



Oscillatory Solution of a Higher Order Neutral Dynamic Equation with Positive and Negative Coefficients

Ramireddy Pasupula

Associate Professor of Mathematics Govt. Degree College for Women - Sangareddy, Sangareddy, India-502001.

ARTICLE INFO	ABSTRACT
Published Online: 02 June 2026	In this paper, the author studied, the sufficient conditions for every solution of the dynamic equation of the form
Corresponding Author: Ramireddy Pasupula	$[x(t) - p(t)x(\alpha(t))]^{\Delta^n} + q(t)G(x(\beta(t))) - h(t)H(x(\gamma(t))) = f(t) \quad (NH)$ for $t \in [t_0, \infty)_{\mathbb{T}}$, where $\mathbb{T}$ is a time scale with $\sup \mathbb{T} = +\infty$, $t_0 (\geq 0) \in \mathbb{T}$; to oscillate or to tends to zero as t tends to infinity.
KEYWORDS: Dynamic equation, delay, oscillation, positive and negative coefficients, bounded solution.	

1. INTRODUCTION

This paper concerned the oscillation of solutions of the neutral dynamic equation of the form

$$[x(t) - p(t)x(\alpha(t))]^{\Delta^n} + q(t)G(x(\beta(t))) - h(t)H(x(\gamma(t))) = f(t), \quad \text{for } t \in [t_0, \infty)_T$$

where $n(> 1)$ is a positive integer, T is a timescale, $q \in C_{rd}([t_0, \infty), (0, \infty))$, $p, h, f \in C_{rd}([t_0, \infty), R)$, $G, H \in C(R, R)$, $xG(x) > 0$ for $x \neq 0$, H is a bounded function and $\alpha, \beta, \gamma \in C_{rd}(T, T)$ are increasing functions such that $\alpha(t) \leq t, \beta(t) \leq t, \gamma(t) \leq t$ with $\lim_{t \rightarrow \infty} \alpha(t) = \lim_{t \rightarrow \infty} \beta(t) = \lim_{t \rightarrow \infty} \gamma(t) = \infty$.

The function G in (NH) is said to have linear growth (or to be linear) at infinity, if $\lim_{x \rightarrow \infty} \frac{|G(x)|}{x}$ is a positive constant, G is super-linear if $\lim_{x \rightarrow \infty} \frac{|G(x)|}{x} = \infty$, and G is sublinear if $\lim_{x \rightarrow \infty} \frac{|G(x)|}{x} = 0$.

In [3] and [4], Parhi and Tripathy have discussed the oscillatory and asymptotic properties of solutions of fourth order differential equation of the form

$$(r(t)(y(t) + p(t)y(t - \alpha)))'''' + q(t)G(y(t - \sigma)) = 0, \quad t \in [0, \infty)$$

where $r, q \in C([0, \infty), (0, \infty))$, $G \in C(R, R)$, $p \in C((0, \infty), R)$ and α, σ are positive real constants; are studied under the assumption $\int_0^\infty \frac{t}{r(t)} dt = \infty$ or $\int_0^\infty \frac{t}{r(t)} dt < \infty$.

In [5] and [6], Panigrahi, Graef and Ramireddy have discussed the oscillatory and asymptotic properties of solutions of fourth order dynamic equation of the form

$$(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q(t)G(x(\beta(t))) - h(t)H(x(\gamma(t))) = 0$$

and

$$(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q(t)G(x(\beta(t))) - h(t)H(x(\gamma(t))) = f(t), \quad t \in \mathbb{T}; \text{ are studied under the assumption}$$

$$\int_{t_0}^\infty \frac{t}{r(t)} \Delta t = \infty \quad \text{or} \quad \int_{t_0}^\infty \frac{\sigma(t)}{r(t)} \Delta t < \infty.$$

We extend the results of Karpuz, Rath and Padhy to the case when G has sublinear growth at infinity. Our results may apply the equation

$$(y(t) - p(t)y(\alpha(t)))^{(n)} + q(t)G(y(g(t))) = f(t)$$

where $q(t)$ has sign changes. Oscillation and nonoscillation of neutral differential equations have been studied by many authors in [7-11]. A time scale is an arbitrary nonempty closed subset of real number. We denote a timescale by T , the forward jump operator σ is defined as $\sigma(t) = \inf \{s \in T: s > t\}$, the delta derivative of f is denoted by f^Δ , the useful Cauchy's resolut on timescales

$$\int_v^u \int_v^{t_{n-1}} \dots \int_v^{t_1} f(\theta) \Delta\theta \Delta t_1 \Delta t_2 \dots \Delta t_{n-1} = \frac{1}{(n-1)!} \int_v^u (u - \sigma(t))^{n-1} f(t) \Delta t$$

and

$$\int_v^u \int_{t_{n-1}}^u \dots \int_{t_1}^u f(\theta) \Delta\theta \Delta t_1 \Delta t_2 \dots \Delta t_{n-1} = \frac{(-1)^{(n-1)}}{(n-1)!} \int_v^u (\sigma(t) - v)^{n-1} f(t) \Delta t$$

For more basics in time scale refer the books [1, 2]. In this work, we assume the following assumptions on G , α and β .

$$(H_0) \int_{t_0}^{\infty} [\sigma(t)]^{n-1} h(t) \Delta t < \infty;$$

$$(H_1) \int_{t_0}^{\infty} q(t) \Delta t = \infty;$$

$$(H_2) \text{ there exists a bounded function } F \text{ such that } F^{\Delta^n}(t) = f(t);$$

$$(H_3) \liminf_{x \rightarrow \infty} G(x) > 0, \limsup_{x \rightarrow -\infty} G(x) < 0$$

$$(H_4) \text{ there exists a bounded function } F \text{ such that } F^{\Delta^n}(t) = f(t) \text{ such that } \lim_{t \rightarrow \infty} F(t) = 0;$$

Remark 1. The prototype of G satisfying (H_0) and (H_1) could be of the form $G(x) = |x|^\lambda \text{sgn}(x)$ and $G(x) = (\beta + |x|^\mu)|x|^\lambda \text{sgn}(x)$ with $\lambda + \mu \geq 1, \beta \geq 1$. For verifying these conditions, we may use the well known inequality

$$x^p + y^p = \begin{cases} (x+y)^p, & 0 \leq p < 1 \\ 2^{1-p}(x+y)^p, & p \geq 1 \end{cases}$$

Note: The above function G also satisfying, for $x, y, z > 0$, we have $G(x+y) \leq \lambda[G(x) + G(y)]$ and $G(xy) \leq G(x)G(y)$; for $x, y < 0$ and $z > 0$, we have $G(x+y) \geq \delta[G(x) + G(y)]$ and $G(xy) \leq G(x)G(y)$.

Let $t_{-1} = \inf_{t \in [t_0, \infty)_{\mathbb{T}}} \{\alpha(t), \beta(t), \gamma(t)\}$. By a solution of (NH) , we mean a function $x \in C_{rd}([t_{-1}, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $x(t) -$

$p(t)x(\alpha(t)) \in C_{rd}^n([t_0, \infty)_T, \mathbb{R})$ and satisfies (NH) identically on $[t_0, \infty)_T$.

. A solution of (NH) is said to be oscillatory if there exists a sequence $\{s_n\}$ in $[t_0, \infty)_{\mathbb{T}}$ such that $x(s_n)x(\sigma(s_n)) \leq 0$; Otherwise, it is called non oscillatory.

2. RESULTS FOR UNBOUNDED SOLUTIONS

Theorem 2.1. Let $-1 < p_1 \leq p(t) \leq 0$ or $0 \leq p(t) \leq p_2 < 1$ for all $t \in [t_0, \infty)_T$. If (H_0) , (H_1) , (H_2) and (H_3) hold, then every unbounded solution of (NH) oscillates.

Proof: Assume the contrary that, $x(t)$ be an unbounded solution of (NH) , such that $x(t) > 0$ for $t \geq t_0$. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$ and $x(\gamma(t))$ are all positive for $t \geq t_1$. Setting

$$k(t) = \frac{(-1)^{n-1}}{(n-1)!} \int_t^{\infty} (\sigma(s) - t)^{(n-1)} h(s) H(x(\gamma(s))) \Delta s \quad (2.1)$$

Since $H(t)$ is bounded and due to (H_0) , then $k(t)$ is well defined and $\lim_{t \rightarrow \infty} k(t) = 0$. Note that, $k^{\Delta^n}(t) = -h(t)H(y(\gamma(t)))$.

Again, define

$$y(t) = x(t) - p(t)x(\alpha(t)) + k(t) - F(t) \quad (2.2)$$

From (NH) , we have

$$y^{\Delta^n}(t) = -q(t)G(x(\beta(t))) \leq 0 \quad (2.3)$$

for all $t \geq t_1$. By (H_1) , $y^{\Delta^n}(t)$ is not identically zero on $t \in [t_1, \infty)_{\mathbb{T}}$. Then there exists $t_2 \in [t_1, \infty)_T$ such that $y^{\Delta^i}, i = 0, 1, 2, 3, \dots, (n-1)$ are monotonic and eventually of one sign on $[t_2, \infty)_T$. Then, $y > 0$ or $y < 0$ for $t \in [t_2, \infty)_T$.

Since x is positive and unbounded, there exists an increasing sequence (s_j) such that $\lim_{j \rightarrow \infty} s_j = \infty$, $\lim_{j \rightarrow \infty} x(s_j) = \infty$, where $x(s_j) =$

$\max_{t_2 \leq s \leq s_j} x(s)$. Since $\lim_{t \rightarrow \infty} k(t) = 0$, for each $\epsilon > 0$, there exists N_0 such that

$$k(a_j) < \epsilon \text{ for } j \geq N_0$$

Since $\lim_{t \rightarrow \infty} \alpha(t) = \lim_{t \rightarrow \infty} \beta(t) = \lim_{t \rightarrow \infty} \gamma(t) = \infty$, then there exists $N_1 \geq N_0$ such that $s_j, \alpha(s_j), \beta(s_j), \gamma(s_j) > t_1$ for $j \geq N_1$. By (H_2) , there is an upper bound m for $|F|$. For the sequence (s_j) , and for the case, $0 \leq p(t) \leq p_2 < 1$, we have

$$y(s_j) = x(s_j) - p(s_j)x(\alpha(s_j)) + k(s_j) - F(s_j) \geq (1 - p_2)x(s_j) - \epsilon - m, j \geq N_1;$$

and for the case $-1 < p_1 \leq p(t) \leq 0$, we have

$$y(s_j) = x(s_j) - p(s_j)x(\alpha(s_j)) + k(s_j) - F(s_j) \geq x(s_j) - \epsilon - m, j \geq N_1;$$

Clearly, $\lim_{j \rightarrow \infty} y(a_j) = \infty$. Since y is monotonic and $y^n(t) \leq 0$ for all $t \geq t_2$, then $y^{\Delta(n-1)}(t) > 0$ for $t \geq t_2$.

Since y is positive and increasing, so for the case $-1 < p_1 \leq p(t) \leq 0$, we have

$$\begin{aligned} (1 + p_1)y(t) &\leq y(t) + p_1y(\alpha(t)) \leq y(t) + p(t)y(\alpha(t)) \\ &= x(t) + k(t) - F(t) + p(t)[-p(\alpha(t))x(\alpha(t)) + k(\alpha(t)) - F(\alpha(t))] \end{aligned}$$

and for the case, $0 \leq p(t) \leq p_2 < 1$,

$$y(t) \leq y(t) + p(t)y(\alpha(t)) = x(t) + k(t) - F(t) + p(t)[p(\alpha(t))x(\alpha(t)) + k(\alpha(t)) - F(\alpha(t))]$$

Since $p(t)$ and $p(\alpha(t))$ have the same sign and $x > 0$ in each of the two inequalities above, and $1 - p_2 \leq 1$, we have

$$(1 - p_2)y(t) \leq x(t) + \epsilon + m + \epsilon p_2 + m p_2 \text{ for } t \geq t_2$$

Since $\lim_{t \rightarrow \infty} y(t) = \infty$, so $\lim_{t \rightarrow \infty} x(t) = \infty$. So, $G(y(\beta(t)))$ is bounded below by a positive constant l . By integrating (2.3), we obtain

$$y^{\Delta(n-1)}(t) = y^{\Delta(n-1)}(t_2) + \int_{t_2}^t -q(s)G(x(\beta(s)))\Delta s \leq y^{\Delta(n-1)}(t_2) - \alpha \int_{t_2}^t q(s)\Delta s$$

Note that by (H_1) , the right-hand side of approaches $-\infty$, while the left hand side is positive. This contradiction implies that, the solution cannot be unbounded and eventually positive.

Next, we assume that, the solution x is unbounded and eventually negative. Define, $y(t)$ as in (2.2), we have $y^{\Delta n}(t) \geq 0$ is not identically zero on $t \in [t_1, \infty)_{\mathbb{T}}$. Then there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $y^{\Delta i}, i = 0, 1, 2, 3, \dots, (n-1)$ are monotonic and eventually of one sign on $[t_2, \infty)_{\mathbb{T}}$. Then, $y > 0$ or $y < 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$.

Since x is negative and unbounded, there exists an increasing sequence (s_j) such that $\lim_{j \rightarrow \infty} s_j = \infty$, $\lim_{j \rightarrow \infty} x(s_j) = \infty$, where $x(s_j) = \max_{t_2 \leq s \leq s_j} x(s)$.

Since $\lim_{t \rightarrow \infty} \alpha(t) = \lim_{t \rightarrow \infty} \beta(t) = \lim_{t \rightarrow \infty} \gamma(t) = \infty$, then there exists $N_1 \geq N_0$ such that $s_j, \alpha(s_j), \beta(s_j), \gamma(s_j) > t_1$ for $j \geq N_1$. By (H_2) , there is an upper bound m for $|F|$. For the sequence (s_j) , and for the case, $0 \leq p(t) \leq p_2 < 1$, we have

$$y(s_j) = x(s_j) - p(s_j)x(\alpha(s_j)) + k(s_j) - F(s_j) \leq (1 - p_2)x(s_j) + \epsilon + m, j \geq N_1;$$

and for the case $-1 < p_1 \leq p(t) \leq 0$, we have

$$y(s_j) = x(s_j) - p(s_j)x(\alpha(s_j)) + k(s_j) - F(s_j) \leq x(s_j) + \epsilon + m, j \geq N_1;$$

Clearly, $\lim_{j \rightarrow \infty} y(a_j) = -\infty$. Since y is monotonic and $y^n(t) \geq 0$ for all $t \geq t_2$, then $y^{(n-1)}(t) < 0$ for $t \geq t_2$.

Since w is negative and increasing, so for the case $-1 < p_1 \leq p(t) \leq 0$, we have

$$\begin{aligned} (1 + p_1)y(t) &\geq y(t) + p_1y(\alpha(t)) = y(t) + p(t)y(\alpha(t)) \\ &= y(t) + k(t) - F(t) + p(t)[-p(\alpha(t))x(\alpha(t)) + k(\alpha(t)) - F(\alpha(t))] \end{aligned}$$

and for the case, $0 \leq p(t) \leq p_2 < 1$,

$$y(t) \geq y(t) + p(t)y(\alpha(t)) = x(t) + k(t) - F(t) + p(t)[-p(\alpha(t))x(\alpha(t)) + k(\alpha(t)) - F(\alpha(t))]$$

Since $p(t)$ and $p(\alpha(t))$ have the same sign and $x < 0$ in each of the two inequalities above, and $0 < 1 + p_1 \leq 1$, we have

$$y(t) \geq x(t) - \epsilon - m + \epsilon p_1 + m p_1 \text{ for } t \geq t_2$$

Since $\lim_{t \rightarrow \infty} y(t) = -\infty$, so $\lim_{t \rightarrow \infty} x(t) = -\infty$. So, $G(y(\beta(t)))$ is bounded below by a negative constant l . By integrating (2.3), we obtain

$$y^{\Delta(n-1)}(t) = y^{\Delta(n-1)}(t_2) + \int_{t_2}^t -q(s)G(x(\beta(s)))\Delta s \geq y^{\Delta(n-1)}(t_2) - l \int_{t_2}^t q(s)\Delta s$$

Note that by (H_1) , the right-hand side of approaches $+\infty$, while the left-hand side is negative. This contradiction implies that, the solution cannot be unbounded and eventually negative. Thus, the theorem is proved.

Theorem 2.2. Let p_1, p_2, p_3, p_4, p_5 and p_6 are constants such that, $p_4 \leq p(t) \leq p_3 < -1$ or $-1 < p_1 \leq p(t) \leq 0$ or $0 \leq p(t) \leq p_2 < 1$ or $1 < p_5 \leq p(t) \leq p_6$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. If $(H_0), (H_2), (H_4)$ and $(H_5) \int_{t_0}^{\infty} [\sigma(t)]^{n-1} v(t) \Delta t = \infty$,

then every unbounded solution of (NH) oscillates.

Proof: Assume the contrary that, x is an eventually positive solution of (NH) , which does not tends to zero as $t \rightarrow \infty$. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t), x(\alpha(t)), x(\beta(t))$ and $x(\gamma(t))$ are all positive for $t \geq t_1$. Define $y(t)$ as in (2.2).

for all $t \geq t_1$. By (H_1) , $y^n(t)$ is not identically zero on $t \in [t_1, \infty)_{\mathbb{T}}$. Then there exists $t_2 \in [t_1, \infty)_T$ such that $y^i, i = 0, 1, 2, 3, \dots, (n-1)$ are monotonic and eventually of one sign on $[t_2, \infty)_T$. Let $\lambda := \lim_{t \rightarrow \infty} y(t)$, which exists as a finite number, because y is monotonic and bounded. Integrating (2.3), n times we obtain

$$y(t) - \lambda = \frac{(-1)^{n-1}}{(n-1)!} \int_t^\infty (\sigma(s) - t)^{(n-1)} q(s) G(x(\beta(s))) \Delta s$$

The above integral is convergent, due to y is bounded. From (H_5) , the above integral implies

$$\liminf_{t \rightarrow \infty} G(y(\beta(t))) = 0$$

As $G(x) \neq 0$ for $x \neq 0$, we have $\liminf_{t \rightarrow \infty} y(\beta(t)) = 0$ and also $\liminf_{t \rightarrow \infty} y(t) = 0$.

Since $\lim_{t \rightarrow \infty} y(t)$ exists, $k(t)$, $F(t)$ approach zero and $p(t)$ is bounded, it follows that $\lim_{t \rightarrow \infty} [x(t) - p(t)x(\alpha(t))] = 0$ for every range of $p(t)$ mentioned in the hypothesis. So, $\lambda = \lim_{t \rightarrow \infty} y(t)$. For $p(t) \leq 0$, we have $y(t) \geq x(t) + k(t) - F(t)$ and

$$0 = \lim_{t \rightarrow \infty} y(t) \geq \limsup_{t \rightarrow \infty} x(t) \geq 0$$

For $0 \leq p(t) \leq p_2 < 1$, we have $y(t) \geq x(t) - p_2 x(\alpha(t)) + k(t) - F(t)$ and

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} y(t) \geq \lim_{t \rightarrow \infty} \sup [x(t) - p_2 x(\alpha(t))] \\ &\geq \lim_{t \rightarrow \infty} \sup x(t) + \liminf_{t \rightarrow \infty} [-p_2 x(\alpha(t))] \\ &= (1-p_2) \lim_{t \rightarrow \infty} \sup x(t) \geq 0. \end{aligned}$$

For $1 < p_5 \leq p(t) \leq p_6$, we have $y(t) \leq x(t) - p_5 x(\alpha(t)) + k(t) - F(t)$ and

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} y(t) \leq \lim_{t \rightarrow \infty} \inf [x(t) - p_5 x(\alpha(t))] \\ &\leq \lim_{t \rightarrow \infty} \sup x(t) + \liminf_{t \rightarrow \infty} [-p_5 x(\alpha(t))] \\ &= (1-p_5) \lim_{t \rightarrow \infty} \sup x(t) \leq 0. \end{aligned}$$

From the four ranges of $p(t)$, $\lim_{t \rightarrow \infty} \sup x(t) = 0$. Therefore, under all the ranges of $p(t)$ bounded eventually positive solutions must approach zero. The proof for bounded eventually negative solutions is similar to the proof above. Hence, the theorem is proved.

Theorem 2.3. Let $0 \leq p(t) \leq p_2 < 1$ for all $t \in [t_0, \infty)_T$. If (H_0) , (H_1) , (H_3) , (H_3) and (H_4) hold, then every solution of (NH) is oscillatory or tends to zero as $t \rightarrow \infty$.

Proof: Assume the contrary that, x is an eventually positive solution of (NH) , which does not tend to zero as $t \rightarrow \infty$. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$ and $x(\gamma(t))$ are all positive for $t \geq t_1$ $\limsup_{t \rightarrow \infty} x(t) > 0$. Define $y(t)$ as in (2.2).

for all $t \geq t_1$. By (H_1) , $y^{\Delta^n}(t)$ is not identically zero on $t \in [t_1, \infty)_{\mathbb{T}}$. Then there exists $t_2 \in [t_1, \infty)_T$ such that $y^{\Delta^i}, i = 0, 1, 2, 3, \dots, (n-1)$ are monotonic and of constant sign on $[t_2, \infty)_T$. Since $0 \leq p(t) \leq p_2 < 1$ and $x > 0$, we have

$$y(t) \geq x(t) - p_2 x(\alpha(t)) + k(t) - F(t)$$

implies

$$\lim_{t \rightarrow \infty} y(t) \geq (1-p_2) \limsup_{t \rightarrow \infty} y(t) > 0.$$

This shows that, $y(t)$ is positive for large t . By (2.3), $y^{\Delta^n} \leq 0$ and $y > 0$ imply the existence of $t_2 \in [t_1, \infty)_T$ such that $y^{\Delta^{n-1}}(t) > 0$ for $t \geq t_2$. Now, we show that, $\liminf_{t \rightarrow \infty} x(t) > 0$, which will be used for bounding $G(x(\beta(t)))$ from below by a positive constant.

Since $p(t)$ is nonnegative and $x > 0$, we have

$$y(t) \leq x(t) + k(t) - F(t)$$

Implies

$$0 < \lim_{t \rightarrow \infty} y(t) \leq \liminf_{t \rightarrow \infty} y(t)$$

Then there exists a $t_3 > t_2$, such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$ and $x(\gamma(t))$ are bounded below by a positive constant l . Integrating (2.3), we obtain

$$y^{\Delta^{(n-1)}}(t) = y^{\Delta^{(n-1)}}(t_2) + \int_{t_2}^t -v(s) G(y(\beta(s))) \Delta s \leq y^{\Delta^{(n-1)}}(t_2) - l \int_{t_2}^t v(s) \Delta s$$

By (H_1) , $y^{\Delta^{(n-1)}}(t)$ tends to $-\infty$ as t tends to ∞ , which is a contradiction. So, our assumption is wrong. Thus, the solution cannot be eventually positive without approaching zero.

Next, assume that, x is eventually negative and does not tend to zero as $t \rightarrow \infty$. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$ and $x(\gamma(t))$ are all negative and $\liminf_{t \rightarrow \infty} x(t) < 0$. Define $y(t)$ as in (2.2). By (2.3), we have $y^{\Delta^n} \geq 0$.

$$y(t) \leq x(t) + k(t) - F(t)$$

Implies

$$0 < \lim_{t \rightarrow \infty} y(t) \leq \liminf_{t \rightarrow \infty} y(t)$$

Then there exists a $t_3 > t_2$, such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$ and $x(\gamma(t))$ are bounded $y^{\Delta^{(n-1)}}(t) = y^{\Delta^{(n-1)}}(t_2) + \int_{t_2}^t -v(s)G(y(\beta(s)))\Delta s \leq y^{\Delta^{(n-1)}}(t_2) - l \int_{t_2}^t v(s)\Delta s$

By (H_1) , $y^{\Delta^{(n-1)}}(t)$ tends to $-\infty$ as $t \rightarrow \infty$, which is a contradiction. So, our assumption is wrong. Thus, the solution cannot be eventually positive without approaching zero.

Next, assume that, x is eventually negative and does not tend to zero as $t \rightarrow \infty$. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$ and $x(\gamma(t))$ are all negative and $\liminf_{t \rightarrow \infty} x(t) < 0$. Define $y(t)$ as in (2.2). By (2.3), $y^{\Delta^n} \geq 0$, so there exists $t_2 \in [t_1, \infty)_T$ such that $y^i, i = 0, 1, 2, 3, \dots, (n-1)$ are monotonic and of constant sign on $[t_2, \infty)_T$. Since $0 \leq p(t) \leq p_2 < 1$ and $x < 0$, we have

$$y(t) \leq x(t) - p_2 x(\alpha(t)) + k(t) - F(t)$$

implies

$$\lim_{t \rightarrow \infty} y(t) \leq (1 - p_2) \liminf_{t \rightarrow \infty} x(t) < 0.$$

This shows that, $y(t)$ is negative for large t . By (2.3), $y^{\Delta^n} \geq 0$ and $y < 0$ imply the existence of $t_2 \in [t_1, \infty)_T$ such that $y^{\Delta^{n-1}}(t) < 0$ for $t \geq t_2$. Now, we show that, $\limsup_{t \rightarrow \infty} x(t) < 0$, which will be used for bounding $G(x(\beta(t)))$ from above by a negative constant.

Since $p(t)$ is nonnegative and $x < 0$, we have

$$y(t) \geq x(t) + k(t) - F(t)$$

Implies

$$0 > \lim_{t \rightarrow \infty} y(t) \geq \limsup_{t \rightarrow \infty} y(t)$$

Then there exists a $t_3 > t_2$, such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$ and $x(\gamma(t))$ are bounded by a negative constant l . Integrating (2.3), we have

$$y^{\Delta^{(n-1)}}(t) = y^{\Delta^{(n-1)}}(t_2) + \int_{t_2}^t -v(s)G(y(\beta(s)))\Delta s \geq y^{\Delta^{(n-1)}}(t_2) - l \int_{t_2}^t v(s)\Delta s$$

By (H_1) , $y^{\Delta^{(n-1)}}(t)$ tends to $-\infty$, while the left hand side is negative. This contradiction implies that an eventually negative solution must approach zero as t tends to ∞ . Thus, the theorem is proved.

Theorem 2.4: Let $p_4 \leq p(t) \leq 0$ for all $t \in [t_0, \infty)_T$. If (H_0) -(H_4),

$$(H_6) \int_{t_0}^{\infty} Q(t)\Delta t = \infty, \text{ where } \min\{q(t), q(\alpha(t))\}$$

And (H_7) there exists a positive constant l such that for $x, y, z > 0$, we have $G(x+y) \leq l[G(x) + G(y)]$, $G(xz) \leq G(x)G(z)$ and for $x, y < 0, z > 0$ we have $G(x+y) \geq l[G(x) + G(y)]$, $G(xz) \geq G(x)G(z)$, then every solution of (NH) is oscillatory or tends to zero as $t \rightarrow \infty$.

Proof: Assume the contrary that, x is an eventually positive solution of (NH) , which does not tend to zero as $t \rightarrow \infty$. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$ and $x(\gamma(t))$ are all positive for $t \geq t_1$ and $\limsup_{t \rightarrow \infty} x(t) > 0$. Define $y(t)$ as in (2.2).

for all $t \geq t_1$. By (H_1) , $y^{\Delta^n}(t)$ is not identically zero on $t \in [t_1, \infty)_{\mathbb{T}}$. Then there exists $t_2 \in [t_1, \infty)_T$ such that $y^{\Delta^i}, i = 0, 1, 2, 3, \dots, (n-1)$ are monotonic and of constant sign on $[t_2, \infty)_T$. Since $p(t) \leq 0$ and $x > 0$, we have

$$y(t) \geq x(t) + k(t) - F(t)$$

implies

$$\lim_{t \rightarrow \infty} y(t) \geq \limsup_{t \rightarrow \infty} y(t) > 0.$$

Since $k(t)$ and $F(t)$ tends to zero and $x(t) - p(t)x(\alpha(t))$ is bounded below by a positive constant δ .

Since $x(t) - p_4 x(\alpha(t)) \geq x(t) - p_4 x(\alpha(t))$, so $x(\beta(t)) - p_4 x(\alpha(\beta(t)))$ is also bounded below by a positive constant. Using (H_7) , we obtain $x(\beta(t)) - p_4 x(\alpha(\beta(t)))$ is also bounded below by a positive constant δ . Using (H_7) , we have

$$\begin{aligned} l &\leq G(x(\beta(t)) - p_4 x(\alpha(\beta(t)))) \\ &\leq \delta[G(x(\beta(t))) + G(-p_4)G(x(\alpha(\beta(t))))] \\ &\leq \delta[G(x(\beta(t))) + G(-p_4)G(x(\alpha(\beta(t))))]. \end{aligned}$$

From (2.3), we have

$$y^{\Delta^n}(t) + G(-p_4)y^{\Delta^n}(\alpha(t)) \leq -\min\{q(t), q(\alpha(t))\}G(x(\beta(t))) + G(-p_4)G(x(\alpha(\beta(t))))$$

$$\leq -Q(t)(l/\delta)$$

By integrating, the above inequality from t_2 to t , we obtain

$$y^{\Delta(n-1)}(t) + G(-p_4)y^{\Delta(n-1)}(\alpha(t)) \leq y^{\Delta(n-1)}(t_2) + G(-p_4)y^{\Delta(n-1)}(\alpha(t_2)) - (l/\delta) \int_{t_2}^t Q(s)\Delta s$$

By (H_6) , we obtain that the right hand side approaches $-\infty$, while the left hand side is positive. This shows that, eventually positive solutions must converge to zero. The proof for eventually negative solution it is similar. Thus the theorem is proved.

Example 1. Consider $T = R$,

$$[y(t) - e^{-\frac{5}{3}t}y(\frac{t}{3})]''' + (e^{-t} + 1)G\left(y\left(\frac{t}{2}\right)\right) - 8e^{-t}(e^{-t} + 1)H\left(y\left(\frac{t}{4}\right)\right) = 0,$$

Where $G(x) = x^2 \frac{\text{sgn}(x)}{(1+x^2)}$ and $H(x) = x^4 \frac{\text{sgn}(x)}{(1+x^4)}$. Clearly, all the conditions of Theorems 2.1, 2.2 and 2.3 are satisfied by the neutral equation (). In particular, $x(t) = e^{-t}$ is a solution of () converges to zero as $t \rightarrow \infty$.

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