

An Optimized Four-Step Three-Off grid third derivative Methods based on the Volterra integral equation of the second kind for Solving third order Initial Value Problems

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ARTICLE INFO	ABSTRACT
Published Online: 29 April 2026 Corresponding Author: Benard Alechenu KEYWORDS: Optimize off-grid; Local Truncation error; Free- Parameter	In this article, an optimize four-step three off hybrid point third derivative method based on the Volterra integral equation of the second kind for solving third Order Ordinary Differential Equations, the method proposes an exponential function and power series as the basis function for a selected three hybrid points which suitably optimizes two of the off-grid points by equating the principal term of the local truncation error to zero and using the local truncation error to determine the approximate values of the unknown parameter, the basic properties of the method was scrutinize and the develop method of the Volterra integral equation of the second kind is apply to work out solve third order initial value problems of ordinary differential equations and from the numerical results obtained, it is observed that our new methods gives better approximation than the existing method compared.

I. INTRODUCTION

In this manuscript we reflect on third order three-step with two optimized hybrid point approach form

$$\phi^{(m)} = \gamma(t, \phi(t), \phi'(t), \phi''(t)), \phi(t_0) = \phi_0, \phi'(t_0) = \phi'_0, \phi''(t_0) = \phi''_0 \quad (1)$$

Third order ordinary differential equations are a crucial mathematical tool for simulating a variety of physical phenomena that arise in a variety of scientific and engineering domains, including fluid dynamics, vibration analysis, control systems, and structural mechanics.. According to [1], the third derivative can be optimized to predict, improve, and enhance computational efficiency and solution accuracy with superior accuracy and a good region of absolute stability. Equation (1) is used to solve problems in a number of real-world domains that frequently occur in physical systems in science and engineering, such as mechanics, control theory, beam deflection, and ship dynamics, among others. However, the majority of these physical issues are complex systems for which an analytical solution is exceedingly challenging, if not possible. Numerical techniques, which solve (1), are therefore

essential tools. As a result, third order ordinary differential equations have garnered a lot of attention from researchers, and as a result, theoretical and numerical studies addressing (1) have recently surfaced in literature. Reducing (1) to a system of differential equations of first order is the well-known and proven approach to solving it which then leads to a computational burden as in Kuboye et al (2020) and Kayode and Adeyeye (2011). Different method have been proposed for the solving (1) ranging from predictor-corrector method to hybrid methods and also recently optimized methods Among those who have recently embraced the hybrid approach as an alternative to the direct method for approximating (1) are Adesanya et al (2013), Adebayo and Adebola (2016), Adoghe and Omole (2019). Optimized approach have been proposed by Akinnukawe et al (2024) who present a numerical solution of fourth-order initial value problems using novel fourth-order block algorithm, Bothayna and Sadeemr (2019) proposes optimization of two-step block method with three hybrid points for solving third order initial value problems, Joshua (2022) uses the optimize appraoch to derive a two step second derivative methods for the solution of stiff systems,

Raymond et al (2023) proposes an optimized half-step scheme third derivative methods for testing higher order initial value problem also Raymond et al (2025) proposes an optimizes hybrid block two-step four off-grid methods for direct solution of general third order ordinary differential equations. By setting the principal term of the local truncation error to zero and utilizing the local truncation error to estimate the values of the unknown parameter, the three-step with three off-grid point derivative approaches that we developed for this study optimizes two of the three off-grid positions. The structure of the paper is as follows: The techniques and resources used to build the method are covered in section 2. Section 4 provides the numerical experiments where the effectiveness of the derived method is tested on some stiff problems; the results were discussed and a conclusion was made in the section. Section 3 examines the analysis of the method's fundamental properties, such as convergence and stability region.

II DERIVATION OF THE METHOD

This section describes the Development of an optimized numerical hybrid techniques with exponentially fitting for general second kind volterra integral equations with three off-grid points using the linear multistep Algorithm. The collocation approach is by using the combination of power series and exponential fitted function as the approximate solution and the algorithm shall be in the form

$$\varphi(x) = \phi(x) + \sum_{j=0}^s \mu_j x^j + \sum_{j=0}^r \psi_j e^{x^j} \quad (2)$$

Let the approximate power series and exponentially fitted function be of the form

$$\sum_{i=0}^k \tau_i \left(\varphi_{n+i} - \phi_{n+i} \right) = h^3 \sum_{i=0}^s \varsigma_i(t) \mu_{n+i} + h^3 \sum_{j=i}^r \psi_j(t) e^{x^j} \quad (3)$$

Where $\phi_n = \phi(x_n)$, $\mu_n = \mu(x_n)$ and $\tau_i, \varsigma_i, \psi_j$ ($i = 0, 1, \dots, m$; $j = i, \dots, k$) are constant coefficient of (3) to be determined which is considered the solution to the second kind volterra integral equation of the form

$$\sum_{i=0}^m \tau_i \left(\varphi_{n+i} - \phi_{n+i} \right) = h^3 \sum_{i=0}^k \sum_{j=1}^k \beta_i^{(j)}(k) \left(x_{n+j}, x_{n+i}, \varphi_{n+i} \right) \quad (4)$$

it is important to note that (3) can be written as (2) subject to the following condition

$$\eta(x) = \varphi(x) - \phi(x) \quad (5)$$

by differentiating equation (2) thrice gives

$$\varphi'(t) - \phi'(t) = \sum_{i=0}^s i x^{i-1} \mu_j(t) + \sum_{j=1}^r j x^{j-1} \psi_j(t) e^{x^j} \quad (6)$$

$$\varphi''(t) - \phi''(t) = \sum_{i=0}^s i(i-1) x^{i-2} \mu_j(t) + \sum_{j=1}^r j x^{j-1} \psi_j(t) e^{x^j} \quad (7)$$

$$\varphi'''(t) = \sum_{i=0}^s i(i-1)(i-2) x^{i-3} \mu_j(t) + \sum_{j=1}^r j x^{j-3} \mu_i(t) e^{x^j} \left(j^2 x^{2j-3} j-3 j x^j + 3 j^2 x^j + j^2 + 2 \right) \quad (8)$$

consider three hybrid points u, v, w such that $0 < u < v < w < \dots < N$ hold (Akinnukwe, 2024).

Evaluate (2), (5) and (7) at 0 and collocating (8) at $0, u, v, w, 1, 2$ points. The continuous approximation is then constructed by imposing three conditions which are

$$\left. \begin{aligned} \varphi_{n+j} &= \phi(x_{n+j}), \quad j=0, u, v, w, 1, 2 \\ \phi_{n+j} &= \phi(x_{n+j}), \quad j=0, u, v, w, 1, 2 \\ \varphi^{iii}(x_{n+j}) &= g_{n+j}, \quad j=0, u, v, w, 1, 2 \\ \xi &= \frac{t-t_{n+3}}{h} \end{aligned} \right\} \quad (9)$$

Equation (2) result to system of non linear equation of the form

$$MD = V \quad (10)$$

$$\begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \\ 0 & 0 & 0 & 0 & 12\frac{u}{1!} & 12\frac{v}{1!} & 12\frac{w}{1!} & 12\frac{1}{1!} & 12\frac{2}{1!} & 12\frac{3}{1!} & 12\frac{4}{1!} \\ 0 & 0 & 0 & 0 & 12\frac{u^2}{2!} & 12\frac{v^2}{2!} & 12\frac{w^2}{2!} & 12\frac{1^2}{2!} & 12\frac{2^2}{2!} & 12\frac{3^2}{2!} & 12\frac{4^2}{2!} \\ 0 & 0 & 0 & 0 & 12\frac{u^3}{3!} & 12\frac{v^3}{3!} & 12\frac{w^3}{3!} & 12\frac{1^3}{3!} & 12\frac{2^3}{3!} & 12\frac{3^3}{3!} & 12\frac{4^3}{3!} \\ 0 & 0 & 0 & 0 & 12\frac{u^4}{4!} & 12\frac{v^4}{4!} & 12\frac{w^4}{4!} & 12\frac{1^4}{4!} & 12\frac{2^4}{4!} & 12\frac{3^4}{4!} & 12\frac{4^4}{4!} \\ 0 & 0 & 0 & 0 & 12\frac{u^5}{5!} & 12\frac{v^5}{5!} & 12\frac{w^5}{5!} & 12\frac{1^5}{5!} & 12\frac{2^5}{5!} & 12\frac{3^5}{5!} & 12\frac{4^5}{5!} \\ 0 & 0 & 0 & 0 & 12\frac{u^6}{6!} & 12\frac{v^6}{6!} & 12\frac{w^6}{6!} & 12\frac{1^6}{6!} & 12\frac{2^6}{6!} & 12\frac{3^6}{6!} & 12\frac{4^6}{6!} \\ 0 & 0 & 0 & 0 & 12\frac{u^7}{7!} & 12\frac{v^7}{7!} & 12\frac{w^7}{7!} & 12\frac{1^7}{7!} & 12\frac{2^7}{7!} & 12\frac{3^7}{7!} & 12\frac{4^7}{7!} \end{bmatrix}^{-1} \begin{bmatrix} \delta_0 \\ \delta_1(m) \\ \delta_2(m) \\ \delta_3(m) \\ \delta_4(m) \\ \delta_5(m) \\ \delta_6(m) \\ \delta_7(m) \\ \delta_8(m) \\ \delta_9(m) \\ \delta_{10}(m) \end{bmatrix} = \begin{bmatrix} (\phi_n - \phi'_n) \\ (\phi'_n - \phi''_n) \\ (\phi''_n - \phi'''_n) \\ g_n \\ g_{n+u} \\ g_{n+v} \\ g_{n+w} \\ g_{n+1} \\ g_{n+2} \\ g_{n+3} \\ g_{n+4} \end{bmatrix}$$

$$\tau_3 = -\frac{1}{5040} m^4 \left(\frac{-2016m+5040u+5670m^2u-1920m^3u+315m^4u}{v(v-3)(v-1)(v-2)(v-4)(2v-1)(u-v)} \right)$$

$$\tau_4 = -\frac{1}{30240} m^4 \left(\frac{-m^2(1464m-285m^2+20m^3-3108)(u+v) - 168(12mu+12mv-30uv+37muv) + m^2(-1776m-915m^2-190m^3+14m^4+1008) + 6m^2uv(-76m+5m^2+427)}{(u-1)(v-1)} \right)$$

$$\tau_5 = \frac{1}{60480} m^4 \left(\frac{-m^2(1104m-255m^2+20m^3+1806)(u+v) - 84(12mu+12mv-30uv+43muv) + 2m^2(-516m+345m^2-85m^3+7m^4+252) + 6m^2uv(-68m+5m^2+322)}{(u-2)(v-2)} \right)$$

Solving (10) for δ_i 's by Gaussian elimination method to obtain the unknown values in term of the parameter u, v, w , gives a continuous hybrid linear multistep method of the form

$$\eta(x) = \sum_{i=0}^k \mu_i \eta_{n+i} + h^3 \left[\sum_{j=0}^k \tau_j g_{n+j} + \sum_{j=\varsigma} \tau_j g_{n+\varsigma} \right], \varsigma = u, \frac{1}{2}, v$$

(11)

Where $\mu_0 = 1$ $\mu_1 = m$ $\mu_2 = \frac{1}{2} m^2$

$$\tau_0 = -\frac{1}{120960} m^3 \left(\frac{-m(8232m-5670m^2+1920m^3-315m^4+20m^5-5040)(u+v) + 2m^2(2058m-1620m^2+600m^3-105m^4+7m^5-1008) + 6uv(3430m-1890m^2+560m^3-84m^4+5m^5-3360)}{uv} \right)$$

$$\tau_1 = \frac{1}{5040} m^4 \left(\frac{-2016m+5040v+5670m^2v-1920m^3v+315m^4v-20m^5v-8232mv+4116m^2-3240m^3+1200m^4-210m^5+14m^6}{u(u-3)(u-1)(u-2)(u-4)(2u-1)(u-v)} \right)$$

$$\tau_2 = -\frac{8}{33075} m^4 \left(\frac{2100m^2u-840m^3u+2100m^2v-840m^3v+150m^4u+150m^4v-10m^5u-10m^5v-2016mu-2016mv+5040uv-1008m^2-1200m^3+525m^4-100m^5+7m^6+1470m^2uv-240m^3uv+15^4uv-4200muv}{(2u-1)(2v-1)} \right)$$

$$\tau_6 = -\frac{1}{151200} m^4 \left(\frac{-5m^2(m-6)(-21m+4m^2+42)(u+v) - 168(4mu+4mv-10uv+15muv) + m^2(-720m+525m^2-150m^3+14m^4+336) + 30m^2uv(-12m+m^2+49)}{(u-3)(v-3)} \right)$$

$$\tau_7 = -\frac{1}{846720} m^4 \left(\frac{-m^2(672m-195m^2+20m^3-966)(u+v) - 84(6mu+6mv-15uv+23muv) + 2m^2(m-3)(78m-44m^2+7m^3-42) + 6m^2uv(-52m+5m^2+196)}{(u-4)(v-4)} \right)$$

by substituting $m=1$ in (11), a multistep formula to approximate the solution of (1.1) at the point t_{n+1} yields

$$t_{n+1} = \eta_n + h\eta'_n + \frac{1}{2}\eta''_n h^2 + h^3 \left[\left(\frac{1}{120960} \frac{(-853u-853v+8034uv+136)}{uv} \right) g_n + \left(\frac{1}{5040} \frac{(853v-136)}{u(u-3)(u-1)(u-2)(u-4)(u-v)} \right) g_{n+u} \right.$$

$$+ \left(\frac{8}{33075} \frac{(-616u-616v+2085uv+240)}{(2u-1)(2v-1)} \right) g_{n+\frac{1}{2}} - \left(\frac{1}{5040} \frac{(853u-136)}{v(v-3)(v-1)(v-2)(v-4)(u-v)} \right) g_{n+v}$$

$$- \left(\frac{1}{30240} \frac{(-107u-107v+960rs-29)}{(s-1)(r-1)} \right) g_{n+1} - \left(\frac{1}{60480} \frac{(-71u-71v+462uv+6)}{(u-2)(v-2)} \right) g_{n+2}$$

$$\left. - \left(\frac{1}{151200} \frac{(-47u-47v+300uv+5)}{(u-3)(v-3)} \right) g_{n+3} + \left(\frac{1}{846720} \frac{(-35u-35v+222uv+4)}{(u-4)(v-4)} \right) g_{n+4} \right]$$

(12)

Also by substituting $m=1$ in the first derivative of (11), a multistep formula to approximate the solution of (1) at the

point t'_{n+1} yields

$$ht'_{n+1} = h\eta'_n + \eta''_n h^2 + h^3 \left[\left(\frac{1}{60480} \frac{(-960u-960v+8994uv+107)}{uv} \right) g_n + \left(\frac{1}{2520} \frac{(960v-107)}{u(u-3)(u-1)(u-2)(u-4)(u-v)} \right) g_{n+u} \right. \\ \left. + \left(\frac{16}{33075} \frac{(-1125u-1125v+3210-509)}{(2u-1)(2v-1)} \right) g_{n+\frac{1}{2}} - \left(\frac{1}{2520} \frac{(960u-107)}{v(v-3)(v-1)(v-2)(v-4)(u-v)} \right) g_{n+v} \right. \\ \left. - \left(\frac{1}{15120} \frac{(210u+210v+750rs+317)}{(u-1)(v-1)} \right) g_{n+1} + \left(\frac{1}{30240} \frac{(-36u-36v+498uv-35)}{(u-2)(v-2)} \right) g_{n+2} \right. \\ \left. - \left(\frac{1}{75600} \frac{(-30u-30v+330uv-17)}{(u-3)(v-3)} \right) g_{n+3} + \left(\frac{1}{423360} \frac{(-24u-24v+246uv-11)}{(u-4)(v-4)} \right) g_{n+4} \right] \quad (13)$$

Finally by substituting $m=1$ in the second derivative of (11), a multistep formula to approximate the solution of (1)

at the point t''_{n+1} yields

$$h^2 t''_{n+1} = \eta''_n h^2 + h^3 \eta'''_n + h^3 \left[\left(\frac{1}{10080} \frac{(-125u-125v+1624uv-35)}{uv} \right) g_n + \left(\frac{1}{84} \frac{(25v+7)}{u(u-3)(u-1)(u-2)(u-4)(u-v)} \right) g_{n+u} \right. \\ \left. + \left(\frac{16}{2205} \frac{(-164u-164v+378uv+89)}{(2u-1)(2v-1)} \right) g_{n+\frac{1}{2}} - \left(\frac{1}{84} \frac{(25u+7)}{v(v-3)(v-1)(v-2)(v-4)(u-v)} \right) g_{n+v} \right. \\ \left. + \left(\frac{1}{840} \frac{(-496u-496v+371uv+461)}{(v-1)(u-1)} \right) g_{n+1} + \left(\frac{1}{5040} \frac{(41u+41v+42uv-47)}{(u-2)(v-2)} \right) g_{n+2} \right. \\ \left. - \left(\frac{1}{2520} \frac{(4u+4v+7uv+5)}{(u-3)(v-3)} \right) g_{n+3} + \left(\frac{1}{70560} \frac{(13u+13v+28uv-17)}{(u-4)(v-4)} \right) g_{n+4} \right] \quad (14)$$

III ALGORITHM FOR THE DERIVATION OF THE OPTIMIZE METHOD

The algorithm for the implementation of the optimize three-step method is given by the following steps;

Step 1: Expanding (12) using the Taylor series to obtain the corresponding Local truncation error given as

$$\ell[\eta(x);h] = \frac{-1}{372556800} (-1496u-1496v+9383uv+196) + o(h^{11}) \quad (15)$$

$$\ell[\eta(x);h] = \frac{-1}{16934400} (-107u-107v+960uv-26) + o(h^{10}) \quad (16)$$

$$\ell[\eta(x);h] = \frac{-1}{16934400} (210u+210v+750uv-317) + o(h^9) \quad (17)$$

Step 2: Equate the principal term of the local truncation errors in (16) and (17) to zero, to obtain the values

$$u = \frac{8303}{50000} \quad \& \quad v = \frac{84331}{100000}$$

Step 3: Substitute the values of $u = \frac{8303}{50000}$ & $v = \frac{84331}{100000}$ as obtain in step 2 into (12) – (14) gives the equations as shown below.

$$\left(\varphi_n + \frac{8303}{50000} - \varphi_n + \frac{8303}{50000} \right) = (\varphi_n - \varphi_n) + \frac{8303}{50000} h(\varphi'_n - \varphi'_n) + \frac{68939809}{5000000000} h^2 (\varphi''_n - \varphi''_n) \\ + 199231987500 00000000 00000000 00000000 00000000 h^3 g_n \\ + 47365 829078831 868597096 061854751 883141843 h^3 g_{n+\frac{1}{2}} \\ + 131185191 970441369 321691259 937500000 000000000 h^3 g_{n+\frac{8303}{50000}} \\ - 52 444938059 929959025 721317407 230033599 534135571 h^3 g_{n+\frac{1}{2}} \\ - 370300 921562988 281250000 000000000 000000000 h^3 g_{n+\frac{1}{2}} \\ + 260079995 689062176 387002045 951474937 644385629 h^3 g_{n+\frac{84331}{100000}} \\ + 2089 086211009 045551938 702932339 083984375 000000000 h^3 g_{n+\frac{84331}{100000}} \\ - 8248738283 613938143 746219703 021312211 216286857 h^3 g_{n+1} \\ - 128628 338934375 000000000 000000000 000000000 h^3 g_{n+1} \\ + 256254247 609278063 644884229 621013236 848474693 h^3 g_{n+2} \\ + 266572 946193750 000000000 000000000 000000000 h^3 g_{n+2} \\ - 59846182 781044423 939853121 197822532 857953087 h^3 g_{n+3} \\ - 678544 866703125 000000000 000000000 000000000 h^3 g_{n+3} \\ + 1981314993 805030865 053982426 200586949 445736857 h^3 g_{n+4} \\ + 333576811 615162500 000000000 000000000 000000000 h^3 g_{n+4} \\ \left(\varphi_{n+\frac{1}{2}} - \varphi_{n+\frac{1}{2}} \right) = (\varphi_n - \varphi_n) + \frac{1}{2} h(\varphi'_n - \varphi'_n) + \frac{1}{8} h^2 (\varphi''_n - \varphi''_n) + \\ 1031 273503339 h^3 g_n + 35960 639938354 492187500 000000000 h^3 g_{n+\frac{8303}{50000}} \\ 172081 224007680 h^3 g_n + 2457304 343987376 499462373 710524957 h^3 g_{n+\frac{8303}{50000}} \\ - 104 756963863 h^3 g_{n+\frac{1}{2}} + 57495 091796875 000000000 000000000 h^3 g_{n+\frac{8303}{50000}} \\ - 485360 823911040 h^3 g_{n+\frac{1}{2}} + 60967891 982087985 387779106 377383827 h^3 g_{n+\frac{84331}{100000}} \\ - 1338 858375727 h^3 g_{n+1} + 758 029015619 h^3 g_{n+2} - 34 583621321 h^3 g_{n+3} \\ - 2528936 046120960 h^3 g_{n+1} + 82109585 628241920 h^3 g_{n+2} - 39429283 194040320 h^3 g_{n+3} \\ + 393 447716251 h^3 g_{n+4} + 6558386977 803386880 h^3 g_{n+4} \\ \left(\varphi_n + \frac{84331}{100000} - \varphi_n + \frac{84331}{100000} \right) = (\varphi_n - \varphi_n) + \frac{84331}{100000} h(\varphi'_n - \varphi'_n) + \frac{7111717561}{20 000000000} h^2 (\varphi''_n - \varphi''_n) \\ + 169 171183797 736887238 772353100 218644408 481233883 h^3 g_n \\ + 10043 308800000 000000000 000000000 000000000 h^3 g_n \\ + 6396460 664105961 378489466 579137697 257458822 782566199 h^3 g_{n+\frac{8303}{50000}} \\ + 104844985 343461397 310394611 649064832 000000000 000000000 h^3 g_{n+\frac{8303}{50000}} \\ + 4306608 877393405 693799053 139087391 643533717 736637273 h^3 g_{n+\frac{1}{2}} \\ + 189594071 840250000 000000000 000000000 000000000 h^3 g_{n+\frac{1}{2}} \\ - 612843659 085117757 712785056 734836508 121581621 h^3 g_{n+\frac{84331}{100000}} \\ - 481 972955631 879027386 363309478 000000000 000000000 h^3 g_{n+\frac{84331}{100000}} \\ + 43662 601901928 034538592 005967233 276113569 846229091 h^3 g_{n+1} \\ + 65857709 534400000 000000000 000000000 000000000 h^3 g_{n+1} \\ - 18871 982832592 692902741 665311293 484609408 142787273 h^3 g_{n+2} \\ - 6414811377 206400000 000000000 000000000 000000000 h^3 g_{n+2} \\ + 394 697758419 148652181 798375077 683773960 963841039 h^3 g_{n+3} \\ + 6600884463 288000000 000000000 000000000 000000000 h^3 g_{n+3} \\ + 453 375432848 474592815 624127641 650078111 549670909 h^3 g_{n+4} \\ + 170 791327546 963200000 000000000 000000000 000000000 h^3 g_{n+4} \\ (\varphi_{n+1} - \varphi_{n+1}) = (\varphi_n - \varphi_n) + h(\varphi'_n - \varphi'_n) + \frac{1}{2} h^2 (\varphi''_n - \varphi''_n) + \frac{1000 223051981}{42348 113720640} h^3 g_n \\ + 75956 175781250 000000000 000000000 h^3 g_{n+\frac{8303}{50000}} + 179 576795638 h^3 g_{n+\frac{1}{2}} \\ + 819101 447995792 166487457 903508319 h^3 g_{n+\frac{8303}{50000}} + 3791 881436805 h^3 g_{n+\frac{1}{2}} \\ + 47076 500000000 000000000 000000000 h^3 g_{n+\frac{84331}{100000}} + 53419453 h^3 g_{n+1} \\ + 20322630 660695995 129259702 125794609 h^3 g_{n+\frac{84331}{100000}} + 82 322136918 h^3 g_{n+1} \\ - 2416907317 h^3 g_{n+2} + 3059803 h^3 g_{n+3} - 199214159 h^3 g_{n+4} \\ - 320740 568860320 h^3 g_{n+2} + 6600 884463288 h^3 g_{n+3} - 8539566 377348160 h^3 g_{n+4} \\ (\varphi_{n+2} - \varphi_{n+2}) = (\varphi_n - \varphi_n) + 2h(\varphi'_n - \varphi'_n) + 2h^2 (\varphi''_n - \varphi''_n) + \frac{7987993709}{63 018026370} h^3 g_n \\ + 18300 500000000 000000000 000000000 h^3 g_{n+\frac{8303}{50000}} + 2461 279704832 h^3 g_{n+\frac{1}{2}} \\ + 52283 071148667 585094944 121500531 h^3 g_{n+\frac{8303}{50000}} + 3791 881436805 h^3 g_{n+\frac{1}{2}} \\ - 25138496 000000000 000000000 000000000 h^3 g_{n+\frac{84331}{100000}} + 374 988845708 h^3 g_{n+1} \\ - 60967891 982087985 387779106 377383827 h^3 g_{n+\frac{84331}{100000}} + 617 416026885 h^3 g_{n+1} \\ + 552582628 h^3 g_{n+2} - 1789173724 h^3 g_{n+3} + 24 235706153 h^3 g_{n+4} \\ + 42 651671391 h^3 g_{n+2} - 3208 763280765 h^3 g_{n+3} + 800584 347876390 h^3 g_{n+4} \\ (\varphi_{n+3} - \varphi_{n+3}) = (\varphi_n - \varphi_n) + 3h(\varphi'_n - \varphi'_n) + \frac{9}{2} h^2 (\varphi''_n - \varphi''_n) + \frac{532 447907571}{1568 448656320} h^3 g_n \\ + 743 425781250 000000000 000000000 h^3 g_{n+\frac{8303}{50000}} + 313 774417602 h^3 g_{n+\frac{1}{2}} \\ + 1123 595950611 511888185 813310711 h^3 g_{n+\frac{8303}{50000}} + 140 440053215 h^3 g_{n+\frac{1}{2}} \\ - 70939 500000000 000000000 000000000 h^3 g_{n+\frac{84331}{100000}} + 941426709 h^3 g_{n+1} \\ + 27877 408313711 927474979 015261721 h^3 g_{n+\frac{84331}{100000}} + 282311855 h^3 g_{n+1} \\ + 5517 920688597 h^3 g_{n+2} + 88 134949863 h^3 g_{n+3} - 16 409022363 h^3 g_{n+4} \\ + 11879 280328160 h^3 g_{n+2} + 8556 702082040 h^3 g_{n+3} - 105426 745399360 h^3 g_{n+4} \\ (\varphi_{n+4} - \varphi_{n+4}) = (\varphi_n - \varphi_n) + 4h(\varphi'_n - \varphi'_n) + 8h^2 (\varphi''_n - \varphi''_n) + \frac{57 165767528}{94 527039555} h^3 g_n \\ + 1015024 000000000 000000000 000000000 h^3 g_{n+\frac{8303}{50000}} + 15866 913759232 h^3 g_{n+\frac{1}{2}} \\ + 819101 447995792 166487457 903508319 h^3 g_{n+\frac{8303}{50000}} + 3791 881436805 h^3 g_{n+\frac{1}{2}} \\ - 96856064 000000000 000000000 000000000 h^3 g_{n+\frac{84331}{100000}} + 1424 742662528 h^3 g_{n+1} \\ + 20322630 660695995 129259702 125794609 h^3 g_{n+\frac{84331}{100000}} + 205 805342295 h^3 g_{n+1} \\ + 19920 14146944 h^3 g_{n+2} + 14030 510192512 h^3 g_{n+3} + 72 711787448 h^3 g_{n+4} \\ + 10023 142776885 h^3 g_{n+2} + 28878 869526885 h^3 g_{n+3} + 8895 381643071 h^3 g_{n+4}$$

“An Optimized Four-Step Three-Off grid third derivative Methods based on the Volterra integral equation of the second kind for Solving third order Initial Value Problems”

$$\begin{aligned} & \left(\phi''_{n+2} - \frac{8303}{50000} \phi''_{n+1} - \frac{8303}{50000} \phi''_n \right) = (\phi''_n - \phi''_{n+1}) + \frac{8303}{50000} h (\phi''_n - \phi''_{n+1}) + \frac{2959}{39585} 177566360 \cdot 813209474 \cdot 136100694 \cdot 311272053 \cdot h^2 g_{n+2} \\ & + 1172 \cdot 865416071 \cdot 839222721 \cdot 54470536 \cdot 300712381 \cdot h^2 g_{n+1} - 7406018431 \cdot 125976582 \cdot 500000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+1} \\ & + 1493143321173 \cdot 362762742 \cdot 059649329 \cdot 106697143 \cdot h^2 g_{n+1} - 945213 \cdot 55117026 \cdot 955738560 \cdot 146687004 \cdot 126273941 \cdot h^2 g_{n+1} \\ & + 8267258 \cdot 838227136 \cdot 858316789 \cdot 9790717251 \cdot 953123600 \cdot h^2 g_{n+1} - 77177033 \cdot 606250000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+1} \\ & + 427517 \cdot 939361322 \cdot 808395230 \cdot 408744236 \cdot 129253941 \cdot h^2 g_{n+1} - 15886 \cdot 34690987 \cdot 984258631 \cdot 535395713 \cdot 71222239 \cdot h^2 g_{n+1} \\ & + 35 \cdot 057859942 \cdot 212500000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+2} - 9499628133 \cdot 843750000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+2} \\ & + 235214 \cdot 914250115 \cdot 808789396 \cdot 468390066 \cdot 735213941 \cdot h^2 g_{n+4} \\ & + 3001 \cdot 468696907 \cdot 750000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+4} \\ & \left(\phi''_{n+1} - \frac{1}{2} \phi''_{n+1} + \frac{1}{2} \phi''_n \right) = (\phi''_n - \phi''_{n+1}) + \frac{1}{2} h (\phi''_n - \phi''_{n+1}) + \frac{8186}{338784} 751941209 \cdot h^2 g_{n+2} - 213457 \cdot 190246582 \cdot 011250000 \cdot 000000000 \cdot h^2 g_{n+2} \\ & + 98 \cdot 418587881 \cdot h^2 g_{n+1} - 215751 \cdot 054687500 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+1} + 203 \cdot 768389103 \cdot h^2 g_{n+1} - 4880087423 \cdot h^2 g_{n+2} \\ & + 6067 \cdot 01026888 \cdot h^2 g_{n+1} - 60676891 \cdot 983087985 \cdot 387779106 \cdot 377383827 \cdot h^2 g_{n+1} + 158058 \cdot 502882560 \cdot h^2 g_{n+1} - 1282962 \cdot 275441280 \cdot h^2 g_{n+2} \\ & + 1090999 \cdot h^2 g_{n+4} \\ & + 1478598 \cdot 119776512 \cdot h^2 g_{n+3} + 204049593 \cdot 056355840 \cdot h^2 g_{n+4} \\ & \left(\phi''_{n+1} - \frac{84331}{100000} \phi''_n + \frac{84331}{100000} \phi''_{n+1} \right) = (\phi''_n - \phi''_{n+1}) + \frac{84331}{100000} h (\phi''_n - \phi''_{n+1}) + 99211 \cdot 827691170 \cdot 230381256 \cdot 001903223 \cdot 094048521 \cdot h^2 g_{n+2} \\ & + 2510827 \cdot 200000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+2} \\ & + 48 \cdot 563199921 \cdot 456336622 \cdot 879526714 \cdot 13762017 \cdot 298887629 \cdot h^2 g_{n+1} + 6046377503 \cdot 641729777 \cdot 260809903 \cdot 045170765 \cdot 247801951 \cdot h^2 g_{n+1} \\ & + 157 \cdot 26749015 \cdot 130295955 \cdot 591931473 \cdot 597248000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+1} \\ & + 7757 \cdot 979172585 \cdot 586499713 \cdot 839929637 \cdot 534678093 \cdot h^2 g_{n+1} - 76767343 \cdot 599263173 \cdot 339888629 \cdot 066363449 \cdot 289979451 \cdot h^2 g_{n+1} \\ & + 145918 \cdot 966895637 \cdot 082159089 \cdot 926434000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+1} \\ & + 49 \cdot 393282150 \cdot 800000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+1} \\ & + 58853796 \cdot 54630510 \cdot 540546268 \cdot 807638006 \cdot 374134851 \cdot h^2 g_{n+1} - 58304626 \cdot 88881428 \cdot 634842754 \cdot 69190515 \cdot 48829843 \cdot h^2 g_{n+1} \\ & + 1603 \cdot 7025443201 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+1} \\ & + 11557 \cdot 547810754 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+1} \\ & + 24773620 \cdot 808737414 \cdot 927578809 \cdot 170593476 \cdot 542444451 \cdot h^2 g_{n+4} \\ & + 128093 \cdot 49560222 \cdot 400000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+4} \\ & \left(\phi''_{n+1} - \phi''_{n+1} \right) = (\phi''_n - \phi''_{n+1}) + h (\phi''_n - \phi''_{n+1}) + \frac{993}{21174} 812717621 \cdot h^2 g_{n+2} - 548888 \cdot 750000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+2} \\ & + 243704 \cdot 34398376 \cdot 499462373 \cdot 710524957 \cdot h^2 g_{n+2} + 708 \cdot 050647528 \cdot h^2 g_{n+1} \\ & + 2628080 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+1} \\ & + 60676891 \cdot 983087985 \cdot 387779106 \cdot 377383827 \cdot h^2 g_{n+1} + 112 \cdot 8075 \cdot 3993427043 \cdot h^2 g_{n+1} + 33940331 \cdot h^2 g_{n+1} \\ & + 193756361 \cdot h^2 g_{n+4} \\ & + 1280934 \cdot 566022240 \cdot h^2 g_{n+4} \\ & \left(\phi''_{n+2} - \frac{1}{2} \phi''_{n+2} + \frac{1}{2} \phi''_{n+1} \right) = (\phi''_{n+1} - \phi''_{n+2}) + 2h (\phi''_{n+1} - \phi''_{n+2}) + \frac{2927}{1323} 783553798 \cdot h^2 g_{n+2} - 22950 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+2} \\ & + 2857804 \cdot 339987376 \cdot 49942374 \cdot 710524957 \cdot h^2 g_{n+2} + 5497 \cdot 399398808 \cdot h^2 g_{n+2} \\ & + 153136400 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+2} \\ & + 1269 \cdot 405361716 \cdot h^2 g_{n+1} + 139 \cdot 138909660 \cdot h^2 g_{n+2} - 71 \cdot 680407032 \cdot h^2 g_{n+1} + 105 \cdot 621211427 \cdot h^2 g_{n+1} \\ & + 60967891 \cdot 982087985 \cdot 387779106 \cdot 377383827 \cdot h^2 g_{n+1} + 6174 \cdot 6026885 \cdot h^2 g_{n+1} + 2004 \cdot 628553377 \cdot h^2 g_{n+2} - 28878 \cdot 869526885 \cdot h^2 g_{n+2} \\ & + 800584 \cdot 347876390 \cdot h^2 g_{n+4} \\ & \left(\phi''_{n+3} - \phi''_{n+3} \right) = (\phi''_{n+1} - \phi''_{n+3}) + 3h (\phi''_{n+1} - \phi''_{n+3}) + \frac{163161461479}{112 \cdot 035046080} \cdot h^2 g_{n+2} - 15330 \cdot 468780000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+2} \\ & + 10112 \cdot 598355500 \cdot 606994672 \cdot 319796399 \cdot h^2 g_{n+2} + 29 \cdot 680106843 \cdot h^2 g_{n+2} \\ & + 316000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+2} \\ & + 2688 \cdot 408859 \cdot h^2 g_{n+1} + 6399 \cdot 270187227 \cdot h^2 g_{n+2} + 124 \cdot 810290878 \cdot h^2 g_{n+3} - 81 \cdot 516997077 \cdot h^2 g_{n+4} \\ & + 5039 \cdot 640164080 \cdot h^2 g_{n+1} + 1711 \cdot 340416408 \cdot h^2 g_{n+3} + 52713 \cdot 372699680 \cdot h^2 g_{n+4} \\ & + 1336 \cdot 0640392 \cdot h^2 g_{n+1} + 1548020 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+1} \\ & + 601 \cdot 689276885 \cdot h^2 g_{n+1} + 2457804 \cdot 34398376 \cdot 499462373 \cdot 710524957 \cdot h^2 g_{n+1} + 23971 \cdot 881438085 \cdot h^2 g_{n+1} \\ & + 151086920 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h^2 g_{n+1} \\ & + 6697891 \cdot 982087985 \cdot 387779106 \cdot 377383827 \cdot h^2 g_{n+1} + 11574 \cdot 899818816 \cdot h^2 g_{n+1} + 13373 \cdot 080576624 \cdot h^2 g_{n+2} + 66925 \cdot 253818256 \cdot h^2 g_{n+3} \\ & + 80058 \cdot 34787639 \cdot h^2 g_{n+4} \\ & \left(\phi''_{n+1} - \frac{8303}{50000} \phi''_{n+1} - \frac{8303}{50000} \phi''_n \right) = (\phi''_n - \phi''_{n+1}) + \frac{82942670}{1328213250} 970987442 \cdot 956274455 \cdot 136709863 \cdot h_{n+2} \\ & + 59976709 \cdot 824499221 \cdot 437167487 \cdot 587830439 \cdot h_{n+1} - 76 \cdot 956182290 \cdot 648260955 \cdot 259589049 \cdot 038237511 \cdot h_{n+2} \\ & + 599765221 \cdot 808859548 \cdot 649559137 \cdot 365000000 \cdot h_{n+1} - 2468 \cdot 672801919 \cdot 921875000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} \\ & + 2228661862 \cdot 821214665 \cdot 757828461 \cdot 284788943 \cdot h_{n+1} + 35 \cdot 146765148 \cdot 867440978 \cdot 25353593 \cdot 428467511 \cdot h_{n+1} \\ & + 83 \cdot 563448440 \cdot 361822077 \cdot 548117325 \cdot 5633593175 \cdot h_{n+1} - 2572 \cdot 56673687 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+1} \\ & + 16 \cdot 868570055 \cdot 511009588 \cdot 042511167 \cdot 958927511 \cdot h_{n+1} - 5841 \cdot 75906 \cdot 73380001 \cdot 594577985 \cdot 44680969 \cdot h_{n+3} \\ & + 35326 \cdot 18907375 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} \\ & + 31665 \cdot 427112812 \cdot 500000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+3} \\ & + 8270940281 \cdot 967693091 \cdot 893795260 \cdot 019847511 \cdot h_{n+4} \\ & + 6671536 \cdot 232303250 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+4} \\ & \left(\phi''_{n+1} - \frac{1}{2} \phi''_{n+1} + \frac{1}{2} \phi''_n \right) = (\phi''_n - \phi''_{n+1}) + \frac{1545}{37642} 893298377 \cdot h_{n+2} - 79017 \cdot 907714843 \cdot 750000000 \cdot 000000000 \cdot h_{n+2} \\ & + 37642 \cdot 767751688 \cdot h_{n+2} + 273033 \cdot 315985937 \cdot 388829152 \cdot 634502773 \cdot h_{n+2} \\ & + 344 \cdot 512814837 \cdot h_{n+2} - 427873 \cdot 437500000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+1} \\ & + 1685 \cdot 280638580 \cdot h_{n+1} + 6774210 \cdot 220231998 \cdot 376419900 \cdot 708598203 \cdot h_{n+1} + 4390 \cdot 513968960 \cdot h_{n+1} \\ & - 90 \cdot 405327679 \cdot h_{n+2} - 136706429 \cdot h_{n+3} + 13 \cdot 042958167 \cdot h_{n+4} \\ & + 285102 \cdot 727875840 \cdot h_{n+2} - 5134 \cdot 021492224 \cdot h_{n+3} + 7990725 \cdot 668753920 \cdot h_{n+4} \\ & \left(\phi''_{n+1} - \frac{84331}{100000} \phi''_{n+1} - \frac{84331}{100000} \phi''_n \right) = (\phi''_n - \phi''_{n+1}) + \frac{1599276443}{33 \cdot 47696000} 714515848 \cdot 362357249 \cdot 431920689 \cdot h_{n+2} \\ & + 241178 \cdot 994263398 \cdot 815338748 \cdot 430993366 \cdot 529744059 \cdot h_{n+2} \\ & + 1048449 \cdot 853434613 \cdot 973103946 \cdot 116496048 \cdot 3200000000 \cdot h_{n+2} + 331980 \cdot 239467500 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} \\ & + 866347036 \cdot 009376536 \cdot 948092713 \cdot 963633094 \cdot 780000000 \cdot h_{n+2} + 20450 \cdot 305614065 \cdot 304921967 \cdot 501790894 \cdot 244135941 \cdot h_{n+2} \\ & + 18917929556 \cdot 3187962073 \cdot 963633094 \cdot 780000000 \cdot h_{n+2} + 638577 \cdot 095344000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} \\ & + 2563 \cdot 900232180 \cdot 1484992010 \cdot 987586767 \cdot 701895941 \cdot h_{n+2} - 2572 \cdot 56673687 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} \\ & + 13526704 \cdot 594688000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+3} \\ & + 129 \cdot 440561172 \cdot 556478750 \cdot 212685777 \cdot 339655941 \cdot h_{n+3} \\ & + 15402637 \cdot 476720000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+3} \\ & + 795 \cdot 498195969 \cdot 221721505 \cdot 419789222 \cdot 887415941 \cdot h_{n+4} \\ & + 1707913275 \cdot 469632000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+4} \\ & \left(\phi''_{n+1} - \phi''_{n+1} \right) = (\phi''_n - \phi''_{n+1}) + \frac{41}{882} 40862822 \cdot h_{n+2} + 219396 \cdot 484375000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} + 472 \cdot 030756984 \cdot h_{n+2} \\ & + 882 \cdot 252569180 \cdot h_{n+2} + 819101 \cdot 447995792 \cdot 166487457 \cdot 903508319 \cdot h_{n+2} + 1263 \cdot 960478935 \cdot h_{n+2} \\ & + 55575750 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} \\ & + 61 \cdot 536708703 \cdot h_{n+1} + 1646 \cdot 442738360 \cdot h_{n+1} + 26728 \cdot 380738360 \cdot h_{n+2} - 77010 \cdot 318738360 \cdot h_{n+3} \\ & + 52664551 \cdot h_{n+4} \\ & + 106 \cdot 445 \cdot 797168520 \cdot h_{n+4} \\ & \left(\phi''_{n+2} - \phi''_{n+2} \right) = (\phi''_n - \phi''_{n+2}) + \frac{10358}{7739053870} 750000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} + 1066 \cdot 875645824 \cdot h_{n+2} \\ & + 7739053870 \cdot h_{n+2} + 14370 \cdot 2002340031 \cdot 4431517323 \cdot 822868567 \cdot h_{n+2} + 421 \cdot 320159645 \cdot h_{n+2} \\ & + 32462240 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} \\ & + 295 \cdot 643828714 \cdot h_{n+1} + 336 \cdot 992598374 \cdot h_{n+2} - 22 \cdot 955268034 \cdot h_{n+3} \\ & + 6774210 \cdot 220231998 \cdot 376419900 \cdot 708598203 \cdot h_{n+1} + 68 \cdot 601780765 \cdot h_{n+1} + 1113 \cdot 682530765 \cdot h_{n+2} - 3208 \cdot 763280765 \cdot h_{n+3} \\ & + 10 \cdot 927284017 \cdot h_{n+4} \\ & + 29651 \cdot 272143570 \cdot h_{n+4} \\ & \left(\phi''_{n+3} - \phi''_{n+3} \right) = (\phi''_n - \phi''_{n+3}) - 134 \cdot 856575097 \cdot h_{n+2} + 131120 \cdot 703125400 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} - 1010 \cdot 887249224 \cdot h_{n+2} \\ & + 98 \cdot 028041020 \cdot h_{n+2} + 30337 \cdot 094666510 \cdot 302981076 \cdot 959389197 \cdot h_{n+2} + 140 \cdot 44053215 \cdot h_{n+2} \\ & + 10084850 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} \\ & + 491 \cdot 814558563 \cdot h_{n+1} + 4759 \cdot 763729601 \cdot h_{n+2} + 548 \cdot 893236153 \cdot h_{n+3} \\ & + 752590 \cdot 024470222 \cdot 041824433 \cdot 412066467 \cdot h_{n+1} - 60 \cdot 979360880 \cdot h_{n+1} + 2969 \cdot 820082040 \cdot h_{n+2} + 1771 \cdot 340416408 \cdot h_{n+3} \\ & - 198 \cdot 882414551 \cdot h_{n+4} \\ & - 39535 \cdot 029524760 \cdot h_{n+4} \\ & \left(\phi''_{n+4} - \phi''_{n+4} \right) = (\phi''_n - \phi''_{n+4}) + 2061 \cdot 123628714 \cdot h_{n+2} - 21253520 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} + 12249 \cdot 620480000 \cdot h_{n+2} \\ & + 220 \cdot 563092295 \cdot h_{n+2} + 819101 \cdot 447995792 \cdot 166487457 \cdot 903508319 \cdot h_{n+2} + 252 \cdot 792095787 \cdot h_{n+2} \\ & - 170697280 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot 000000000 \cdot h_{n+2} \\ & + 11462 \cdot 716931264 \cdot h_{n+1} - 5403 \cdot 115950776 \cdot h_{n+2} + 17204 \cdot 726531264 \cdot h_{n+3} \\ & + 2032630 \cdot 666095995 \cdot 129259702 \cdot 125794609 \cdot h_{n+1} + 205 \cdot 805342295 \cdot h_{n+1} - 3341 \cdot 047592295 \cdot h_{n+2} + 9626 \cdot 289842295 \cdot h_{n+3} \\ & + 36319 \cdot 277000998 \cdot h_{n+4} \\ & + 133430 \cdot 724646065 \cdot h_{n+4} \end{aligned}$$

(15)

IV ANALYSIS OF BASIC PROPERTIES OF THE METHOD

Order of the Block

Let the linear operator $L\{y(x):h\}$ associated with the discrete block method (15) be defined as

$$L\{\phi(t):h\} = A^{(0)} \phi_m^i - \sum_{j=0}^3 h^i e_i \phi_n^i - h^4 (g_i \gamma(\phi_n) + p_i \gamma(\phi_m)) \quad (16)$$

Expanding (12) in Taylor series and comparing the coefficient of h gives

$$L\{y(x):h\} = C_0 y(x) + C_1 y'(x) + \dots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + C_{p+3} h^{p+3} y^{p+3}(x) + \dots$$

Definition: Linear operator L and associated block formula are said to be of order

$$p, \text{ if } C_0 = C_1 = \dots = C_p = C_{p+1} = C_{p+2} = 0. \text{ and } C_{p+3} \neq 0. C_{p+3}$$

is called the error constant and implies that the truncation error is given by

$$t_{n+k} = C_{p+3} h^{p+3} y^{p+3}(x) + O(h^{p+4}).$$

For our method, expanding (15) in Taylor series Comparing the coefficient of h gives

$$C_0 = C_1 = C_2 = C_3 = \dots = C_9 = 0 \text{ and } C_{10} = \begin{bmatrix} -\frac{1242821607424}{351921678624755859375}, & -\frac{15217}{87887055375}, & -\frac{193}{571536000}, & -\frac{53}{4465125}, & -\frac{159}{784000}, & -\frac{4986438656}{74480778544921875} \\ -\frac{1258}{1953045675}, & -\frac{257}{12700800}, & -\frac{467}{99225}, & -\frac{17131}{156800}, & -\frac{5906887936}{6384066732421875}, & -\frac{19}{217005075}, & -\frac{23}{12700800}, \\ \frac{886320180988492}{39318633167698828125}, & -\frac{144842602584717907}{18639944612835000000}, & -\frac{16780608}{2251875390625}, & -\frac{16}{1476225}, & -\frac{23}{88905600}, & -\frac{8}{14175}, & -\frac{8934287}{2622352320} \end{bmatrix}$$

Hence our method is of order five (7).

Consistency

The optimize hybrid block method (15) is said to be consistent if

$$Y_m = \mu(z) Y_{m-1}, z = \lambda h \quad (18)$$

Where $\mu(z)$ is the amplification matrix given by

$$\mu(z) = -\left(\xi^1 - \zeta^{(1)} - z^4 \eta^{(1)}\right)^{-1} \left(\xi^{(0)} - \zeta^{(0)} - z^4 \eta^{(0)}\right) \quad (19)$$

The matrix $\mu(z)$ has Eigen values $(0, 0, \dots, \xi_k)$ where

ξ_k is called the stability function. Thus, the stability function for three-step optimize third derivative method with three off grid hybrid points based on the Volterra integral equation of the second kind is given by

$$\xi = \frac{3559170761614363211316z^7 - 61177826371777287130170z^6 + 232512117478282416668877z^5 - 1343794418406246970615008z^4 + 2746207325002929068984972z^3 - 4678197774730915644611280z^2 + 2276680107484104332000000z - 14116037906880000000000}{5883365403807430188z^7 - 132858824500543213202z^6 + 1489574607384253468845z^5 - 10312612430082415415040z^4 + 45938133307789620169740z^3 - 127897416332604943811280z^2 + 203083379005384332000000z - 141160379068800000000000}$$

Regions of Absolute Stability

The stability polynomial for optimize three step with three off step point based on the Volterra integral equation of the second kind gives

$$h(w) = -\frac{53036225293}{21000000000}h^7w^7 + h^6\left(\frac{19759857127554541889}{137851932684375000000}w^7 - \frac{163002347212944362893}{56016023440000000000}w^6\right) - h^5\left(\frac{51722006925743204960047}{4705345968960000000000}w^7 - \frac{1251359862038683416787}{224064093760000000000}w^6\right) - h^4\left(\frac{10645984923211418387787}{196056082040000000000}w^6 - \frac{24483171515212802587891}{588168246120000000000}w^7\right) - h^3\left(\frac{398097350208157976049979}{3529009476720000000000}w^7 - \frac{42848430624217385176957}{504144210960000000000}w^6\right) - h^2\left(\frac{3557669935828043930849}{196056082040000000000}w^6 - \frac{3117462273982401330849}{196056082040000000000}w^7\right) + h\left(\frac{239390246276786083}{35290094767200000}w^6 - \frac{380550625345586083}{35290094767200000}w^7\right) + w^7 - w^6$$

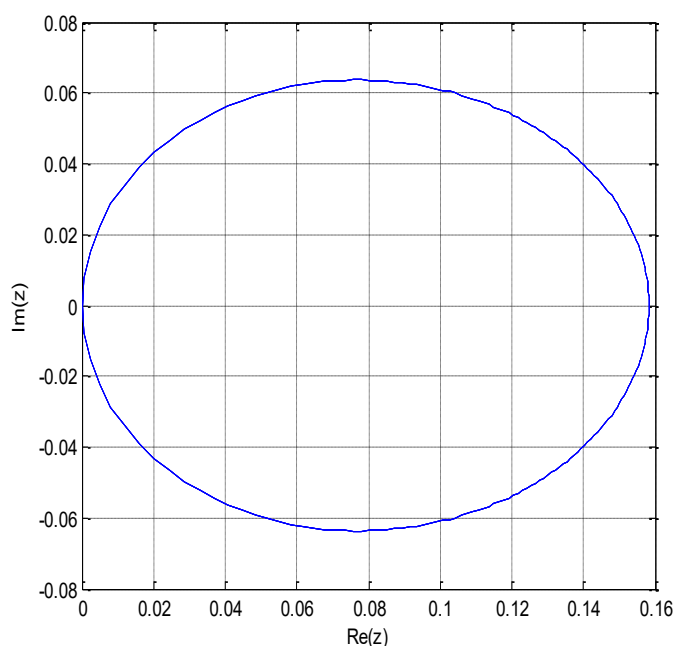


Figure1: Region of absolute stability

V NUMERICAL RESULT AND DISCUSSION

Problem 1 We consider the initial value problem

$$\mu''' - \mu'' + \mu' - \mu = 0, \quad h = 0.01:$$

$$\mu(0) = 1, \mu'(0) = 0, \mu''(0) = -1.$$

with Exact Solution: $\mu(x) = \cos(x)$

Table I. Showing the Result meant for Problem 1

x-values	Error in our method	Error in Atabo <i>et al</i> (2022)
0.1	2.00000e-30	0.000000e+00
0.2	7.00000e-30	1.110223e-16
0.3	4.00000 e-30	1.332268e-15
0.4	4.81000 e-29	7.105427e-15
0.5	1.97000 e-27	2.489617e-14
0.6	6.37500 e-27	
0.7	1.32460 e-26	
0.8	2.28080 e-26	
0.9	4.11980 e-26	
1.0	7.04830e-26	

Problem 2 We consider the initial value problem of third-order

$$\mu''' + \mu' = 0$$

$$\mu(0) = 0, \mu'(0) = 1, \mu''(0) = 2, h = \frac{1}{100}$$

with Exact Solution: $\mu(x) = 2(1 - \cos(x)) + \sin(x)$,

Table II. Showing the Result meant for Problem 2

x-values	Error in our method	Error in Kayode & Adebisi (2025)
0.01	3.35698e-24	4.38e-18
0.02	3.93630e-23	2.04e-17
0.03	1.52751e-22	4.82e-17
0.04	3.93699e-22	8.72e-17
0.05	8.17272e-22	1.33e-16
0.06	1.48179e-21	1.84e-16
0.07	2.44819e-21	2.42e-16
0.08	3.77810e-21	3.05e-16
0.09	5.53320e-21	3.69e-16
0.10	7.77317e-21	4.33e-16

Problem 3 We consider the initial value problem of third-order

$$\mu''' + \mu = 0$$

$$\mu(0) = 1, \mu'(0) = -1, \mu''(0) = 1, h = \frac{1}{10}$$

with Exact Solution: $\mu(x) = \exp(-x)$

Table III. Showing the Result meant for Problem

x-values	Error in our method	Error in Kayode <i>et al</i> (2025)
0.01	3.15157e-24	4.6730e-22
0.02	3.47261e-23	3.24300e-21
0.03	1.264789e-22	1.44730e-20
0.04	3.06360e-22	6.08299e-20
0.05	6.01969e-22	1.17224e-19
0.06	1.03449e-21	2.02888e-19
0.07	1.62500e-21	3.53857e-19
0.08	2.39138e-21	5.36240e-19
0.09	3,34903e-21	7.76000e-19
0.10	4.51082e-21	1.12175e-18

VI CONCLUSIONS

Several third derivative numerical problems from Akinnukawe (2024), Mohammed (2010), Victor & Solomon (2021), and Ukpebr et al. (2020) are used to test the created optimized three-step third derivative hybrid block approach. The optimal hybrid methods were derived using Scientific Workplace 5.5 software, whereas the implementation was done using Maple 18. The MATLAB 2021a programming language was also used to create the graphical representation, which displays an A-stable zone as seen above. The data above make it clear that our suggested optimized approaches can handle stiff equations and converge more quickly than the current approach. It is evident that the new approach outperforms the current one when compared to the results obtained in tables 1, 2, and 3, respectively. If feasible, this approach could be expanded to higher derivatives that provide solutions for both the stiff and oscillatory general ordinary differential equations.

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