = s_b$. Now recall that the elements of S_m are defined as distinct orbital positional states, which means when s_a and s_b represent the same element of S_m if and only if $a = b$. So, g is injective. This proved, g is an isomorphism. Therefore, $(S_m, \circ) \cong (\mathbb{Z}_m, +_m)$. ■

The actual state of a planet is determined simultaneously by its rotational and orbital position. We model this combined motion using the direct product.

Proposition 3.5. *Let a planet have rotation period n and revolution period m . Define the combined positional set $R_n \times S_m = \{(r_a, s_p) | a \in \mathbb{Z}_n, p \in \mathbb{Z}_m\}$ and define the operation $' \otimes '$ by $(r_a, s_p) \otimes (r_b, s_q) = (r_a * r_b, s_p \circ s_q) = (r_{(a+b) \pmod n}, s_{(p+q) \pmod m})$ for all $(r_a, s_p), (r_b, s_q) \in R_n \times S_m$. Then $(R_n \times S_m, \otimes)$ is an Abelian group.*

Proof. Let $(r_a, s_p), (r_b, s_q) \in R_n \times S_m$. By definition of operation $' \otimes '$ we have

$$\begin{aligned} (r_a, s_p) \otimes (r_b, s_q) &= (r_a * r_b, s_p \circ s_q) \\ &= (r_{(a+b) \pmod n}, s_{(p+q) \pmod m}) \end{aligned}$$

Since R_n and S_m is a group, therefore $r_{(a+b) \pmod n} \in R_n$ and $s_{(p+q) \pmod m} \in S_m$. So, $(r_{(a+b) \pmod n}, s_{(p+q) \pmod m}) \in R_n \times S_m$. Hence, $R_n \times S_m$ is closed under $' \otimes '$. Furthermore, the operation $' * '$ is well-defined on R_n and the operation $' \circ '$ is well-defined on S_m . So, $r_{(a+b) \pmod n} \in R_n$ and $s_{(p+q) \pmod m} \in S_m$ is unique. Therefore, $\otimes$ is well-defined.

Let $(r_a, s_p), (r_b, s_q), (r_c, s_r) \in R_n \times S_m$. By definition of $' \otimes '$ and because $' * '$ and $' \circ '$ are associative we have

$$\begin{aligned} &((r_a, s_p) \otimes (r_b, s_q)) \otimes (r_c, s_r) \\ &= (r_a * r_b, s_p \circ s_q) \otimes (r_c, s_r) \\ &= ((r_a * r_b) * r_c, (s_p \circ s_q) \circ s_r) \\ &= (r_a * (r_b * r_c), s_p \circ (s_q \circ s_r)) \end{aligned}$$

$$\begin{aligned} &= (r_a, s_p) \\ &\otimes (r_b * r_c, s_q \circ s_r) \\ &= (r_a, s_p) \\ &\otimes ((r_b, s_q) \otimes (r_c, s_r)) \end{aligned}$$

So, $' \otimes '$ is associative. We claim that $(r_0, s_0) \in R_n \times S_m$ is the identity element. Since r_0 is the identity element of R_n and s_0 is the identity element of S_m . For any $(r_a, s_p) \in R_n \times S_m$,

$$\begin{aligned} (r_a, s_p) \otimes (r_0, s_0) &= (r_a * r_0, s_p \circ s_0) = (r_a, s_p) \\ (r_0, s_0) \otimes (r_a, s_p) &= (r_0 * r_a, s_0 \circ s_p) = (r_a, s_p) \end{aligned}$$

So, the set $R_n \times S_m$ under the operation $' \otimes '$ have an identity element.

Let $(r_a, s_p) \in R_n \times S_m$.

Case 1. For $a = p = 0$, the inverse of $(r_0, s_0) \in R_n \times S_m$ is $(r_0, s_0) \in R_n \times S_m$, because $r_0^{-1} = r_0$ in R_n and $s_0^{-1} = s_0$ in S_m . Therefore

$$(r_0, s_0) \otimes (r_0, s_0) = (r_0 * r_0, s_0 \circ s_0) = (r_0, s_0)$$

So, $(r_0, s_0)^{-1} = (r_0, s_0)$.

Case 2. For $a, p \in \{1, 2, \dots, m-1\}$, the inverse of $(r_a, s_p) \in R_n \times S_m$ is $(r_{n-a}, s_{m-p}) \in R_n \times S_m$, because $r_a^{-1} = r_{n-a}$ in R_n and $s_p^{-1} = s_{m-p}$ in S_m . Therefore

$$\begin{aligned} (r_a, s_p) \otimes (r_{n-a}, s_{m-p}) &= (r_a * r_{n-a}, s_p \circ s_{m-p}) = (r_0, s_0) \\ (r_{n-a}, s_{m-p}) \otimes (r_a, s_p) &= (r_{n-a} * r_a, s_{m-p} \circ s_p) = (r_0, s_0) \end{aligned}$$

So, $(r_a, s_p)^{-1} = (r_{n-a}, s_{m-p})$. Therefore, every element in $R_n \times S_m$ has an inverse element under the operation $' \otimes '$.

Let $(r_a, s_p), (r_b, s_q) \in R_n \times S_m$. By definition of operation $' \otimes '$ and because the operation $' * '$ in R_n and the operation $' \circ '$ in S_m are commutative we have

$$\begin{aligned} (r_a, s_p) \otimes (r_b, s_q) &= (r_a * r_b, s_p \circ s_q) \\ &= (r_b * r_a, s_q \circ s_p) \\ &= (r_b, s_q) \otimes (r_a, s_p) \end{aligned}$$

Hence, the operation $' \otimes '$ is commutative. This proved $(R_n \times S_m, \otimes)$ is an Abelian group. ■

Example 3.6. Planet Earth has a rotation period of approximately 23,93 hours, which we round to $n = 24$. This means that we model Earth's rotational motion as completing one full cycle every 24-time steps, where each time step represent one hour. We define the set of positional states of Earth's rotation as

$$R_{24} = \{r_0, r_1, r_2, \dots, r_{23}\}$$

where r_k represents Earth's positional state at the end of hour k , measured from a fixed reference point r_0 . For example, r_0 is the reference point at UTC+00:00, r_6 is the Earth's position after rotating for 6 hours from the reference point, r_{12} is the Earth's position after rotating for 12 hours from the reference point, and r_{23} is the position one hour before completing a full rotation. After 24 hours, Earth returns to r_0 , so there is no state r_{24} , it coincides with r_0 .

The binary operation $' * '$ on R_{24} is defined by $r_a * r_b = r_{(a+b) \pmod{24}}$. To see what this means concretely,

consider two examples. First, suppose Earth is at position r_7 and we advance it by 5 more hours, then

$$r_7 * r_5 = r_{(7+5) \bmod 24} = r_{12}$$

This means at first Earth already rotating for 7 hours then advancing 5 hours gives Earth rotating for 12 hours. Second, suppose Earth is at position r_{20} and we advance it by 8 more hours, then

$$r_{20} * r_8 = r_{(20+8) \bmod 24} = r_4$$

This means at first Earth already rotating for 20 hours then advancing 8 hours passes through midnight (r_0) and arrives at hour 4 of the next day, which means Earth finish one cycle and continue rotating for 4 hours.

By Proposition 3.1, the tuple $(R_{24}, *)$ satisfy all group axioms. The identity element is r_0 , which is the Earth's reference starting point. The inverse of each element r_k is r_{24-k} , which means after advancing k hours, the Earth needs $24 - k$ more hours to complete exactly one full rotation and returns to r_0 . By Proposition 3.3, the group $(R_{24}, *)$ is isomorphic to $(\mathbb{Z}_{24}, +_{24})$ via the map $f: \mathbb{Z}_{24} \rightarrow R_{24}$ defined by $f(k) = r_k$ for each $k \in \mathbb{Z}_{24}$. This isomorphism tells us that the algebraic structure of Earth's rotational positional model is exactly the same as the structure of modular arithmetic modulo 24, the same system used in everyday timekeeping.

Example 3.7. Earth has a rotation period of $n = 24$ hours and a revolution period of $m = 365$ days. Following the construction of Proposition 3.5, we define the combined positional set as

$$R_{24} \times S_{365} = \{(r_a, s_p) | a \in \mathbb{Z}_{24}, p \in \mathbb{Z}_{365}\}$$

where r_a represents Earth's rotational positional state at hour a within a single day, and s_p represents Earth's orbital positional state at day p within a single year. For example, the element (r_6, s_{90}) represents the state where Earth is at hour 6 of its daily rotation and simultaneously at day 90 of its yearly orbit around the Sun. The binary operation ' $\otimes$ ' on $R_{24} \times S_{365}$ is defined by

$$(r_a, s_p) \otimes (r_b, s_q) = (r_{(a+b) \bmod 24}, s_{(p+q) \bmod 365})$$

for each $(r_a, s_p), (r_b, s_q) \in R_{24} \times S_{365}$. By Proposition 3.5, the tuple $(R_{24} \times S_{365}, \otimes)$ is an Abelian group. For example, take $(r_{20}, s_{300}), (r_{10}, s_{80}) \in R_{24} \times S_{365}$, by definition of the operation ' $\otimes$ ' we have

$$\begin{aligned} (r_{20}, s_{300}) \otimes (r_{10}, s_{80}) &= (r_{(20+10) \bmod 24}, s_{(300+80) \bmod 365}) \\ &= (r_6, s_{15}) \end{aligned}$$

The first component $r_{(20+10) \bmod 24} = r_6$ means that advancing 20 hours and then 10 more hours passes through one full rotation and arrives at hour 6 of the next cycle. The second component $s_{(300+80) \bmod 365} = s_{15}$ means that advancing 300 days and then 80 more days passes through one complete revolution around the Sun and arrives at day 15 of the next year.

The identity element is $(r_0, s_0) \in R_{24} \times S_{365}$ representing the reference rotational position and the reference orbital position simultaneously. The inverse of any element $(r_a, s_p) \in R_{24} \times S_{365}$ is $(r_{24-a}, s_{365-p}) \in R_{24} \times S_{365}$, because advancing a hours then $24 - a$ more hours

complete one full rotation, and advancing p days then $365 - p$ more days complete one full revolution.

IV. CONCLUSION

From the results above, we can conclude that the discrete positional model of planetary rotation and revolution, constructed by defining a finite set of positional states and a modular composition operation, forms an Abelian group isomorphic to the group $(\mathbb{Z}_n, +_n)$. As a concrete example, the 24-hour rotation of the Earth follows the group $\mathbb{Z}_{24}$, similar constructions apply to all eight planets of the solar system by analogy. This research shows that the elementary concepts from abstract algebra can serve as a meaningful modelling language for periodic phenomena in the natural world, providing a concrete example of how group theory extends beyond purely theoretical settings.

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