

A Study on Strong and Weak Schauder Bases in Neutrosophic Normed Spaces

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Abstract

In this study, we explore the concept of Strong and Weak Neutrosophic Schauder bases in the context of functional analysis. Neutrosophic logic, which generalizes fuzzy and intuitionistic fuzzy logic, allows for handling indeterminacies. We extend this to Schauder bases, which are fundamental tools in the study of Banach spaces, by incorporating neutrosophic components to capture uncertainty, imprecision, and vagueness in the approximation of elements within these spaces. We define both strong and weak Neutrosophic Schauder bases and investigate their properties, convergence behavior, and relationships with classical Schauder bases. Our results contribute to the understanding of approximation theory in spaces where uncertainty plays a critical role, offering new insights into both theoretical and applied mathematics, particularly in areas such as decision theory, optimization, and computational analysis where neutrosophic logic is increasingly relevant.

Keywords: Neutrosophic Normed Space, Strong and Weak Schauder bases, Separability, Approximation Property.

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1. Introduction and Preliminaries

Lotfi A. Zadeh[12] invented fuzzy set theory in 1965. In order to address the idea of partial truth where truth values vary from entirely true to entirely false he put out this

hypothesis. When handling uncertain or imprecise data, which traditional binary logic was inadequate to handle, this method proved especially helpful. Krassimir T. Atanassov[1][2] presented intuitionistic fuzzy set theory in 1986. With the addition of a degree of membership and a degree of non-membership, this theory expands upon the traditional fuzzy set theory. Florentin Smarandache[10] introduced the concept of neutrosophic sets as an expansion of the intuitionistic fuzzy set. He introduced the idea of the neutrosophic set (NS) and the degree of indeterminacy as independent components in his 1995 manuscript, which was published in 1998. This makes it more effective in applications where judgements need to take incomplete or uncertain information into account, as it permits a representation of uncertainty and imprecision compared to normal fuzzy sets. Fuzzy norms on a linear space were later developed by Chun- Kai Cheng and John N. Moderson. Saadati and Park[8] presented the idea of intuitionistic fuzzy normed space in 2006. Lately, Jeyaraman and Jenifer[3][4][5] developed the idea of neutrosophic normed space. Palomares et al. and Yılmaz[7][11] have recently investigated the idea of Schauder basis and its applications. This idea was developed by Mursaleen, Karakaya and Mohiuddine[6] in intuitionistic fuzzy normed spaces.

In this work, we define and examine the notions of approximation property, separability, and Schauder basis in neutrosophic normed spaces, and we prove some associated findings. Here, we also present several fascinating instances that make use of classical sequence spaces $\ell_p (1 \leq p < \infty)$.

The fundamental definitions used in this article are listed below.

Definition 1.1.[4] The 7-tuple $(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$ is called Neutrosophic Normed space $[\mathcal{NN}\mathfrak{S}]$ if $\mathcal{X}$ is a linear space, $*$ is a continuous t-norm, $\diamond$ and $\odot$ are continuous t-conorm, ϱ, v and ϖ are fuzzy sets on $\mathcal{X} \times (0, \infty)$ fulfils the following conditions:

For every $x, y \in \mathcal{X}$ and $s, \lambda > 0$;

- a) $0 \leq \varrho(x, \lambda) \leq 1; 0 \leq v(x, \lambda) \leq 1; 0 \leq \varpi(x, \lambda) \leq 1;$
- b) $\varrho(x, \lambda) > 0,$
- c) $\varrho(x, \lambda) = 1$ iff if $x = 0,$
- d) $\varrho(\alpha x, \lambda) = \varrho\left(x, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0,$
- e) $\varrho(x, \lambda) * \varrho(y, s) \leq \varrho(x + y, \lambda + s),$
- f) $\varrho(x, \lambda) : (0, \infty) \rightarrow [0, 1]$ is continuous,
- g) $\lim_{\lambda \rightarrow \infty} \varrho(x, \lambda) = 1$ and $\lim_{\lambda \rightarrow 0} \varrho(x, \lambda) = 0,$

- h) $v(x, \lambda) < 1$,
- i) $v(x, \lambda) = 0$ iff $x = 0$,
- j) $v(\alpha x, \lambda) = v\left(x, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$,
- k) $v(x, \lambda) \diamond v(y, s) \geq v(x + y, \lambda + s)$,
- l) $v(x, \lambda) : (0, \infty) \rightarrow [0, 1]$ is continuous,
- m) $\lim_{\lambda \rightarrow \infty} v(x, \lambda) = 0$ and $\lim_{\lambda \rightarrow 0} v(x, \lambda) = 1$,
- n) $\varpi(x, \lambda) < 1$,
- o) $\varpi(x, \lambda) = 0$ iff $x = 0$,
- p) $\varpi(\alpha x, \lambda) = \varpi\left(x, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$,
- q) $\varpi(x, \lambda) \odot \varpi(y, s) \geq \varpi(x + y, \lambda + s)$
- r) $\varpi(x, \lambda) : (0, \infty) \rightarrow [0, 1]$ is continuous,
- s) $\lim_{\lambda \rightarrow \infty} \varpi(x, \lambda) = 0$ and $\lim_{\lambda \rightarrow 0} \varpi(x, \lambda) = 1$,

Then (ϱ, v, ϖ) is known as Neutrosophic Normed $[NN]$.

Example 1.2.[4] Let $(X, \|\cdot\|)$ be a NN , $p * q = pq$, $p \diamond q = \min\{p + q, 1\}$ and $p \odot q = \min\{p + q, 1\}$ for all $p, q \in [0, 1]$. For all $x \in X$ and every $\lambda > 0$, consider

$$\varrho(x, \lambda) = \begin{cases} \frac{\lambda}{\lambda + \|x\|} & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda \leq 0; \end{cases} \quad v(x, \lambda) = \begin{cases} \frac{\|x\|}{\lambda + \|x\|} & \text{if } \lambda > 0, \\ 1 & \text{if } \lambda \leq 0; \end{cases} \quad \varpi(x, \lambda) = \begin{cases} \frac{\|x\|}{\lambda} & \text{if } \lambda > 0, \\ 1 & \text{if } \lambda \leq 0; \end{cases}$$

Then $(X, \varrho, v, \varpi, *, \diamond, \odot)$ is a NN .

Definition 1.3. Let $(X, \varrho, v, \varpi, *, \diamond, \odot)$ be a NN with the property that $\varrho(x, \lambda) > 0$, $v(x, \lambda) < 1$ and $\varpi(x, \lambda) < 1$ implies $x = 0$ for all $\lambda \in \mathbb{R}$. Also, let $\|x\|_\alpha = \inf\{\lambda > 0 : \varrho(x, \lambda) \geq \alpha, v(x, \lambda) \leq 1 - \alpha \text{ and } \varpi(x, \lambda) \leq 1 - \alpha\}$, for all $\alpha \in (0, 1)$. Then $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms on X . These are referred to as α -norms on X corresponding to $NN(\varrho, v, \varpi)$.

2. Topological Concepts with Weak and Strong in NN

Definition 2.1. Let $(X, \varrho, v, \varpi, *, \diamond, \odot)$ be a NN . Then, a sequence (x_m) is said to be

- (i) Weakly neutrosophic convergent($\mathcal{WNC}$) to $x \in X$ iff, for every $\varepsilon > 0$ and $\alpha \in (0,1)$, $\exists$ some $j_0 = j_0(\alpha, \varepsilon) : \varrho(x_m - x, \varepsilon) \geq 1 - \alpha$, $v(x_m - x, \varepsilon) \leq \alpha$ and $\varpi(x_m - x, \varepsilon) \leq \alpha$ for all $j \geq j_0$.

In particular, $x_m \xrightarrow{\mathcal{WNC}} x$ is written.

- (ii) Strongly neutrosophic convergent($\mathcal{SNC}$) to $x \in X$ iff, for every $\alpha \in (0,1)$, $\exists$ some $j_0 = j_0(\alpha) : \varrho(x_m - x, \lambda) \geq 1 - \alpha$, $v(x_m - x, \lambda) \leq \alpha$ and $\varpi(x_m - x, \lambda) \leq \alpha$ for all $\lambda > 0$.

In particular, $x_m \xrightarrow{\mathcal{SNC}} x$ is written.

Through α -norms, the ($\mathcal{WNC}$) and ($\mathcal{SNC}$)-limit are characterised by the following finding.

Proposition 2.2. Let $(\mathfrak{A}, \varrho_1, v_1, \varpi_1, *, \diamond, \odot)$ and $(\mathfrak{B}, \varrho_2, v_2, \varpi_2, *, \diamond, \odot)$ be two $\mathfrak{NN}\mathfrak{S}$ s and $f: \mathfrak{A} \rightarrow \mathfrak{B}$ be a mapping. Then

- (i) $(\mathcal{WNC}) - \lim_{x \rightarrow x_0} f(x) = \mathfrak{L}$ iff, for every $\alpha \in (0,1)$ $\lim_{\|x-x_0\|_\alpha^{(1)} \rightarrow 0} \|f(x) - \mathfrak{L}\|_\alpha^{(2)} = 0$
- (ii) $(\mathcal{SNC}) - \lim_{x \rightarrow x_0} f(x) = \mathfrak{L}$ iff, $\lim_{\|x-x_0\|_\alpha^{(1)} \rightarrow 0} \|f(x) - \mathfrak{L}\|_\alpha^{(2)} = 0$ uniformly in α ,

where $\|\cdot\|_\alpha^{(1)}$ and $\|\cdot\|_\alpha^{(2)}$ are α -norms of the $\mathfrak{NN}\mathfrak{S}$ s $(\varrho_1, v_1, \varpi_1)$ and $(\varrho_2, v_2, \varpi_2)$, respectively.

Proof. Now, we establish case (ii). Assume that $(\mathcal{SNC}) - \lim_{x \rightarrow x_0} f(x) = \mathfrak{L}$. For a given $\varepsilon > 0$, $\exists$ some $\vartheta = \vartheta(\varepsilon) > 0$:

$$\varrho_2(f(x) - \mathfrak{L}, \varepsilon) \geq \varrho_1(x - x_0, \vartheta),$$

$$v_2(f(x) - \mathfrak{L}, \varepsilon) \leq v_1(x - x_0, \vartheta),$$

$$\varpi_2(f(x) - \mathfrak{L}, \varepsilon) \leq \varpi_1(x - x_0, \vartheta), \text{ for all } x \in \mathfrak{A}.$$

For each $\alpha \in (0,1)$, if

$$\|x - x_0\|_\alpha^{(1)} = \inf\{\lambda > 0: \varrho_1(x - x_0, \lambda) \geq \alpha, v_1(x - x_0, \lambda) \leq 1 - \alpha \text{ and } \varpi_1(x - x_0, \lambda) \leq 1 - \alpha\} < \vartheta \text{ then } \varrho_1(x - x_0, \vartheta) \geq \alpha, v_1(x - x_0, \vartheta) \leq 1 - \alpha \text{ and } \varpi_1(x - x_0, \vartheta) \leq 1 - \alpha.$$

Hence $\varrho_2(f(x) - \mathfrak{L}, \varepsilon) \geq \alpha$, $v_2(f(x) - \mathfrak{L}, \varepsilon) \leq 1 - \alpha$ and $\varpi_2(f(x) - \mathfrak{L}, \varepsilon) \leq 1 - \alpha$ and so that $\|f(x) - \mathfrak{L}\|_\alpha^{(2)} < \varepsilon$.

Because ϑ is independent of α , we obtain $\lim_{\|x-x_0\|_\alpha^{(1)}} \|f(x) - \mathfrak{L}\|_\alpha^{(2)} = 0$ uniformly in α .

Conversely, let $\lim_{\|x-x_0\|_\alpha^{(1)}} \|f(x) - \mathfrak{L}\|_\alpha^{(2)} = 0$ uniformly in α .

Given $\varepsilon > 0$, $\exists$ some $\vartheta = \vartheta(\varepsilon) > 0$: $\|x - x_0\|_\alpha^{(1)} < \vartheta$ implies $\|f(x) - \mathfrak{L}\|_\alpha^{(2)} < \varepsilon \forall x \in \mathcal{X}$ and $\alpha \in (0,1)$. Decide on some $\lambda < \varrho_1(x - x_0, \vartheta)$, $\lambda > \nu_1(x - x_0, \vartheta)$ and $\lambda > \varpi_1(x - x_0, \vartheta)$.

Since $\varrho_1(x - x_0, \vartheta) = \sup\{\alpha \in (0,1) : \|x - x_0\|_\alpha^{(1)} < \vartheta\}$, there are some $\alpha_0 \in (0,1)$ such that $\lambda < \alpha_0$ and $\|x - x_0\|_{\alpha_0}^{(1)} < \vartheta$. Hence $\|f(x) - \mathfrak{L}\|_{\alpha_0}^{(2)} < \varepsilon$ by the hypothesis, that is,

$$\varrho_2(f(x) - \mathfrak{L}, \varepsilon) \geq \alpha_0 > \lambda, \nu_2(f(x) - \mathfrak{L}, \varepsilon) \leq 1 - \alpha_0 < 1 - \lambda,$$

$$\varpi_2(f(x) - \mathfrak{L}, \varepsilon) \leq 1 - \alpha_0 < 1 - \lambda.$$

So, we get $\varrho_2(f(x) - \mathfrak{L}, \varepsilon) \geq \varrho_1(x - x_0, \vartheta)$, $\nu_2(f(x) - \mathfrak{L}, \varepsilon) \leq \nu_1(x - x_0, \vartheta)$ and $\varpi_2(f(x) - \mathfrak{L}, \varepsilon) \leq \varpi_1(x - x_0, \vartheta)$.

Proposition 2.3. Let $(\mathcal{X}, \varrho, \nu, \varpi, *, \diamond, \odot)$ be a $\mathfrak{NN}\mathfrak{S}$ and let (x_i) be a sequence in the $\mathfrak{NN}\mathfrak{S}$ $(\mathcal{X}, \varrho, \nu, \varpi, *, \diamond, \odot)$. Then

$$(i) \quad x_i \xrightarrow{\mathcal{WNC}} x \text{ iff } \lim_{i \rightarrow \infty} \|x_i - x\|_\alpha = 0 \text{ for every } \alpha \in (0,1)$$

$$(ii) \quad x_i \xrightarrow{\mathcal{SN}} x \text{ iff } \lim_{i \rightarrow \infty} \|x_i - x\|_\alpha = 0 \text{ uniformly in } \alpha,$$

where $\|\cdot\|_\alpha$ are α -norms of the $\mathfrak{NN}\mathfrak{S}$ (ϱ, ν, ϖ) .

The proof is obvious from Proposition (2.2).

Definition 2.4. Let $(\mathcal{X}, \varrho, \nu, \varpi, *, \diamond, \odot)$ be a $\mathfrak{NN}\mathfrak{S}$. The $\mathcal{SN}(\mathcal{WN})$ -closure of a subset $\mathcal{Y}$ in $\mathfrak{NN}\mathfrak{S}(\mathcal{X}, \varrho, \nu, \varpi, *, \diamond, \odot)$ is the set of all $x \in \mathcal{X}$, $\exists$ a sequence $(x_n) \in \mathcal{Y} : x_n \xrightarrow{\mathcal{SN}(\mathcal{WN})} x$. Here, we write $\bar{\mathcal{Y}}^{\mathcal{SN}}(\bar{\mathcal{Y}}^{\mathcal{WN}})$. $\mathcal{Y}$ is said to be $\mathcal{SN}(\mathcal{WN})$ -closed whenever $\bar{\mathcal{Y}}^{\mathcal{SN}}(\bar{\mathcal{Y}}^{\mathcal{WN}}) = \mathcal{Y}$.

It's simple to observe that $\bar{\mathcal{Y}}^{\mathcal{SN}} \subseteq \bar{\mathcal{Y}}^{\mathcal{WN}}$.

Example 2.5. Let $(\mathcal{X}, \varrho, \nu, \varpi, *, \diamond, \odot)$ be a $\mathfrak{NN}\mathfrak{S}$. Let $\mathcal{X} = \mathbb{R}$ and

$$\varrho(\mathfrak{x}, \lambda) = \begin{cases} \frac{\lambda - |\mathfrak{x}|}{\lambda + |\mathfrak{x}|} & \text{if } \lambda > |\mathfrak{x}|, \\ 0 & \text{if } \lambda \leq |\mathfrak{x}|; \end{cases} \quad v(\mathfrak{x}, \lambda) = \begin{cases} \frac{2|\mathfrak{x}|}{\lambda + |\mathfrak{x}|} & \text{if } \lambda > |\mathfrak{x}|, \\ 1 & \text{if } \lambda \leq |\mathfrak{x}|, \end{cases}$$

$$\varpi(\mathfrak{x}, \lambda) = \begin{cases} \frac{2|\mathfrak{x}|}{\lambda - |\mathfrak{x}|} & \text{if } \lambda > |\mathfrak{x}|, \\ 1 & \text{if } \lambda \leq |\mathfrak{x}|, \end{cases}$$

on $\mathcal{X}$. Let $T_{\mathcal{X}} = \{\mathfrak{x} \in \mathcal{X} : |\mathfrak{x}| < 1\}$ and we prove that $\bar{T}_{\mathcal{X}}^{\mathcal{WN}} = \mathcal{Y}_{\mathcal{X}} = \{\mathfrak{x} \in \mathcal{X} : |\mathfrak{x}| \leq 1\}$. For every $\mathfrak{x} \in \mathcal{Y}_{\mathcal{X}}$, $\exists$ a sequence $(\mathfrak{x}_n) \subset T_{\mathcal{X}} : \|\mathfrak{x}_n - \mathfrak{x}\|_{\alpha} \rightarrow 0$ as $n \rightarrow \infty$, for each $\alpha \in (0,1)$.

To achieve this, take $\mathfrak{x}_n = \left(1 - \frac{1}{n+1}\right)\mathfrak{x}$ since each $\mathfrak{x}_n \in T_{\mathcal{X}}$ and

$$\|\mathfrak{x}_n - \mathfrak{x}\|_{\alpha} = \frac{1+\alpha}{1-\alpha} |\mathfrak{x}_n - \mathfrak{x}| < \left(\frac{1+\alpha}{1-\alpha}\right) \frac{1}{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for each } \alpha \in (0,1).$$

However, $\bar{T}_{\mathcal{X}}^{\mathcal{SN}} = T_{\mathcal{X}}$. In fact for $\mathfrak{x} \in \bar{T}_{\mathcal{X}}^{\mathcal{SN}}$, $\exists (\mathfrak{x}_n) \subset T_{\mathcal{X}} : \|\mathfrak{x}_n - \mathfrak{x}\|_{\alpha} \rightarrow 0$ as $n \rightarrow \infty$, uniformly in α . As a result, given $\varepsilon > 0$, $\exists$ an integer $n_0(\varepsilon) > 0 : \forall \alpha \in (0,1)$ and $n \geq n_0$, $\|\mathfrak{x}_n - \mathfrak{x}\|_{\alpha} < \varepsilon$.

However, $|\mathfrak{x}| \leq |\mathfrak{x}_n - \mathfrak{x}| + |\mathfrak{x}_n| < |\mathfrak{x}_n - \mathfrak{x}| + 1 = \left(\frac{1-\alpha}{1+\alpha}\right) \|\mathfrak{x}_n - \mathfrak{x}\|_{\alpha} + 1 < \left(\frac{1-\alpha}{1+\alpha}\right) \varepsilon + 1$, $\forall \alpha \in (0,1)$ and $n \geq n_0$. Let $\varepsilon \rightarrow 0$ and hence $\alpha \rightarrow 1$, we get $|\mathfrak{x}| < 1$.

$$\therefore \bar{T}_{\mathcal{X}}^{\mathcal{SN}} \subseteq T_{\mathcal{X}}.$$

Definition 2.6. A subset $\mathfrak{G}$ of a $\mathfrak{NN}\mathfrak{S}(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$ is called dense ($\mathfrak{D}$) in

$$(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot) \text{ iff } \overline{\mathfrak{G}}^{(\mathcal{SN})}(\overline{\mathfrak{G}}^{(\mathcal{WN})}) = \mathcal{X}.$$

Definition 2.7. Let $(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$ be a $\mathfrak{NN}\mathfrak{S}$. If the $\mathfrak{NN}\mathfrak{S}$ has a countable dense ($\mathcal{CD}$) subset then it is called separable.

That is, $\exists$ a countable set $\{\mathfrak{x}_i\}$: for each $\varepsilon > 0$ and $\mathfrak{x} \in \mathcal{X}$, there is at least one $\mathfrak{x}_n$ with $\varrho(\mathfrak{x}_n - \mathfrak{x}, \varepsilon) \geq 1 - \alpha$, $v(\mathfrak{x}_n - \mathfrak{x}, \varepsilon) \leq \alpha$ and $\varpi(\mathfrak{x}_n - \mathfrak{x}, \varepsilon) \leq \alpha$, for $\alpha \in (0,1)$.

Theorem 2.8. There is separability for all finite dimension $\mathfrak{NN}\mathfrak{S}$.

Proof. Let be a finite dimension $\mathfrak{NN}\mathfrak{S}(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$ and let be a basis of $\{b_1, b_2, \dots, b_n\}$.

Since $\mathfrak{Q}_{\mathbb{K}}$ is a countable subset of $\mathbb{K}$, as a result, $\mathfrak{D}_n = \{\sum_{j=1}^n \alpha_j b_j : \alpha_j \in \mathfrak{Q}_{\mathbb{K}}\}$ is countable subset $(\mathcal{CS})$ of $\text{span}\{b_1, b_2, \dots, b_n\} = \mathcal{X}$. $\mathfrak{D}$ is dense in $(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$ as well. Observing this, let $\mathfrak{x} \in \mathcal{X}$ and $\varepsilon > 0$. Let $\gamma_1, \dots, \gamma_n$ be scalars : $\mathfrak{x} = \sum_{j=1}^n \gamma_j b_j$.

Due to denseness of $\mathfrak{Q}_{\mathbb{K}}$ in $\mathbb{K}$, $\exists \alpha_1, \alpha_2, \dots, \alpha_n$ in $\mathfrak{Q}_{\mathbb{K}}$:

$$\varrho\left(\gamma_j - \alpha_j, \frac{\varepsilon}{|\sum_{j=1}^n b_j|}\right) \geq 1 - \alpha, \quad v\left(\gamma_j - \alpha_j, \frac{\varepsilon}{|\sum_{j=1}^n b_j|}\right) \leq \alpha \text{ and } \varpi\left(\gamma_j - \alpha_j, \frac{\varepsilon}{|\sum_{j=1}^n b_j|}\right) \leq \alpha$$

$\forall j \in \{1, 2, \dots, n\}$. It consequently follows that

$$\varrho(\mathfrak{x} - \sum_{j=1}^n \alpha_j b_j, \varepsilon) \geq \varrho\left((\gamma_j - \alpha_j) \sum_{j=1}^n b_j, \varepsilon\right) = \varrho\left(\gamma_j - \alpha_j, \frac{\varepsilon}{|\sum_{k=1}^n b_k|}\right) \geq 1 - \alpha.$$

Similarly, $v(\mathfrak{x} - \sum_{j=1}^n \alpha_j b_j, \varepsilon) \leq \alpha$ and $\varpi(\mathfrak{x} - \sum_{j=1}^n \alpha_j b_j, \varepsilon) \leq \alpha$.

This indicates that $\mathfrak{D}$ is dense in $(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$.

Theorem 2.9. Any $\mathfrak{NN}\mathfrak{S}$ with $\mathcal{WN}$ -basis is separable.

Proof. Let $\mathcal{X}$ be $\mathfrak{NN}\mathfrak{S}$ with $\mathcal{WN}$ -basis $\{b_1, b_2, \dots\}$. Since $\mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{X}_n$ with

$\mathcal{X}_n = \text{span}\{b_1, b_2, \dots, b_n\}$ for every $n \in \mathbb{N}$ is dense in $\mathfrak{X}$, it suffices to prove this $\mathcal{X}$ has a countable dense subset $(\mathcal{CDS})$. Let $\mathfrak{D}_n = \{\sum_{j=1}^n \alpha_j b_j : \alpha_j \in \mathfrak{Q}_{\mathbb{K}}\}$ for all $n \in \mathbb{N}$. Then $\mathfrak{D}_n$ will be a $\mathcal{CDS}$ of $\mathcal{X}_n$. Thus $\bigcup_{n=1}^{\infty} \mathfrak{D}_n$ is a $\mathcal{CDS}$ of $\mathcal{X}$.

3. Schauder bases in the Banach Spaces

This section defines $\mathcal{SWN}$ $\mathfrak{schr}$ bases.

Definition 3.1.[9] A sequence $(\mathfrak{a}^{(n)})_{n=0}^{\infty}$ in $[\mathfrak{NL}\mathfrak{S}]$ normed linear space $\mathcal{X}$ is called a Schauder($\mathfrak{schr}$) basis if for every $\mathfrak{x} \in \mathcal{X}$, there is a unique sequence $(\mathfrak{P}_n)_{n=0}^{\infty}$ of scalars : $\mathfrak{x} = \sum_{n=0}^{\infty} \mathfrak{P}_n \mathfrak{a}^{(n)}$, that is, $\lim_{m \rightarrow \infty} \|\mathfrak{x} - \sum_{n=0}^m \mathfrak{P}_n \mathfrak{a}^{(n)}\|_{\mathcal{X}} = 0$.

Definition 3.2. Let a sequence $(\mathfrak{x}_n)$ in the $\mathfrak{NN}\mathfrak{S} (\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$. Then, the sequence $(\mathfrak{x}_n)$ is called

- (i) $\mathcal{SN}$ - $\mathfrak{schr}$ basis of $\mathcal{X}$ iff $\forall \mathfrak{x} \in \mathcal{X} \exists$ a sequence (a_n) of scalars : $\sum_{t=1}^n a_t \mathfrak{x}_t \xrightarrow{\mathcal{SN}} \mathfrak{x}$.

Hence, for $\alpha \in (0, 1) \exists n_0 = n_0(\alpha) : n \geq n_0$ implies

$$\varrho(x - \sum_{i=1}^n a_i x_i, \lambda) \geq 1 - \alpha, \quad v(x - \sum_{i=1}^n a_i x_i, \lambda) \leq \alpha \quad \text{and} \quad \varpi(x - \sum_{i=1}^n a_i x_i, \lambda) \leq \alpha \quad \forall \lambda > 0,$$

- (ii) $\mathcal{WN}$ - schr basis of $\mathcal{X}$ iff for all $x \in \mathcal{X} \exists$ a sequence (a_n) of scalars: $\sum_{i=1}^n a_i x_i \xrightarrow{\mathcal{WN}} x$. Hence, for each $\alpha \in (0,1)$ and $\varepsilon > 0$, $\exists n_0 = n_0(\alpha, \varepsilon) : n \geq n_0$ implies $\varrho(x - \sum_{i=1}^n a_i x_i, \varepsilon) \geq 1 - \alpha, \quad v(x - \sum_{i=1}^n a_i x_i, \varepsilon) \leq \alpha \quad \text{and} \quad \varpi(x - \sum_{i=1}^n a_i x_i, \varepsilon) \leq \alpha.$

Proposition 3.3. Let (x_i) be a sequence in the $\mathfrak{NS}(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$. Then

- (i) (x_i) is a $\mathcal{WN}$ -basis of $\mathcal{X}$ iff $\forall x \in \mathcal{X} \exists$ a sequence (a_n) of scalars : for each $\alpha \in (0,1)$, $\lim_{n \rightarrow \infty} \|x - \sum_{i=1}^n a_i x_i\|_\alpha = 0$;
- (ii) (x_i) is a $\mathcal{SN}$ -basis of $\mathcal{X}$ iff $\forall x \in \mathcal{X} \exists$ a sequence (a_n) of scalars : $\lim_{n \rightarrow \infty} \|x - \sum_{i=1}^n a_i x_i\|_\alpha = 0$ uniformly in α ,
where $\|\cdot\|_\alpha$ denotes α -norms of the $\mathfrak{NS}(\varrho, v, \varpi)$.

Definition 3.4. The mappings $p_n: \mathcal{X} \rightarrow \mathbb{R}$ such that $p_n(x) = p_n(\sum_{i=1}^\infty a_i x_i) = a_n$, $q_n: \mathcal{X} \rightarrow \mathcal{X} : q_n(x) = q_n(\sum_{i=1}^\infty a_i x_i) = \sum_{i=1}^n a_i x_i, n = 1, 2, \dots$ are referred coordinate functional and natural projections, respectively, with relation to the $\mathcal{SN}(\mathcal{WN})$ -basis (x_n) in $\mathcal{X}$.

Proposition 3.5. Let (x_n) be a basis in $\mathcal{WN}$ - complete $\mathfrak{NS}(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$. Then each p_n and q_n is $\mathcal{WN}$ - continuous.

Proof. Due to Proposition (3.3), (x_n) is also a schr basis in the Banach space $(\mathcal{X}, \|\cdot\|_\alpha)$ for each $\alpha \in (0,1)$. Thus $p_n: (\mathcal{X}, \|\cdot\|_\alpha) \rightarrow \mathbb{R} : p_n(x) = p_n(\sum_{i=1}^\infty a_i x_i) = a_n, q_n: (\mathcal{X}, \|\cdot\|_\alpha) \rightarrow (\mathcal{X}, \|\cdot\|_\alpha) : q_n(x) = q_n(\sum_{i=1}^\infty a_i x_i) = \sum_{i=1}^n a_i x_i$ are continuous. Consequently, for any n , the mappings are $\mathcal{WN}$ -continuous.

Remark 3.6. Clearly, if (x_n) is an $\mathcal{SN}$ - basis of $\mathcal{X}$ then it is $\mathcal{WN}$ -basis of $\mathcal{X}$, nevertheless, not in the other way. Let's take the following case as an illustration of the converse part.

Example 3.7. Let $(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$ be a $\mathfrak{NS}$ and let $\mathcal{X} = \ell_p (1 \leq p < \infty)$, the Banach space of all absolutely p -summable sequences having the norm $\|x\|_p = (\sum_k |x_k|^p)^{\frac{1}{p}}$. With the norm $\|x\|_p = (\sum_k |x_k|^p)^{\frac{1}{p}}, \mathcal{X} = \ell_p (1 \leq p < \infty)$, be the Banach space of all absolutely p - summable sequences, and let $(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$ be a $\mathfrak{NS}$.

Take into consideration

$$\varrho(\mathfrak{x}, t) = \begin{cases} \frac{\lambda - \|\mathfrak{x}\|_p}{\lambda + \|\mathfrak{x}\|_p} & \text{if } \lambda > \|\mathfrak{x}\|_p, \\ 0 & \text{if } \lambda \leq \|\mathfrak{x}\|_p; \end{cases} \quad v(\mathfrak{x}, t) = \begin{cases} \frac{2\|\mathfrak{x}\|_p}{\lambda + \|\mathfrak{x}\|_p} & \text{if } \lambda > \|\mathfrak{x}\|_p, \\ 1 & \text{if } \lambda \leq \|\mathfrak{x}\|_p, \end{cases}$$

$$\varpi(\mathfrak{x}, t) = \begin{cases} \frac{2\|\mathfrak{x}\|_p}{\lambda - \|\mathfrak{x}\|_p} & \text{if } \lambda > \|\mathfrak{x}\|_p, \\ 1 & \text{if } \lambda \leq \|\mathfrak{x}\|_p, \end{cases} \quad \text{on } \mathcal{X}.$$

Since it meets condition of definition (1.1), we can find α -norms for the $\mathfrak{NN}$ (ϱ, v, ϖ) .

Thus

$$\varrho(\mathfrak{x}, \lambda) \geq \alpha \Leftrightarrow \frac{\lambda - \|\mathfrak{x}\|_p}{\lambda + \|\mathfrak{x}\|_p} \geq \alpha \Leftrightarrow \frac{1+\alpha}{1-\alpha} \|\mathfrak{x}\|_p \leq \lambda,$$

$$v(\mathfrak{x}, \lambda) \leq 1 - \alpha \Leftrightarrow \frac{2\|\mathfrak{x}\|_p}{\lambda + \|\mathfrak{x}\|_p} \leq 1 - \alpha \Leftrightarrow \frac{1+\alpha}{1-\alpha} \|\mathfrak{x}\|_p \leq \lambda$$

$$\varpi(\mathfrak{x}, \lambda) \leq 1 - \alpha \Leftrightarrow \frac{2\|\mathfrak{x}\|_p}{\lambda - \|\mathfrak{x}\|_p} \leq 1 - \alpha \Leftrightarrow \frac{3-\alpha}{1-\alpha} \|\mathfrak{x}\|_p \geq \lambda$$

This is clear that

$$\|\mathfrak{x}\|_\alpha = \inf\{\lambda > 0: \varrho(\mathfrak{x}, \lambda) \geq \alpha, v(\mathfrak{x}, \lambda) \leq 1 - \alpha \text{ and } \varpi(\mathfrak{x}, \lambda) \leq 1 - \alpha\} = \frac{1+\alpha}{1-\alpha} \|\mathfrak{x}\|_p.$$

The sequence $(e_{\mathfrak{f}})_{\mathfrak{f}=1}^\infty$ is now shown to be $\mathcal{WN}$ -basis rather than $\mathcal{SN}$ -basis.

Consider any $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{f}}) \in \ell_p$.

To get $\mathfrak{x}_n = (0, 0, \dots, \mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}, \dots)$, put $\mathfrak{x}_n = \mathfrak{x} - (\mathfrak{x}_1, \mathfrak{x}_2, \dots, \mathfrak{x}_n, 0, 0, \dots)$.

Thus, $\lim_n \|\mathfrak{x}_n\|_\alpha = \frac{1+\alpha}{1-\alpha} \lim_n \|\mathfrak{x}_n\|_p = \frac{1+\alpha}{1-\alpha} \lim_n (\sum_{\mathfrak{f}=n+1}^\infty |\mathfrak{x}_{\mathfrak{f}}|^p)^{\frac{1}{p}} = 0$, where $(e_{\mathfrak{f}})$ is $\mathcal{WN}$ -basis for ℓ_p . Nevertheless, when α approaches 1, this convergence is not uniform in α since $\frac{(1+\alpha)}{(1-\alpha)} \epsilon \rightarrow \infty$.

Although, this convergence is not uniform in α since $\frac{(1+\alpha)}{(1-\alpha)} \epsilon \rightarrow \infty$ as $\alpha \rightarrow 1$.

Although, if we placed

$$\varrho_1(\mathfrak{x}, \lambda) = \begin{cases} 1 & \text{if } \lambda > \|\mathfrak{x}\|_p, \\ 0 & \text{if } \lambda \leq \|\mathfrak{x}\|_p; \end{cases} \quad v_1(\mathfrak{x}, \lambda) = \begin{cases} 0 & \text{if } \lambda > \|\mathfrak{x}\|_p, \\ 1 & \text{if } \lambda \leq \|\mathfrak{x}\|_p, \end{cases} \quad \text{and}$$

$\varpi_1(x, \lambda) = \begin{cases} 0 & \text{if } \lambda > \|x\|_p, \\ 1 & \text{if } \lambda \leq \|x\|_p, \end{cases}$ on $\mathcal{X}$, then $(\ell_p, \varrho_1, \nu_1, \varpi_1, *, \diamond, \odot)$ is a $\mathfrak{NN}\mathfrak{S}$, and Since $\|x\|_\alpha = \|x\|_p$ for any $\alpha \in (0,1)$, (e_i) is an $\mathcal{SN}$ -basis for ℓ_p .

Remark 3.8. Since every $\mathfrak{NN}\mathfrak{S}$ induces a topology τ : for some $\mathfrak{U} \subset \mathcal{X}$, $\mathfrak{U} \in \tau$ iff for each $x \in \mathfrak{U} \exists$ some $\lambda > 0$ and $\alpha \in (0,1) : \mathfrak{B}(x, \alpha, \lambda) \subset \mathfrak{U}$, where

$$\mathfrak{B}(x, \alpha, \lambda) := \{x : \varrho(x - y, \lambda) \geq 1 - \alpha, \nu(x - y, \lambda) \leq \alpha \text{ and } \varpi(x - y, \lambda) \leq \alpha\}.$$

Proposition 3.9. The vector space operations are continuous in the topology τ .

Proof. τ is the first countable topology of $\mathcal{X}$, since the family $\mathfrak{B}(x, \frac{1}{n}, \frac{1}{n}) : n = 1, 2, 3, \dots$ is a countable basis at x . Thus, demonstrating the sequential continuity of the vector space operations in τ suffices. Assume that in the topological space $(\mathcal{X}, \tau)$, $x_n \rightarrow x$ and $z_n \rightarrow z$.

This means $\varrho(x_n - x, \frac{\lambda}{2}), \varrho(z_n - z, \frac{\lambda}{2}), \nu(x_n - x, \frac{\lambda}{2}), \nu(z_n - z, \frac{\lambda}{2}), \varpi(x_n - x, \frac{\lambda}{2})$ and

$$\varpi(z_n - z, \frac{\lambda}{2}) \rightarrow 1 \text{ as } n \rightarrow \infty, \text{ for all } \lambda > 0.$$

Now

$$\varrho(x_n + z_n - (x + z), \lambda) \geq \varrho(x_n - x, \frac{\lambda}{2}) * \varrho(z_n - z, \frac{\lambda}{2}) \rightarrow 1 \text{ as } n \rightarrow \infty,$$

$$\nu(x_n + z_n - (x + z), \lambda) \leq \nu(x_n - x, \frac{\lambda}{2}) \diamond \nu(z_n - z, \frac{\lambda}{2}) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$\varpi(x_n + z_n - (x + z), \lambda) \leq \varpi(x_n - x, \frac{\lambda}{2}) \odot \varpi(z_n - z, \frac{\lambda}{2}) \rightarrow 0 \text{ as } n \rightarrow \infty, \forall \lambda > 0.$$

Additionally, if $t_n \rightarrow t$ in $\mathbb{R}$ or $\mathbb{C}$, the scalar field of $\mathcal{X}$, then

$$\mu(t_n x_n - tx, \lambda) = \mu(t_n x_n - tx_n + tx_n, \lambda) = \mu((t_n - t)x_n + t(x_n - x), \lambda)$$

$$\geq \mu(x_n, \frac{\lambda}{2|t_n - t|}) * \mu(x_n - x, \frac{\lambda}{2|t|}) \rightarrow 1 \text{ as } n \rightarrow \infty$$

$$\nu(t_n x_n - tx, \lambda) = \nu(t_n x_n - tx_n + tx_n, \lambda) = \nu((t_n - t)x_n + t(x_n - x), \lambda)$$

$$\leq \nu(x_n, \frac{\lambda}{2|t_n - t|}) \diamond \nu(x_n - x, \frac{\lambda}{2|t|}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\varpi(t_n x_n - tx, \lambda) = \varpi(t_n x_n - tx_n + tx_n, \lambda) = \varpi((t_n - t)x_n + t(x_n - x), \lambda)$$

$$\leq \varpi\left(x_n, \frac{\lambda}{2|t_n - t|}\right) \odot \varpi\left(x_n - x, \frac{\lambda}{2|t|}\right) \rightarrow 0 \text{ as } n \rightarrow \infty$$

Here, we demonstrate the separability of a $\mathfrak{NLS}$ with a Schauder basis in accordance with the classical results.

Theorem 3.10. Let $(X, \varrho, v, \varpi, *, \diamond, \odot)$ be a $\mathfrak{NNS}$ with $\mathcal{WN}$ -basis (x_n) . Then, separability exists in (X, τ) .

Proof. Given a rational number b_f and the condition $\sum_{f=1}^n b_f x_f$, let $\mathfrak{E}$ be the set of all finite linear combinations. $\mathfrak{E}$ is obviously countable. Given an arbitrary $x \in X$, a unique sequence (a_n) of scalars exists. For every $\varepsilon > 0$ and $\alpha \in (0, 1)$, we can find an integer $n_0 = n_0(\alpha, \varepsilon)$:

$$\varrho(x - \sum_{f=1}^n a_f x_f, \varepsilon) \geq 1 - \alpha, \quad v(x - \sum_{f=1}^n a_f x_f, \varepsilon) \leq \alpha \text{ and } \varpi(x - \sum_{f=1}^n a_f x_f, \varepsilon) \leq \alpha$$

That is, $\forall n \geq n_0, \sum_{f=1}^n a_f x_f \in \mathfrak{B}(x, \alpha, \varepsilon)$.

On the other hand, one can build a sequence $(b_f^i)_{i=1}^\infty$ of scalars that converge to a_f for each f .

It follows that the sequence $(\sum_{f=1}^n b_f^i x_f)_{i=1}^\infty$ to $\sum_{f=1}^n a_f x_f$ in τ , converges. It follows that an element $\sum_{f=1}^n b_f^i x_f$ of $\mathfrak{E}$ is included in any x -centered τ -open sphere $\mathfrak{B}(x, \alpha, \varepsilon)$.

Theorem 3.11. Let (x_n) be a basis in X and $(X, \|\cdot\|)$ be a normed space. Then, for $\mathfrak{NS}(X, \varrho, v, \varpi, *, \diamond, \odot)$, (x_n) is a $\mathcal{WN}$ -basis, where

$$\varrho(x, \lambda) = \begin{cases} \frac{\lambda}{\lambda + \|x\|} & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda \leq 0; \end{cases} \quad v(x, \lambda) = \begin{cases} \frac{\|x\|}{\lambda + \|x\|} & \text{if } \lambda > 0, \\ 1 & \text{if } \lambda \leq 0, \end{cases} \quad \text{and}$$

$$\varpi(x, \lambda) = \begin{cases} \frac{\|x\|}{\lambda} & \text{if } \lambda > 0, \\ 1 & \text{if } \lambda \leq 0. \end{cases}$$

Proof. Assuming the norm topology to hold, for any $x \in X$, $\exists$ a unique sequence (a_n) of scalars with $\sum_{f=1}^n a_f x_f \rightarrow 0$ as $n \rightarrow \infty$. When $n \geq n_0$ implies $\|x - \sum_{f=1}^n a_f x_f\| \leq \delta$, then for every $\delta > 0$, $\exists$ an integer $n_0 = n_0(\delta)$.

Now, for each ε and $\alpha \in (0, 1)$, take $\delta = \frac{\alpha\varepsilon}{(1-\alpha)}$. So, $\exists$ an integer $n_0 = n_0(\delta) = n_0(\alpha, \varepsilon)$:

$n \geq n_0$ implies $\|x - \sum_{k=1}^n a_k x_k\| \leq \frac{\alpha\varepsilon}{1-\alpha}$ iff

$$\varrho(x - \sum_{f=1}^n a_f x_f, \varepsilon) = \frac{\varepsilon}{\varepsilon + \|x - \sum_{f=1}^n a_f x_f\|} \geq 1 - \alpha$$

$$v(x - \sum_{i=1}^n \alpha_i x_i, \varepsilon) = \frac{\|x - \sum_{i=1}^n \alpha_i x_i\|}{\varepsilon + \|\eta - \sum_{i=1}^n \alpha_i x_i\|} \leq \alpha$$

$$\varpi(x - \sum_{i=1}^n \alpha_i x_i, \varepsilon) = \frac{\|x - \sum_{i=1}^n \alpha_i x_i\|}{\varepsilon} \leq \alpha$$

4. Weak and strong Neutrosophic Approximation Property

Here, we define the Weak and strong Neutrosophic Approximation Property ($\mathcal{SWNAP}$) and demonstrate a few noteworthy findings.

Definition 4.1. A $\mathcal{WN}$ -complete $\mathfrak{NN}\mathfrak{S}(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$ is said to possess the $\mathcal{WN}$ - $\mathcal{AP}$ condition if $\forall$ $\mathcal{WN}$ -compact set $\mathfrak{K} \subset \mathcal{X}$ and $\exists$ an operator $\mathfrak{T}_\alpha: \mathcal{X} \rightarrow \mathcal{X}$ of finite rank for each $\varepsilon > 0$ and $\alpha \in (0, 1)$:

$$\varrho(\mathfrak{T}_\alpha(x) - x, \varepsilon) \geq 1 - \alpha, \quad v(\mathfrak{T}_\alpha(x) - x, \varepsilon) \leq \alpha \text{ and } \varpi(\mathfrak{T}_\alpha(x) - x, \varepsilon) \leq \alpha \quad \forall x \in \mathfrak{K}.$$

Definition 4.2. For every $\mathcal{SN}$ -compact set $\mathfrak{K} \subset \mathcal{X}$ and $\alpha \in (0, 1) \exists$ an operator $\mathfrak{T}: \mathcal{X} \rightarrow \mathcal{X}$ of finite rank, an $\mathcal{SN}$ -complete $\mathfrak{NN}\mathfrak{S}(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$ is said to have the $\mathcal{SN}$ - $\mathcal{AP}$ feature if: $\varrho(\mathfrak{T}(x) - x, \lambda) \geq 1 - \alpha$, $v(\mathfrak{T}(x) - x, \lambda) \leq \alpha$ and $\varpi(\mathfrak{T}(x) - x, \lambda) \leq \alpha \quad \forall x \in \mathfrak{K}$ and $\lambda > 0$.

Theorem 4.3. Let $(\mathcal{X}, \varrho, v, \varpi, *, \diamond, \odot)$ be a $\mathfrak{NN}\mathfrak{S}$ possessing a $\mathcal{WN}$ -basis (x_n) . Then $\mathcal{X}$ has the $\mathcal{WN}$ - $\mathcal{AP}$.

Proof. Let $\mathfrak{K} \subset \mathcal{X}$ be a compact subset of $\mathcal{X}$ that is $\mathcal{WN}$. Assume that $\varepsilon > 0$ and $\alpha \in (0, 1)$. Based on the hypothesis, $\exists$ a unique sequence (α_n) of scalars for every $x \in \mathfrak{K}$:

$P_n(x) = \sum_{k=1}^n \alpha_k x_k \xrightarrow{\mathcal{WN}} x$ as $n \rightarrow \infty$. Then, $\exists$ some $n_0(\alpha, \varepsilon)$: $\varrho(P_n(x) - x, \varepsilon) \geq 1 - \alpha$, $v(P_n(x) - x, \varepsilon) \leq \alpha$, $\varpi(P_n(x) - x, \varepsilon) \leq \alpha \quad \forall n \geq n_0$. Further, each P_n has a finite rank in the linear space $\mathcal{X}$ since $\dim P_n(x) = n$. Thus, it is possible to interpret each $P_n : n \geq n_0$ as a desirable finite rank operator in the formulation.

Remark 4.4. The $\mathcal{SN}$ -basis can also be demonstrated by Theorem 4.3.

5. Conclusion

In this article we define and investigate the concepts of Schauder basis, separability, and approximation property in neutrosophic normed spaces. This work investigates and characterizes the concepts of strong and weak approximation property in neutrosophic normed spaces, as well as identifies some related results.

References

- [1] Atanassov, K. T., Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, (1986), 20(1), 87-96.
- [2] Atanassov, K. T., More on intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, (1989), 33(1), 37-45.
- [3] Jeyaraman, M., Jenifer, P., Statistical Δ^m convergence in neutrosophic normed spaces, *Journal of Computational Mathematica* 7(1), (2023), 46-60.
- [4] Jeyaraman, M., Jenifer, P., Praveena, U., New Approach in Logarithmic Summability of Sequences in Neutrosophic Normed Spaces, *International Journal of Neutrosophic Science (IJNS)*, (2022), 19(3), 29-39.
- [5] Jeyaraman, M., Jenifer. P., Rough Statistical convergence in Neutrosophic Normed spaces, *Annals of Communications in Mathematics*, 6(3), (2023), 177-190.
- [6] Mursaleen, M., Karakaya, V. and Mohiuddine, S. A., Schauder Basis, Separability, and Approximation Property in Intuitionistic Fuzzy Normed Space, *Abstract and Applied Analysis*, (2010), 1-14.
- [7] Palomares, A., Pasadas, M., Ramirez, V., and Ruiz Galán, M., Schauder bases in Banach spaces: application to numerical solutions of differential equations, *Computers & Mathematics with Applications*, (2002), 44(5-6), 619–622.
- [8] Saadati, R., and Park, J. H., On the intuitionistic fuzzy topological spaces, *Chaos, Solitons & Fractals*, (2006), 27(2), 331–344.
- [9] Schauder, J., Zur Theorie stetiger Abbildungen in Funktionalräumen, *Mathematische Zeitschrift*, (1927), 26(1), 47–65.
- [10] Smarandache, F., Neutrosophy. Neutrosophic Probability, Set and Logic, Proquest Information & Learning, Ann Arbor, Mi. Chigan, USA, (1998).
- [11] Yilmaz, Y., Schauder bases and approximation property in fuzzy normed spaces, *Computers & Mathematics with Applications*, (2010), 59(6), 1957–1964.
- [12] Zadeh, L. A., Fuzzy sets, *Information Control*, (1965), 8(3), 338-353.