



The Infinitesimal Generator of Continuous Semigroup of Linear Operator

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ARTICLE INFO	ABSTRACT
Published Online: 29 June 2026 Corresponding Author: Susilo Hariyanto	In this article, we investigate the problem of determining the infinitesimal generator of a co-semigroup on a Hilbert space. The concept of an infinitesimal generator plays a fundamental role in the theory of semigroups and evolution equations, since it characterizes the behavior of the associated operator family. Motivated by the relationship between bounded linear operators and operator groups, we begin by considering a group $S(\cdot)$ generated by a bounded linear operator (A) we have the mapping $t \rightarrow S(t)$ is differentiable for every (x) in the Hilbert space and satisfies the corresponding differential equation associated $(d/dt)S(t) = AS(t)$. This differentiability property provides a natural way to identify an operator that governs the dynamics of the group. Consequently, we obtain a candidate for the infinitesimal generator and investigate the conditions under which this operator serves as the infinitesimal generator of a co-semigroup. The results presented contribute to a better understanding of the connection between co-semigroups and their generators in Hilbert spaces.
KEYWORDS: Infinitesimal generator, semigroup, semigroup continuous.	

I. INTRODUCTION

In this [14, 15, 16] we have seen that any bounded linear operator on X generates a group of bounded linear operator $S(t)$ on X by the following formula:

$$S(t) \equiv e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \text{ for any } t \in R. \dots (1)$$

Since the series converges absolutely for any $t \in R$, it defines a family of bounded linear operators $S(t)$, $t \in R$. The series representation (1) is a group of bounded linear operator on X because $S(0) = I$ and $S(t+s) = S(t)S(s)$ for any $s, t \in R$. Furthermore

$$\frac{d}{dt} S(t) = AS(t) = S(t)A, \quad t \in R.$$

This show that the function $u(t) = S(t)x$ for any x is the unique solution of the Cauchy problem

$$\frac{du}{dt} = Au, \quad u(0) = x.$$

But the concept of group is not of sufficient generality. It turns out that the solutions partial differential equations are not exponential functions corresponding to a bounded linear operator.

Definition 1.1 [6-10]

Let $S(t)$, $t \geq 0$, be a family of bounded linear operators on X . $S(t)$ is called a semigroup if and only if

1. $S(0) = I$,
2. $S(t+s) = S(t)S(s)$, $t, s \geq 0$.

If, in addition,

3. $\lim_{t \rightarrow 0^+} S(t)x = x$ for any $x \in X$.

then $S(t)$ is called a strongly continuous semigroup or, for short, a Co-semigroup on X .

If $S(t)$ is defined for all $t \in R$, then the family $S(t)$ is called a Co-group on X if and only if it satisfies 1, 2 for all $t, s \in R$ and $\lim_{t \rightarrow 0^+} S(t)x = x$ for any $x \in X$. Co in

the above definition stands for “strongly continuous at zero”.

We shall see below (Proposition 1.2) that instead of 3 in Definition 1 we equally well could have imposed the condition that for any $x \in X$ the mapping $t \rightarrow S(t)x$ is continuous on $[0, \infty)$. We first establish estimate for $\|S(t)\|, t \geq 0$.

Proposition 1.2:

If $S(\cdot)$ is a Co-Semigroup, then there exist two constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}, t \geq 0$.

The constant ω computed in the proof of Proposition 1.2 is always nonnegative even if allow an estimate of the form (2.1) with $\omega < 0$. When using (2.1) in estimates it is often convenient to assume $\omega \geq 0$ which we shall do without mentioning it on each occasion.

Proposition 1.2 show that

$$\tau_{S(\cdot)} = \inf \left\{ \omega \in \mathbb{R} \mid \exists M \geq 1 \text{ s.t. } \|S(t)\| \leq Me^{\omega t}, t \geq 0 \right\} \text{ exist.}$$

As in case of groups generated by a bounded linear operator we call the exponential type of the Co-semigroup $S(\cdot)$.

Proposition 1.3 [1,2, 12]

Let $S(\cdot)$ be a Co-semigroup on X . Then the exponential type of $\tau_{S(\cdot)}$ is given by

$$\tau_{S(\cdot)} = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|S(t)\| = \inf_{t > 0} \frac{1}{t} \ln \|S(t)\|.$$

Proposition 1.4 [3-5, 13]

Let $S(\cdot)$ be a Co-semigroup on X . Then for any $x \in X$ the mapping $t \rightarrow S(t)x$ is continuous on $[0, \infty)$.

II. INFINITESIMAL GENERATOR

For groups $S(\cdot)$ of bounded operators generated by a bounded linear operator A we have seen that the mapping $t \rightarrow S(t)x$ is differentiable and satisfies

$$\frac{d}{dt} S(t) = AS(t).$$

In this section we investigate if analogous properties are true when $S(\cdot)$ is a general Co-Semigroup. We first look at differentiability of the functions $t \rightarrow S(t)x$. As for continuity it turns out that the behavior at $t=0$ is significant.

Lemma 2.1

Let $S(\cdot)$ be a Co-semigroup on X and then let $x, y \in X$. Then the following two statements are equivalent

1. $\lim_{t \rightarrow 0} \frac{1}{h} (S(h)x - x) = y$
2. The function $t \rightarrow S(t)x$ is differentiable for $t > 0$ and

$$\frac{d}{dt} S(t)x = S(t)y, \text{ for all } t \in \mathbb{R}.$$

Proof

Assume that (1) is true. Then for all $t > 0$ and $0 < h < t$ we have

$$\left\| \frac{1}{h} (S(t+h)x - S(t)x) - S(t)y \right\| \leq \|S(t)\| \left\| \frac{1}{h} (S(h)x - x) - y \right\|$$

and

$$\left\| \frac{1}{h} (S(t+h)x - S(t)x) - S(t)y \right\| \leq \|S(t)\| \left\| \frac{1}{h} (S(h)x - x) - y \right\|$$

$$\leq \|S(t-h)\| \left\| \frac{1}{h} (S(h)x - x) - S(h)y \right\|$$

$$\leq Me^{\omega t} \left(\left\| \frac{1}{h} (S(h)x - x) - S(h)y \right\| + \|S(h)y - y\| \right),$$

Where $M \geq 1$ and $\omega \geq 0$ are constant according to Proposition . Taking we get 2.

Conversely, if (2) is true, then is continuous on $t > 0$ with . Then, for $h > 0$,

$$\left\| \frac{1}{h} (S(h)x - x) - y \right\| = \left\| \frac{1}{h} \int_0^h \frac{d}{dt} S(t)x dt - y \right\| = \frac{1}{h} \left\| \int_0^h (S(t)y - y) dt \right\|$$

$$\leq \sup_{0 \leq t \leq h} \|S(t)y - y\|, \text{ which proves (1).}$$

In particular, Lemma 2.1 tell us that right-hand differentiability of the mapping $t \rightarrow S(t)x$ at $t=0$ is equivalent to $t \rightarrow S(t)x$ being continuously differentiable on $t \geq 0$. Later we shall see that Lemma is not true if instead of 2 we just assume that $t \rightarrow S(t)x$ is continuously differentiable on $t > 0$. If $(d/dt) S(t)x$ exist for some $t > 0$, then

$$\frac{d}{dt} S(t)x = \lim_{h \downarrow 0} \frac{1}{h} (S(t+h)x - S(t)x) =$$

$$\lim_{h \downarrow 0} \frac{1}{h} (S(h) - I)S(t)x. \tag{2.1}$$

Therefore a candidate for an operator A such that $u(t) = S(t)x$ satisfies is $u'(t) = Au(t)$ given by

$$\text{Dom } A = \left\{ x \in X \mid \lim_{h \downarrow 0} \frac{1}{h} (S(h)x - x) \text{ exist} \right\},$$

$$Ax = \lim_{h \downarrow 0} (S(h)x - x) \text{ for } x \in \text{dom } A.$$

Obviously $\text{dom } A$ is a linear subspace of X and A is a linear operator.

Definition 2.2 [12,17]

Let be a Co-semigroup on X . The linear operator A given by 10 is called the infinitesimal generator of $S(\cdot)$.

Example 2.3

Let $X = L^p(R; C)$, $1 \leq p < \infty$ and define for $t \in R$ and $\varphi \in X$, is $(S(t)\varphi)(s) = \varphi(t+s)$, $s \in R$. Then is a Co-group on X , usually called the shift group on X .

Since the semigroup property and boundedness of $S(t)$, $t \in R$ are obvious, we only have to verify strong continuity. We use the fact that continuous functions $R \rightarrow C$ with compact support such that $\|\varphi - \psi\|_{L^p} < \varepsilon/3$. Furthermore we choose $\gamma > 0$ with $\text{supp } \psi \subset (-\gamma, \gamma)$. By uniform continuity of ψ we can find a $\delta \in (0,1)$ such that

$$|\psi(t+s) - \psi(s)| < \frac{\varepsilon}{3} \left(\frac{1}{2(1+\gamma)} \right)^{1/p} \text{ for } |t| < \delta.$$

Then

$$\|\psi(t+\cdot) - \psi\|_{L^p} = \left(\int |\psi(t+s) - \psi(s)|^p ds \right)^{1/p} < \frac{\varepsilon}{3} \text{ for}$$

$t < \delta$.

Obviously we have

$$\|\varphi(t+\cdot) - \psi(t+\cdot)\|_{L^p} = \|\varphi - \psi\|_{L^p} \text{ for all } t \in R \text{ and}$$

hence

$$\begin{aligned} \|\varphi(t+\cdot) - \varphi\|_{L^p} &= \|\varphi(t+\cdot) - \psi(t+\cdot)\|_{L^p} \\ &+ \|\psi(t+\cdot) - \psi\|_{L^p} + \|\varphi - \psi\|_{L^p} < \varepsilon \text{ for } t < \delta. \end{aligned}$$

Proposition 2.4

Let $S(\cdot)$ be a Co-Semigroup on X , with infinitesimal generator A . Then $x \in \text{dom } A$ implies

1. $S(t)x \in \text{dom } A$ for all $t \geq 0$.

2. $S(t)x$ is continuously differentiable on $t \geq 0$, with

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax, \quad t \in R. \quad (2.2)$$

Proof:

Let $x \in \text{dom } A$. Then Lemma 2.1 implies that $S(t)x$ is continuously differentiable on $t \geq 0$, and

$$\frac{d}{dt} S(t)x = S(t)Ax.$$

Furthermore, equation 2.1 implies that $S(t)x \in \text{dom } A$

also for $t > 0$ and $\frac{d}{dt} S(t)x = AS(t)x$.

In particular, equation 2.2 states that for any $x \in \text{dom } A$ the function $u(t) = S(t)x$ is a solution of the abstract Cauchy problem $u' = Au$, $u(0) = x$. Therefore one possibility to establish existence of solutions for such a Cauchy problem is to prove that the operator A is infinitesimal generator of Co-semigroup.

III. RESULT AND DISCUSSION

After we discuss this topic, we can result how to find infinitesimal generator of continuous semigroup. The infinitesimal generator A of a Co-semigroup $S(\cdot)$ satisfies:

1. $\text{Dom } A$ is dense in X
2. A is closed
3. Let $t > 0$ and f is absolutely continuous and differentiable

a.e on $[0,t]$, then $\int_0^t S(s)f(s)ds \in \text{dom } A$, and

$$A \int_0^t S(s)f(s)ds = S(t)f(t) - f(0) - \int_0^t S(s)f'(s)ds.$$

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