



Theoretical Study of Various Applications of Gamma Function in Computation

Sudarno Sudarno^{1*}, Moch Abdul Mukid²

^{1,2}Department of Statistics Diponegoro University, Semarang, Indonesia

ARTICLE INFO	ABSTRACT
Published Online: 29 May 2026 Corresponding Author: Sudarno Sudarno KEYWORDS: Gamma function, Positive integer, Negative integer, Fractional number, Improper integral	The gamma function is an improper integral. The gamma function is widely used in statistics and mathematics. In statistics, it is useful for defining the gamma distribution, while in mathematics, it is useful for finding the zero of the factorial and pi. Theoretical study of the gamma function is expected to facilitate readers in learning and applying the function. The various applications of the gamma function include its application to all rational numbers, except zero and negative integers. The gamma function can also be used to solve improper and definite integrals.

I. INTRODUCTION

The gamma function is an improper integral. The lower limit of the integral is zero, while the upper limit of the integral is infinity. Therefore, the domain of definition of the gamma function is the nonnegative set [1], [2].

In the scientific world, the gamma function has many uses. In combinatorics problems, especially the combination of events found $0! = 1$. By finding $0! = 1$, the binomial coefficient is easy to determine [3]. In Mathematical Statistics, there is a gamma distribution. The gamma distribution is a distribution with a continuous random variable. The gamma distribution produces two distributions, namely the exponential distribution and the chi-square distribution [3], [4], [5]. Meanwhile, in mathematics, the gamma function can be found the value of π . In addition, the gamma function can be used to solve integral that is difficult to solve using ordinary integrals [2].

Some research on the gamma function that it has been done is in [6], the gamma function is used to prove that $0! = 1$. While in [7], the gamma function is used to prove the properties of the probability density function of the standard normal distribution and find the moment generating function of a standard normal distribution. Furthermore, the development of the gamma function in the form of a new generalized gamma function, which it is named as p-v-gamma function and provides some properties generalizing those satisfied by the classical gamma function. If the gamma function is a function of x , the p-v-gamma function

is a gamma function with the independent variable divided by v , as p goes to infinity. The numbers x and v are positive numbers, while the number p is a positive integer [8].

This research is a theoretical study of the gamma function. It is hoped that this theoretical study will be easy to understand. This research contains various applications of the gamma function to rational numbers, except for zero and negative integers. Furthermore, the gamma function is used to solve improper and definite integrals. When these integrals are solved using the gamma function, they become easy and quick to solve.

II. LITERATURE REVIEW

The gamma function is an integral function with integral limits from zero to infinity. The independent variable of the gamma function is not an independent variable like other functions in general. The independent variable of the gamma function is more like a parameter of a general function. This variable is a power of the independent variable of a function. For clarity, the definition of the gamma function is presented in Equation 1 below.

Definition of Gamma Function

The gamma function is defined by improper integral in the form

$$\Gamma(n) = \int_{x=0}^{\infty} x^{n-1} e^{-x} dx \quad (1)$$

where $\Gamma(n)$ is convergent for $n > 0$.

If we choose $n = 1$, then

$$\begin{aligned}\Gamma(1) &= \int_{x=0}^{\infty} e^{-x} dx = \lim_{a \rightarrow \infty} \int_{x=0}^a e^{-x} dx \\ &= \lim_{a \rightarrow \infty} -e^{-x} \Big|_{x=0}^a = \lim_{a \rightarrow \infty} (1 - e^{-a}) = 1\end{aligned}\quad (2)$$

Therefore, it be obtained that $\Gamma(1) = 1$.

Furthermore, if a recursive process is carried out on the gamma function formula, it will be obtained as in Theorem 1.

Theorem 1.

$$\text{If } n > 0, \text{ then } \Gamma(n+1) = n\Gamma(n) \quad (3)$$

Proof:

Based on Equation (1) and if $n > 0$,

$$\begin{aligned}\Gamma(n+1) &= \lim_{a \rightarrow \infty} \int_{x=0}^a x^n e^{-x} dx \\ &= \lim_{a \rightarrow \infty} \left[x^n (-e^{-x}) \Big|_{x=0}^a - \int_{x=0}^a n(x^{n-1})(-e^{-x}) dx \right] \\ &= \lim_{a \rightarrow \infty} \left[-a^n e^{-a} + n \int_{x=0}^a x^{n-1} e^{-x} dx \right] \\ &= \lim_{a \rightarrow \infty} \int_{x=0}^a x^{n-1} e^{-x} dx = n\Gamma(n)\end{aligned}$$

Corollary 1.

If n is a positive integer as $n = 1, 2, 3, \dots$, then

$$\Gamma(n+1) = n! \quad (4)$$

Proof:

Based on Equation (3),

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) \\ &= n(n-1)\cdots\Gamma(1) = n(n-1)(n-2)\cdots 1 = n!\end{aligned}$$

Basis for Determining Value of 0!

The value of $0!$ is essential in combinatorics problems, particularly combination problems. Combinations of events are found in probability theory. By finding the value of $0!$, combinatorics problems can proceed logically. There are two ways to determine the value of $0!$, namely

a. Based on factorial definition

In the definition of factorial, it applies that $n! = n(n-1)!$

$$\Leftrightarrow \frac{n!}{n} = (n-1)!. \text{ If we choose } n=1, \text{ then it applies}$$

$1! = 0!$. We have known that $1! = 1$. Therefore, $0! = 1$.

b. Based on the gamma function

Based on gamma function, it is true that $\Gamma(n+1) = n!$. If we choose $n=0$, then $\Gamma(1) = 0!$. We know that $\Gamma(1) = 1$, so $0! = 1$.

III. RESULT AND DISCUSSION

The following discussion will discuss various applications of the gamma function. The gamma function can be used in statistics and mathematics. In statistics, the

gamma function is used on the gamma distribution, producing the exponential distribution and the chi-square distribution. In mathematics, the gamma function is used to solve definite and improper integrals. Specifically, the various applications of the gamma function in computing include:

1. Positive Integers

Based on Corollary 1, if n is a positive integer as $n = 1, 2, 3, \dots$, then $\Gamma(n+1) = n!$. If we apply this gamma function to positive integers from 1 to 5. The results are as tabulated in Table 1.

Table 1. The value of the gamma function on positive integers.

n	$\Gamma(n+1)$	Value
1	$\Gamma(2)$	1
2	$\Gamma(3)$	2
3	$\Gamma(4)$	6
4	$\Gamma(5)$	24
5	$\Gamma(6)$	120

2. Positive Fractional Numbers

Based on Theorem 1, if $n > 0$, then $\Gamma(n+1) = n\Gamma(n)$. This equation is equivalent to new equation $\Gamma(n) = (n-1)\Gamma(n-1)$. If we apply this new equation to several positive fractional numbers. The results are tabulated in Table 2 as follow.

Table 2. The value of the gamma function on positive fractional numbers.

n	$\Gamma(n)$
1/2	$\sqrt{\pi}$
3/2	$\frac{1}{2}\sqrt{\pi}$
5/2	$\frac{3}{4}\sqrt{\pi}$
7/2	$1\frac{7}{8}\sqrt{\pi}$
9/2	$6\frac{9}{16}\sqrt{\pi}$

3. Negative Fractional Numbers

Based on Theorem 1, if $n > 0$, then $\Gamma(n+1) = n\Gamma(n)$. If the equation $\Gamma(n+1) = n\Gamma(n)$ is created for $n < 0$, then

$$\Gamma(n) = \frac{1}{n} \Gamma(n+1). \quad (5)$$

Because with the addition of one, the negative value becomes increasingly lost or it becomes positive. If we

apply this equation $\Gamma(n) = \frac{1}{n} \Gamma(n+1)$ to several negative fractional numbers. The results are tabulated as in Table 3.

Table 3. The value of the gamma function on negative fractional numbers.

n	$\Gamma(n)$
-1/2	$-2\sqrt{\pi}$
-3/2	$\frac{2^2}{3}\sqrt{\pi}$
-5/2	$-\frac{2^3}{15}\sqrt{\pi}$
-7/2	$\frac{2^4}{105}\sqrt{\pi}$
-9/2	$-\frac{2^5}{945}\sqrt{\pi}$

4. The Improper Integral with Positive Exponents Between 0 and 1

Before discussing the material further, it is necessary to provide a theoretical basis.

Theorem 2.

If there is a p such that $0 < p < 1$, then

$$\int_{x=0}^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}.$$

Proof:

We use contour integral and choose contour C such that the integral is formed by

$$\oint_C \frac{z^{p-1}}{1+z} dz.$$

The point $z = 0$ is a branch point, whereas the integrand has the pole $z = -1$ lying within C . Residu at $z = -1$ is

$$\lim_{z \rightarrow -1} (z+1) \frac{z^{p-1}}{1+z} = e^{(p-1)\pi i}.$$

Therefore,

$$\oint_C \frac{z^{p-1}}{1+z} dz = 2\pi i e^{(p-1)\pi i}.$$

The contour integral $\oint_C \frac{z^{p-1}}{1+z} dz$ could be expanded by

$$\oint_C \frac{z^{p-1}}{1+z} dz = A + B + C + D,$$

where

$$A = \int_{x=r}^R \frac{x^{p-1}}{1+x} dx, \quad B = \int_{\theta=0}^{2\pi} \frac{(\text{Re}^{i\theta})^{p-1} i \text{Re}^{i\theta}}{1 + \text{Re}^{i\theta}} d\theta,$$

$$C = \int_{x=R}^r \frac{(xe^{2\pi i})^{p-1}}{1 + xe^{2\pi i}} dx,$$

$$D = \int_{\theta=2\pi}^0 \frac{(re^{i\theta})^{p-1} i re^{i\theta}}{1 + re^{i\theta}} d\theta, \quad \text{if } z = xe^{2\pi i} \text{ and the}$$

radius $r \leq R$. Based on the pole location and angle direction, integrals of B and D approach to zero.

Therefore,

$$\oint_C \frac{z^{p-1}}{1+z} dz = A + C.$$

If we take the limit $r \rightarrow 0$ and $R \rightarrow \infty$, the contour becomes

$$\oint_C \frac{z^{p-1}}{1+z} dz = A + C = \int_{x=0}^{\infty} \frac{x^{p-1}}{1+x} dx + \int_{x=\infty}^0 \frac{(xe^{2\pi i})^{p-1}}{1 + xe^{2\pi i}} dx.$$

Then

$$\int_{x=0}^{\infty} \frac{x^{p-1}}{1+x} dx + \int_{x=\infty}^0 \frac{(xe^{2\pi i})^{p-1}}{1 + xe^{2\pi i}} dx = 2\pi i e^{(p-1)\pi i}$$

or

$$(1 - e^{2\pi i(p-1)}) \int_{x=0}^{\infty} \frac{x^{p-1}}{1+x} dx = 2\pi i e^{(p-1)\pi i}$$

Therefore,

$$\int_{x=0}^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{2\pi i}{e^{p\pi i} - e^{-p\pi i}} = \frac{\pi}{\sin p\pi}.$$

$$\text{Because } \sin p\pi = \frac{e^{p\pi i} - e^{-p\pi i}}{2i}.$$

According to Theorem 2, if we have

$$\int_{x=0}^{\infty} \frac{x^{p-1}}{1+x} dx = A \text{ with } 0 < p < 1, \text{ then the results can be}$$

displayed as in Table 4 below.

Table 4. Integral value to the power of $0 < p < 1$

p	A
1/6	2π
1/4	$\sqrt{2}\pi$
1/3	$\frac{2}{3}\sqrt{3}\pi$
1/2	π

If $p = 0 \vee 1$, then $\sin 0 = 0$ or $\sin \pi = 0$, so the value of integral is undefined. Therefore, the improper integral becomes an undefined or divergent integral.

Theorem 3.

If there is a p such that $0 < p < 1$, then

$$\int_{x=0}^{\infty} \frac{x^{p-1}}{1+x} dx = \Gamma(p)\Gamma(1-p).$$

Proof:

$$\text{We choose } \frac{x}{1+x} = y, \text{ then } x = \frac{y}{1-y}$$

$$\text{so that } dx = \frac{1}{(1-y)^2} dy = (1-y)^{-2} dy.$$

Integral limits change:

$$x=0 \rightarrow y=0 \text{ and } x=\infty \rightarrow y=1. \text{ Then}$$

$$\begin{aligned} \int_{x=0}^{\infty} \frac{x^{p-1}}{1+x} dx &= \int_{y=0}^1 y^{p-1} (1-y)^{1-p} (1-y)(1-y)^{-2} dy \\ &= \int_{y=0}^1 y^{p-1} (1-y)^{-p} dy = B(p, 1-p) \\ &= \frac{\Gamma(p)\Gamma(1-p)}{\Gamma(1)} = \Gamma(p)\Gamma(1-p) \end{aligned}$$

Corollary 2.

If there is a p such that $0 < p < 1$,

$$\text{then } \int_{x=0}^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi} = \Gamma(p)\Gamma(1-p).$$

Proof:

Based on Theorem 2 and Theorem 3 are proven.

If we take several values of p with $0 < p < 1$, then the multiplication of the gamma functions $\Gamma(p)\Gamma(1-p)$ produce values as tabulated in Table 5.

Table 5. Multiplication value of the gamma functions.

p	$\Gamma(p)\Gamma(1-p)$	Value
1/6	$\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})$	2π
1/4	$\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})$	$\sqrt{2}\pi$
1/3	$\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})$	$\frac{2}{3}\sqrt{3}\pi$
1/2	$\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})$	π

The following is a discussion of the gamma function which is used to solve improper integral problems with exponents p such that $0 < p < 1$ as in the two integral examples below.

a. If there is $\int_{x=0}^{\infty} \frac{4}{1+x^4} dx$. We want to determine the value of integral.

Solution:

$$\text{Choose } x^4 = y, \text{ so } x = y^{1/4} \text{ and } dx = \frac{1}{4} y^{-3/4} dy.$$

New integral limits:

$$x=0 \rightarrow y=0 \text{ and } x=\infty \rightarrow y=\infty. \text{ Then}$$

$$\begin{aligned} \int_{x=0}^{\infty} \frac{4}{1+x^4} dx &= \int_{y=0}^{\infty} \frac{4}{1+y} \frac{1}{4} y^{-3/4} dy = \int_{y=0}^{\infty} \frac{y^{-3/4}}{1+y} dy \\ &= \frac{\pi}{\sin(\pi/4)} = \sqrt{2}\pi \end{aligned}$$

b. If there is $\int_{x=0}^{\infty} \frac{3x}{1+x^6} dx$, solve this integral.

Solution:

The trick is to replace that $x^6 = y$ so $x = y^{1/6}$, and $dx = \frac{1}{6} y^{-5/6} dy$. New integral limits are $x=0 \Leftrightarrow y=0$ and $x=\infty \Leftrightarrow y=\infty$, so that the integral becomes

$$\begin{aligned} \int_{x=0}^{\infty} \frac{3x}{1+x^6} dx &= \int_{y=0}^{\infty} \frac{3y^{1/6}}{1+y} \frac{1}{6} y^{-5/6} dy = \frac{1}{2} \int_{y=0}^{\infty} \frac{y^{-4/6}}{1+y} dy \\ &= \frac{1}{2} \frac{\pi}{\sin(\pi/3)} = \frac{1}{3} \sqrt{3}\pi \end{aligned}$$

5. Calculation of Improper Integral and Definite Integral

Application of the gamma function in the calculation of improper integral and definite integral be given in the two integral problems below.

a. If there is $\int_{x=0}^{\infty} x^3 e^{-2x^5} dx$, we want to determine the value of the integral.

Solution:

$$\text{Choose } 2x^5 = y, \text{ so } x = 2^{-1/5} y^{1/5}$$

$$\text{and } dx = 2^{-1/5} \left(\frac{1}{5}\right) y^{-4/5} dy.$$

New integral limits are

$$x=0 \rightarrow y=0 \text{ and } x=\infty \rightarrow y=\infty.$$

Then

$$\begin{aligned} \int_{x=0}^{\infty} x^3 e^{-2x^5} dx &= \int_{y=0}^{\infty} 2^{-3/5} y^{3/5} e^{-y} 2^{-1/5} \left(\frac{1}{5}\right) y^{-4/5} dy \\ &= \left(\frac{1}{5}\right) \int_{y=0}^{\infty} 2^{-4/5} y^{-1/5} e^{-y} dy = \frac{1}{5\sqrt[5]{16}} \Gamma\left(\frac{4}{5}\right) \end{aligned}$$

An improper integral with selected substitutions will become a new improper integral. This new improper integral is a gamma function, so the value of the integral can be determined.

b. If there is $\int_{x=0}^1 \sqrt[3]{\ln\left(\frac{1}{x}\right)} dx$, we want to determine the value of this integral.

Solution:

$$\text{Choose } \ln\left(\frac{1}{x}\right) = y, \text{ so } x = e^{-y} \text{ and } dx = -e^{-y} dy.$$

Integral limits change:

$$x=0 \rightarrow y=\infty \text{ and } x=1 \rightarrow y=0. \text{ Then}$$

$$\int_{x=0}^1 \sqrt[3]{\ln\left(\frac{1}{x}\right)} dx = \int_{y=\infty}^0 y^{1/3} e^{-y} dy = \frac{1}{3} \Gamma\left(\frac{4}{3}\right).$$

A definite integral that undergoes proper substitution will become a new improper integral. This new improper integral is a gamma function, so the value of the integral can be calculated.

Exception for Negative Integers

If we want to understand how the gamma function works with negative integers. We need to first examine to $n=0$.

For $n=0$, then $\Gamma(0) = \int_{x=0}^{\infty} x^{-1} e^{-x} dx$. This integral cannot be solved using partial integration, so that the

integral is divergent or undefined. Therefore, the gamma function cannot be defined at zero.

Furthermore, for $n = -1$, based on Equation (5),

$$\Gamma(-1) = \frac{\Gamma(0)}{(-1)}.$$

Since $\Gamma(0)$ is undefined, then $\Gamma(-1)$ is

undefined, too. Because $\Gamma(-1)$ is undefined, the gamma function cannot be used to calculate value for negative integers.

IV. CONCLUSION

The gamma function is an improper integral. Its integral limit is from zero to infinity. The gamma function is widely used in statistics and mathematics. In general, the gamma function can be used on positive integers, positive fractions, negative fractions, real numbers between zero and one, and to calculate improper integrals and definite integrals. However, the weakness of the gamma function is that it cannot be used on zero and negative integers. By understanding the use of the gamma function, we will be able to easily perform computations related to the gamma function.

REFERENCES

1. Folland, G.B. 2023. *Advanced Calculus*, Second Edition. Department of Mathematics, University of Washington.
2. Wrede, R. and Spiegel, M.R. 2010, *Advanced Calculus*, Third Edition. The McGraw-Hill Companies, Inc. New Delhi.
3. Hogg, R.V., McKean, J.W. and Craig, A.T. 2019. *Introduction to Mathematical Statistics*, Eighth Edition. Pearson Education, Inc., United States of America.
4. Larsen, R.J. and Marx, M.L. 2018. *An Introduction to Mathematical Statistics and Its Applications*, Sixth Edition. Pearson Education, Inc., United States of America.
5. Wackerly, D.D., Mendenhall III, W. and Scheaffer, R. L. 2008. *Mathematical Statistics with Application*, Seventh Edition. Thomson Learning, Inc. Spain.
6. Hapsan, A. 2016. *Penerapan Fungsi Gamma Dalam Pembuktian $0! = 1$* . Jurnal Matematika dan Pembelajarannya, Vol. 2, No. 1. ISSN: 2303-0992.
7. Iddrisu, M.M. and Tetteh, K.I. 2017. *The Gamma Function and Its Analytical Applications*. Journal of Advances in Mathematics and Computer Science 23(3): 1-16, No. JAMCS.34779. ISSN: 2231-0851.
8. Ege, I. 2022. *A New Generalized Gamma Function and Its Properties*. Journal of Advances in Mathematics and Computer Science 37(4): 1-11, No. JAMCS.87037. ISSN: 2456-9968.