



Three Dimensional Generalized Arithmetic Progressions with Multiplicity Three

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ARTICLE INFO	ABSTRACT
Published Online: 18 April 2026	The principle purpose of producing this article is to review, expose and present the advanced concepts of arithmetic progressions originated by combining the concepts of dimension (three) and the multiplicity (three) generally applied on different common differences, which was previously published and mentioned in chapter six in two books on multidimensional arithmetic progressions cited in the reference list and articles. It contains a discussion about many basic properties of three dimensional generalized arithmetic progressions with multiplicity three. It includes the processes to find the general term, sum of finite number of terms and the formulae to find the arithmetic means between any two arbitrary members of generalized progressions. The mathematics teachers and learners will find it fruitful to teach the topic with a new look and a research-oriented approach. This article also opens a new area of research and its applications.
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1. INTRODUCTION

In mathematics, the study of patterns leads to many significant mathematical generalizations. The succession of numbers (real or complex) designated as the first term, the second term, the third term and so on gives rise to what we name it a sequence. A sequence is also known as progression. Such progression is then divided into arithmetic, geometric, harmonic, etc. progressions. 4000 years ago Babylonians knew about arithmetic and geometric sequences. It has been found that the arithmetic, geometric and harmonic sequences were also known to early Greek writers (James & James, 2001; Katz, 2019; Progression - Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020, 2026; Yadav et al., 2026).

A progression is a collection of numbers (may be repeated or different) in a fixed order followed by a specific rule. The terms of the sequence are called elements or terms or members. In a set we don't repeat elements but in a sequence we can repeat it multiple times following a definite order. In progression the order of the terms matter whereas in a set order doesn't matter. It is defined by a function from an index

set. It is finite or infinite as per the number of terms. It is denoted by its n th term, where n denotes the n th element of the progression. We can represent the finite progression (sequence) having its n th term as t_n by $(t_n)_{n=1}^r$, where r is a natural number. The infinite sequence for the same n th term of the sequence is denoted by $(t_n)_{n=1}^{\infty}$, where n is a natural number (James & James, 2001; Progression - Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020, 2026; Yadav et al., 2026). The sum of the terms of a progression makes a series denoted by $\sum_{n=1}^{\infty} t_n = t_1 + t_2 + t_3 + \dots$, where t_n is the n th term of the sequence. The partial sum of the series is obtained by replacing the infinity symbol with a finite number (say N) as $\sum_{n=1}^N t_n = t_1 + t_2 + t_3 + \dots + t_N$. Generally it is denoted by S_N . If the partial sum converges, we say that the infinite series is convergent (Convergent series - Wikipedia contributors, 2025; Progression - Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Series (mathematics) - Wikipedia contributors, 2026; Ulas, 2005; Yadav, 2008a, 2008b, 2010, 2020, 2026; Yadav et al., 2026).

As discussed earlier a progression may be divided into many parts out of which the one is the arithmetic progression (Arithmetic progression - Wikipedia contributors, 2026; Dawson, 2012; Progression - Wikipedia contributors, 2025). It is a progression of numbers based on the fact that the difference from any succeeding term to its preceding term remains a fixed constant throughout the entire progression. The fixed difference is called a common difference of the arithmetic progression. In it we study about the general term, particular term, sum of the first n and infinite terms and some properties. A finite part of an arithmetic progression is called a finite arithmetic progression. The sum of a finite arithmetic progression is called an arithmetic series (Arithmetic progression - Wikipedia contributors, 2026; Convergent series - Wikipedia contributors, 2025; Progression - Wikipedia contributors, 2025; Series (mathematics) - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020, 2026; Yadav et al., 2026).

2. PRELIMINARY IDEAS

To understand the article we should have the basic concepts of the mathematical terms multiplicity, dimension, multiple arithmetic progressions, K -dimensional generalized arithmetic progression, and K -Dimensional Generalized Arithmetic Progression with Multiplicity m discussed by Yadav (2008a, 2008b, 2010, 2020, 2026) and Yadav et al. (2026).

3. DISCUSSION

We know that 3-D.A.P.M.(3) stands for three dimensional arithmetic progressions with multiplicity three. It is also known as three dimensional generalized arithmetic progressions with multiplicity three. Yadav et al. (2026) have discussed the generation of arithmetic means in the middle of two terms of three dimensional arithmetic progressions with multiplicity two. So we can discuss the following arithmetic progression:

3.1. Three Dimensional Arithmetic Progression with Multiplicity Three: If a is the first term, d_1 is the first common difference (c.d.), d_2 is the second c.d., and d_3 is the third c.d., then the general form of 3-D.G.A.P. with multiplicity three (may also be denoted by 3-D.A.P.M.(3)) is given by

$$\begin{aligned} &a \\ &a + d_1 \\ &a + 2d_1 \\ &a + 3d_1 \\ &a + 3d_1 + d_2 \\ &a + 3d_1 + 2d_2 \\ &a + 3d_1 + 3d_2 \\ &a + 3d_1 + 3d_2 + d_3 \\ &a + 3d_1 + 3d_2 + 2d_3 \end{aligned}$$

$$\begin{aligned} &a + 3d_1 + 3d_2 + 3d_3 \\ &a + 4d_1 + 3d_2 + 3d_3 \\ &a + 5d_1 + 3d_2 + 3d_3 \\ &a + 6d_1 + 3d_2 + 3d_3 \\ &a + 6d_1 + 4d_2 + 3d_3 \\ &a + 6d_1 + 5d_2 + 3d_3 \\ &a + 6d_1 + 6d_2 + 3d_3 \\ &a + 6d_1 + 6d_2 + 4d_3 \\ &a + 6d_1 + 6d_2 + 5d_3 \\ &a + 6d_1 + 6d_2 + 6d_3 \\ &a + 7d_1 + 6d_2 + 6d_3 \\ &a + 8d_1 + 6d_2 + 6d_3 \\ &a + 9d_1 + 6d_2 + 6d_3 \\ &a + 9d_1 + 7d_2 + 6d_3 \\ &a + 9d_1 + 8d_2 + 6d_3 \\ &a + 9d_1 + 9d_2 + 6d_3 \\ &a + 9d_1 + 9d_2 + 7d_3 \end{aligned}$$

and so on.

Example 3.1: 3, 5, 7, 9, 12, 15, 18, 22, 26, 30, 32, 34, 36, ... is a 3-D.A.P.M.(3) with 3 as first term and 2, 3, 4 as successive common differences.

3.2. General Term of a 3-D.A.P.M.(3): The n th term of 3-D.A.P.M.(3) cannot be written in a single mathematical expression. There arise nine cases to find the n th term, as discussed below one by one: let a be the first term, d_1 , d_2 , and d_3 be the three c.d.'s and multiplicity $M = 3$. Then

Case I: When n is of the form $(9m-8)$, where $m = 1, 2, 3, \dots$ i.e., $n = 1, 10, 19, 28, \dots = 9m-8$. Then

$$\begin{aligned} t_n = t_{9m-8} &= a + 3(m-1)(d_1 + d_2 + d_3) \\ &= a + 3\left(\frac{n-1}{9}\right)(d_1 + d_2 + d_3) \end{aligned}$$

Case II: When n is of the form $(9m-7)$, where $m = 1, 2, 3, \dots$ i.e., $n = 2, 11, 20, 29, \dots = 9m-7$. Then

$$\begin{aligned} t_n = t_{9m-7} &= (a + d_1) + 3(m-1)(d_1 + d_2 + d_3) \\ &= (a + d_1) + 3\left(\frac{n-2}{9}\right)(d_1 + d_2 + d_3) \end{aligned}$$

Case III: When n is of the form $(9m-6)$, where $m = 1, 2, 3, \dots$ i.e., $n = 3, 12, 21, 30, \dots = 9m-6$. Then

$$\begin{aligned} t_n = t_{9m-6} &= (a + 2d_1) + 3(m-1)(d_1 + d_2 + d_3) \\ &= (a + 2d_1) + 3\left(\frac{n-3}{9}\right)(d_1 + d_2 + d_3) \end{aligned}$$

Case IV: When n is of the form $(9m-5)$, where $m = 1, 2, 3, \dots$ i.e., $n = 4, 13, 22, 31, \dots = 9m-5$. Then

$$t_n = t_{9m-5} = (a + 3d_1) + 3(m-1)(d_1 + d_2 + d_3)$$

$$= (a + 3d_1) + 3\left(\frac{n-4}{9}\right)(d_1 + d_2 + d_3)$$

Case V: When n is of the form $(9m-4)$, where $m = 1, 2, 3, \dots$ i.e., $n = 5, 14, 23, 32, \dots = 9m-4$. Then

$$\begin{aligned} t_n &= t_{9m-4} = (a + 3d_1 + d_2) + 3(m-1)(d_1 + d_2 + d_3) \\ &= (a + 3d_1 + d_2) + 3\left(\frac{n-5}{9}\right)(d_1 + d_2 + d_3) \end{aligned}$$

Case VI: When n is of the form $(9m-3)$, where $m = 1, 2, 3, \dots$ i.e., $n = 6, 15, 24, 33, \dots = 9m-3$. Then

$$\begin{aligned} t_n &= t_{9m-3} = (a + 3d_1 + 2d_2) + 3(m-1)(d_1 + d_2 + d_3) \\ &= (a + 3d_1 + 2d_2) + 3\left(\frac{n-6}{9}\right)(d_1 + d_2 + d_3) \end{aligned}$$

Case VII: When n is of the form $(9m-2)$, where $m = 1, 2, 3, \dots$ i.e., $n = 7, 16, 25, 34, \dots = 9m-2$. Then

$$\begin{aligned} t_n &= t_{9m-2} = (a + 3d_1 + 3d_2) + 3(m-1)(d_1 + d_2 + d_3) \\ &= (a + 3d_1 + 3d_2) + 3\left(\frac{n-7}{9}\right)(d_1 + d_2 + d_3) \end{aligned}$$

Case VIII: When n is of the form $(9m-1)$, where $m = 1, 2, 3, \dots$ i.e., $n = 8, 17, 26, 35, \dots = 9m-1$. Then

$$\begin{aligned} t_n &= t_{9m-1} = (a + 3d_1 + 3d_2 + d_3) \\ &\quad + 3(m-1)(d_1 + d_2 + d_3) \\ &= (a + 3d_1 + 3d_2 + d_3) + 3\left(\frac{n-8}{9}\right)(d_1 + d_2 + d_3) \end{aligned}$$

Case IX: When n is of the form $(9m)$, where $m = 1, 2, 3, \dots$ i.e., $n = 9, 18, 27, 36, \dots = 9m$. Then

$$\begin{aligned} t_n &= t_{9m} = (a + 3d_1 + 3d_2 + 2d_3) \\ &\quad + 3(m-1)(d_1 + d_2 + d_3) \\ &= (a + 3d_1 + 3d_2 + 2d_3) + 3\left(\frac{n-9}{9}\right)(d_1 + d_2 + d_3) \end{aligned}$$

Example 3.2: Find the 10th, 11th, 12th, 13th, 14th, 15th, 16th, 17th, and 18th terms of the sequence of 3-D.A.P.M.(3) with first term 3 and c.d.'s 2, 3, 4 respectively.

Here $a = 3$, $d_1 = 2$, $d_2 = 3$, $d_3 = 4$, and $M = 3$. Therefore

$$\begin{aligned} t_{10} &= t_{18-8} = t_{9m-8}[m = 2] \\ &= a + 3\left(\frac{n-1}{9}\right)(d_1 + d_2 + d_3) \\ &= 3 + 3\left(\frac{10-1}{9}\right)9 = 30 \end{aligned}$$

$$\begin{aligned} t_{11} &= t_{18-7} = t_{9m-7}[m = 2] \\ &= (a + d_1) + 3\left(\frac{n-2}{9}\right)(d_1 + d_2 + d_3) \\ &= (3 + 2) + 3\left(\frac{11-2}{9}\right)9 = 32 \end{aligned}$$

$$\begin{aligned} t_{12} &= t_{18-6} = t_{9m-6}[m = 2] \\ &= (a + 2d_1) + 3\left(\frac{n-3}{9}\right)(d_1 + d_2 + d_3) \\ &= (3 + 4) + 3\left(\frac{12-3}{9}\right)9 = 34 \end{aligned}$$

$$\begin{aligned} t_{13} &= t_{18-5} = t_{9m-5}[m = 2] \\ &= (a + 3d_1) + 3\left(\frac{n-4}{9}\right)(d_1 + d_2 + d_3) \end{aligned}$$

$$= (3 + 6) + 3\left(\frac{13-4}{9}\right)9 = 36$$

$$\begin{aligned} t_{14} &= t_{18-4} = t_{9m-4}[m = 2] \\ &= (a + 3d_1 + d_2) \\ &\quad + 3\left(\frac{n-5}{9}\right)(d_1 + d_2 + d_3) \\ &= (3 + 6 + 3) + 3\left(\frac{14-5}{9}\right)9 = 39 \end{aligned}$$

$$\begin{aligned} t_{15} &= t_{18-3} = t_{9m-3}[m = 2] \\ &= (a + 3d_1 + 2d_2) \\ &\quad + 3\left(\frac{n-6}{9}\right)(d_1 + d_2 + d_3) \\ &= (3 + 6 + 6) + 3\left(\frac{15-6}{9}\right)9 = 42 \end{aligned}$$

$$\begin{aligned} t_{16} &= t_{18-2} = t_{9m-2}[m = 2] \\ &= (a + 3d_1 + 3d_2) \\ &\quad + 3\left(\frac{n-7}{9}\right)(d_1 + d_2 + d_3) \\ &= (3 + 6 + 9) + 3\left(\frac{16-7}{9}\right)9 = 45 \end{aligned}$$

$$\begin{aligned} t_{17} &= t_{18-1} = t_{9m-1}[m = 2] \\ &= (a + 3d_1 + 3d_2 + d_3) \\ &\quad + 3\left(\frac{n-8}{9}\right)(d_1 + d_2 + d_3) \\ &= (3 + 6 + 9 + 4) + 3\left(\frac{17-8}{9}\right)9 = 49 \end{aligned}$$

$$\begin{aligned} t_{18} &= t_{9m}[m = 2] \\ &= (a + 3d_1 + 3d_2 + 2d_3) \\ &\quad + 3\left(\frac{n-9}{9}\right)(d_1 + d_2 + d_3) \\ &= (3 + 6 + 9 + 8) + 3\left(\frac{18-9}{9}\right)9 = 53 \end{aligned}$$

Hence the given 3-D.A.P.M.(3) is 3, 5, 7, 9, 12, 15, 18, 22, 26, 30, 32, 34, 36, 39, 42, 45, 49, 53, 57, ...

3.3. Sum of First n Terms of a 3-D.A.P.M.(3): The sum of all the terms of a 3-D.A.P. with multiplicity three is called a 3-D.A.P. Series with Multiplicity Three. Let a be the first term, d_1 , d_2 , and d_3 be three c.d.'s with multiplicity three and let S_n be the sum of first n terms, then

$$\begin{aligned} S_n &= t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + \dots + t_n \\ &= (t_1 + t_{10} + t_{19} + t_{28} + \dots) + (t_2 + t_{11} + t_{20} + t_{29} + \dots) + (t_3 + t_{12} + t_{21} + t_{30} + \dots) + \\ &\quad (t_4 + t_{13} + t_{22} + t_{31} + \dots) + (t_5 + t_{14} + t_{23} + t_{32} + \dots) + (t_6 + t_{15} + t_{24} + t_{33} + \dots) + \\ &\quad (t_7 + t_{16} + t_{25} + t_{34} + \dots) + (t_8 + t_{17} + t_{26} + t_{35} + \dots) + (t_9 + t_{18} + t_{27} + t_{36} + \dots) \\ &= S_{9m-8} + S_{9m-7} + S_{9m-6} + S_{9m-5} + S_{9m-4} + S_{9m-3} + S_{9m-2} + S_{9m-1} + S_{9m} \end{aligned}$$

$$= S_A + S_B + S_C + S_D + S_E + S_F + S_G + S_H + S_I$$

To find S_A , S_B , S_C , S_D , S_E , S_F , S_G , S_H , S_I we consider nine different cases as follows:

Case A: When only the terms $n = 1, 10, 19, 28, 37, \dots$ of the form $(9m-8)$ for $m = 1, 2, 3, \dots$ are taken into consideration, then we get an arithmetic series with first term a and common difference $3(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$S_A = \frac{m}{2} [2a + (m-1)3(d_1 + d_2 + d_3)] \\ = \frac{n+8}{18} \left[2a + 3 \left(\frac{n-1}{9} \right) (d_1 + d_2 + d_3) \right]$$

Case B: When only the terms $n = 2, 11, 20, 29, 38, \dots$ of the form $(9m-7)$ for $m = 1, 2, 3, \dots$ are taken into consideration, then we get an arithmetic series with first term $(a+d_1)$ and common difference $3(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$S_B = \frac{m}{2} [2(a + d_1) + (m-1)3(d_1 + d_2 + d_3)] \\ = \frac{n+7}{18} \left[2(a + d_1) + 3 \left(\frac{n-2}{9} \right) (d_1 + d_2 + d_3) \right]$$

Case C: When only the terms $n = 3, 12, 21, 30, 39, \dots$ of the form $(9m-6)$ for $m = 1, 2, 3, \dots$ are taken into consideration, then we get an arithmetic series with first term $(a+2d_1)$ and common difference $3(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$S_C = \frac{m}{2} [2(a + 2d_1) + (m-1)3(d_1 + d_2 + d_3)] \\ = \frac{n+6}{18} \left[2(a + 2d_1) + 3 \left(\frac{n-3}{9} \right) (d_1 + d_2 + d_3) \right]$$

Case D: When only the terms $n = 4, 13, 22, 31, 40, \dots$ of the form $(9m-5)$ for $m = 1, 2, 3, \dots$ are taken into consideration, then we get an arithmetic series with first term $(a+3d_1)$ and common difference $3(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$S_D = \frac{m}{2} [2(a + 3d_1) + (m-1)3(d_1 + d_2 + d_3)] \\ = \frac{n+5}{18} \left[2(a + 3d_1) + 3 \left(\frac{n-4}{9} \right) (d_1 + d_2 + d_3) \right]$$

Case E: When only the terms $n = 5, 14, 23, 32, 41, \dots$ of the form $(9m-4)$ for $m = 1, 2, 3, \dots$ are taken into consideration, then we get an arithmetic series with first term $(a+3d_1+d_2)$ and common difference $3(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$S_E = \frac{m}{2} [2(a + 3d_1 + d_2) + (m-1)3(d_1 + d_2 + d_3)] \\ = \frac{n+4}{18} \left[2(a + 3d_1 + d_2) + 3 \left(\frac{n-5}{9} \right) (d_1 + d_2 + d_3) \right]$$

Case F: When only the terms $n = 6, 15, 24, 33, 42, \dots$ of the form $(9m-3)$ for $m = 1, 2, 3, \dots$ are taken into consideration, then we get an arithmetic series with first term $(a+3d_1+2d_2)$ and common difference $3(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$S_F = \frac{m}{2} [2(a + 3d_1 + 2d_2) + (m-1)3(d_1 + d_2 + d_3)] \\ = \frac{n+3}{18} \left[2(a + 3d_1 + 2d_2) + 3 \left(\frac{n-6}{9} \right) (d_1 + d_2 + d_3) \right]$$

Case G: When only the terms $n = 7, 16, 25, 34, 43, \dots$ of the form $(9m-2)$ for $m = 1, 2, 3, \dots$ are taken into consideration, then we get an arithmetic series with first term $(a+3d_1+3d_2)$ and common difference $3(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$S_G = \frac{m}{2} [2(a + 3d_1 + 3d_2) + (m-1)3(d_1 + d_2 + d_3)] \\ = \frac{n+2}{18} \left[2(a + 3d_1 + 3d_2) + 3 \left(\frac{n-7}{9} \right) (d_1 + d_2 + d_3) \right]$$

Case H: When only the terms $n = 8, 17, 26, 35, 44, \dots$ of the form $(9m-1)$ for $m = 1, 2, 3, \dots$ are taken into consideration, then we get an arithmetic series with first term $(a+3d_1+3d_2+d_3)$ and common difference $3(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$S_H = \frac{m}{2} [2(a + 3d_1 + 3d_2 + d_3) + (m-1)3(d_1 + d_2 + d_3)] \\ = \frac{n+1}{18} \left[2(a + 3d_1 + 3d_2 + d_3) + 3 \left(\frac{n-8}{9} \right) (d_1 + d_2 + d_3) \right]$$

Case I: When only the terms $n = 9, 18, 27, 36, 45, \dots$ of the form $(9m)$ for $m = 1, 2, 3, \dots$ are taken into consideration, then we get an arithmetic series with first term $(a+3d_1+3d_2+2d_3)$ and common difference $3(d_1+d_2+d_3)$ having m terms. Their sum is given by

$$S_I = \frac{m}{2} [2(a + 3d_1 + 3d_2 + 2d_3) + (m-1)3(d_1 + d_2 + d_3)] \\ = \frac{n}{18} \left[2(a + 3d_1 + 3d_2 + 2d_3) + 3 \left(\frac{n-9}{9} \right) (d_1 + d_2 + d_3) \right]$$

Therefore the sum of the first n terms of the series of 3-D.A.P.M.(3) is determined as follows by applying the suitable formula depending on the number of terms:

Case I: When $n = 1, 10, 19, 28, 37, \dots$ of the form $(9m-8)$; where $m = 1, 2, 3, \dots$ Then S_A has m terms and each from S_B to S_I has $(m-1)$ terms. Therefore

$$S_n = S_A + (S_B + S_C + S_D + S_E + S_F + S_G + S_H + S_I) \\ = \frac{m}{2} [2a + (m-1)3(d_1 + d_2 + d_3)] \\ + \frac{m-1}{2} [2(8a + 21d_1 + 12d_2 + 3d_3) + 24(m-2)(d_1 + d_2 + d_3)] \\ = \frac{n+8}{18} \left[2a + 3 \left(\frac{n-1}{9} \right) (d_1 + d_2 + d_3) \right] \\ + \frac{n-1}{18} \left[2(8a + 21d_1 + 12d_2 + 3d_3) + 24 \left(\frac{n-10}{9} \right) (d_1 + d_2 + d_3) \right]$$

Case II: When $n = 2, 11, 20, 29, 38, \dots$ of the form $(9m-7)$; where $m = 1, 2, 3, \dots$ Then S_A, S_B have m terms and each from S_C to S_I has $(m-1)$ terms. Therefore

$$\begin{aligned}
 S_n &= (S_A + S_B) + (S_C + S_D + S_E + S_F + S_G + S_H + S_I) \\
 &= \frac{m}{2} [2(2a + d_1) + 6(m-1)(d_1 + d_2 + d_3)] \\
 &\quad + \frac{m-1}{2} [2(7a + 20d_1 + 12d_2 + 3d_3) \\
 &\quad + 21(m-2)(d_1 + d_2 + d_3)] \\
 &= \frac{n+7}{18} \left[2(2a + d_1) + 6 \left(\frac{n-2}{9} \right) (d_1 + d_2 + d_3) \right] \\
 &\quad + \frac{n-2}{18} \left[2(7a + 20d_1 + 12d_2 + 3d_3) \right. \\
 &\quad \left. + 21 \left(\frac{n-11}{9} \right) (d_1 + d_2 + d_3) \right]
 \end{aligned}$$

Case III: When $n = 3, 12, 21, 30, 39, \dots$ of the form $(9m-6)$; where $m = 1, 2, 3, \dots$. Then each of $S_A, S_B,$ and S_C has m terms and each from S_D to S_I has $(m-1)$ terms. Therefore

$$\begin{aligned}
 S_n &= (S_A + S_B + S_C) + (S_D + S_E + S_F + S_G + S_H + S_I) \\
 &= \frac{m}{2} [2(3a + 3d_1) + 9(m-1)(d_1 + d_2 + d_3)] \\
 &\quad + \frac{m-1}{2} [2(6a + 18d_1 + 12d_2 + 3d_3) \\
 &\quad + 18(m-2)(d_1 + d_2 + d_3)] \\
 &= \frac{n+6}{18} \left[2(3a + 3d_1) + 9 \left(\frac{n-3}{9} \right) (d_1 + d_2 + d_3) \right] \\
 &\quad + \frac{n-3}{18} \left[2(6a + 18d_1 + 12d_2 + 3d_3) \right. \\
 &\quad \left. + 18 \left(\frac{n-12}{9} \right) (d_1 + d_2 + d_3) \right]
 \end{aligned}$$

Case IV: When $n = 4, 13, 22, 31, 40, \dots$ of the form $(9m-5)$; where $m = 1, 2, 3, \dots$. Then each of S_A, S_B, S_C and S_D has m terms and each from S_E to S_I has $(m-1)$ terms. Therefore

$$\begin{aligned}
 S_n &= (S_A + S_B + S_C + S_D) + (S_E + S_F + S_G + S_H + S_I) \\
 &= \frac{m}{2} [2(4a + 6d_1) + 12(m-1)(d_1 + d_2 + d_3)] \\
 &\quad + \frac{m-1}{2} [2(5a + 15d_1 + 12d_2 + 3d_3) \\
 &\quad + 15(m-2)(d_1 + d_2 + d_3)] \\
 &= \frac{n+5}{18} \left[2(4a + 6d_1) + 12 \left(\frac{n-4}{9} \right) (d_1 + d_2 + d_3) \right] \\
 &\quad + \frac{n-4}{18} \left[2(5a + 15d_1 + 12d_2 + 3d_3) \right. \\
 &\quad \left. + 15 \left(\frac{n-13}{9} \right) (d_1 + d_2 + d_3) \right]
 \end{aligned}$$

Case V: When $n = 5, 14, 23, 32, 41, \dots$ of the form $(9m-4)$; where $m = 1, 2, 3, \dots$. Then each of S_A, S_B, S_C, S_D and S_E has m terms and each from S_F to S_I has $(m-1)$ terms. Therefore

$$\begin{aligned}
 S_n &= (S_A + S_B + S_C + S_D + S_E) + (S_F + S_G + S_H + S_I) \\
 &= \frac{m}{2} [2(5a + 9d_1 + d_2) + 15(m-1)(d_1 + d_2 + d_3)] \\
 &\quad + \frac{m-1}{2} [2(4a + 12d_1 + 11d_2 + 3d_3) \\
 &\quad + 12(m-2)(d_1 + d_2 + d_3)]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{n+4}{18} \left[2(5a + 9d_1 + d_2) + 15 \left(\frac{n-5}{9} \right) (d_1 + d_2 + d_3) \right] \\
 &\quad + \frac{n-5}{18} \left[2(4a + 12d_1 + 11d_2 + 3d_3) \right. \\
 &\quad \left. + 12 \left(\frac{n-14}{9} \right) (d_1 + d_2 + d_3) \right]
 \end{aligned}$$

Case VI: When $n = 6, 15, 24, 33, 42, \dots$ of the form $(9m-3)$; where $m = 1, 2, 3, \dots$. Then each of S_A, S_B, S_C, S_D, S_E and S_F has m terms and each from S_G to S_I has $(m-1)$ terms. Therefore

$$\begin{aligned}
 S_n &= (S_A + S_B + S_C + S_D + S_E + S_F) + (S_G + S_H + S_I) \\
 &= \frac{m}{2} [2(6a + 12d_1 + 3d_2) + 18(m-1)(d_1 + d_2 + d_3)] \\
 &\quad + \frac{m-1}{2} [2(3a + 9d_1 + 9d_2 + 3d_3) \\
 &\quad + 9(m-2)(d_1 + d_2 + d_3)] \\
 &= \frac{n+3}{18} \left[2(6a + 12d_1 + 3d_2) \right. \\
 &\quad \left. + 18 \left(\frac{n-6}{9} \right) (d_1 + d_2 + d_3) \right] \\
 &\quad + \frac{n-6}{18} \left[2(3a + 9d_1 + 9d_2 + 3d_3) \right. \\
 &\quad \left. + 9 \left(\frac{n-15}{9} \right) (d_1 + d_2 + d_3) \right]
 \end{aligned}$$

Case VII: When $n = 7, 16, 25, 34, 43, \dots$ of the form $(9m-2)$; where $m = 1, 2, 3, \dots$. Then each of $S_A, S_B, S_C, S_D, S_E, S_F$ and S_G has m terms and each from S_H to S_I has $(m-1)$ terms. Therefore

$$\begin{aligned}
 S_n &= (S_A + S_B + S_C + S_D + S_E + S_F + S_G) + (S_H + S_I) \\
 &= \frac{m}{2} [2(7a + 15d_1 + 6d_2) + 21(m-1)(d_1 + d_2 + d_3)] \\
 &\quad + \frac{m-1}{2} [2(2a + 6d_1 + 6d_2 + 3d_3) \\
 &\quad + 6(m-2)(d_1 + d_2 + d_3)] \\
 &= \frac{n+2}{18} \left[2(7a + 15d_1 + 6d_2) \right. \\
 &\quad \left. + 21 \left(\frac{n-7}{9} \right) (d_1 + d_2 + d_3) \right] \\
 &\quad + \frac{n-7}{18} \left[2(2a + 6d_1 + 6d_2 + 3d_3) \right. \\
 &\quad \left. + 6 \left(\frac{n-16}{9} \right) (d_1 + d_2 + d_3) \right]
 \end{aligned}$$

Case VIII: When $n = 8, 17, 26, 35, 44, \dots$ of the form $(9m-1)$; where $m = 1, 2, 3, \dots$. Then each of $S_A, S_B, S_C, S_D, S_E, S_F, S_G$ and S_H has m terms and S_I has $(m-1)$ terms. Therefore

$$\begin{aligned}
 S_n &= (S_A + S_B + S_C + S_D + S_E + S_F + S_G + S_H) + S_I \\
 &= \frac{m}{2} [2(8a + 18d_1 + 9d_2 + d_3) \\
 &\quad + 24(m-1)(d_1 + d_2 + d_3)] \\
 &\quad + \frac{m-1}{2} [2(a + 3d_1 + 3d_2 + 2d_3) \\
 &\quad + 3(m-2)(d_1 + d_2 + d_3)]
 \end{aligned}$$

$$= \frac{n+1}{18} \left[2(8a + 18d_1 + 9d_2 + d_3) \right. \\ \left. + 24 \left(\frac{n-8}{9} \right) (d_1 + d_2 + d_3) \right] \\ + \frac{n-8}{18} \left[2(a + 3d_1 + 3d_2 + 2d_3) \right. \\ \left. + 3 \left(\frac{n-17}{9} \right) (d_1 + d_2 + d_3) \right]$$

$$= \frac{11+7}{18} \left[2(6+2) + 6 \left(\frac{11-2}{9} \right) 9 \right] \\ + \frac{11-2}{18} \left[2(21+40+36+12) \right. \\ \left. + 21 \left(\frac{11-11}{9} \right) 9 \right] \\ = 70 + 109 = 179$$

Case IX: When $n = 9, 18, 27, 36, 45, \dots$ of the form $(9m)$; where $m = 1, 2, 3, \dots$. Then each of $S_A, S_B, S_C, S_D, S_E, S_F, S_G, S_H$ and S_I has (m) terms. Therefore

$$S_n = S_A + S_B + S_C + S_D + S_E + S_F + S_G + S_H + S_I \\ = \frac{m}{2} [2(9a + 21d_1 + 12d_2 + 3d_3) \\ + 27(m-1)(d_1 + d_2 + d_3)] \\ = \frac{n}{18} \left[2(9a + 21d_1 + 12d_2 + 3d_3) \right. \\ \left. + 27 \left(\frac{n-9}{9} \right) (d_1 + d_2 + d_3) \right]$$

Example 3.3: Find the sum of first 9, 10, 11, 12, 13, 14, 15, 16, 17, and 18 terms separately of a 3-D.A.P.M.(3) series $3 + 5 + 7 + 9 + 12 + 15 + 18 + 22 + 26 + 30 + 32 + 34 + 36 + 39 + 42 + 45 + 49 + \dots$

Here $a = 3, d_1 = 2, d_2 = 3, d_3 = 4, M = 3$.

For $n = 9$, we have $n = 9 = 9.m$ ($m = 1$). Therefore

$$S_9 = \frac{n}{18} \left[2(9a + 21d_1 + 12d_2 + 3d_3) \right. \\ \left. + 27 \left(\frac{n-9}{9} \right) (d_1 + d_2 + d_3) \right] \\ = \frac{9}{18} [2(27 + 42 + 36 + 12) + 0] = 117$$

For $n = 10$, we have $n = 10 = 9.m - 8$ ($m = 2$). Therefore

$$S_{10} = \frac{n+8}{18} \left[2a + 3 \left(\frac{n-1}{9} \right) (d_1 + d_2 + d_3) \right] \\ + \frac{n-1}{18} \left[2(8a + 21d_1 + 12d_2 + 3d_3) \right. \\ \left. + 24 \left(\frac{n-10}{9} \right) (d_1 + d_2 + d_3) \right] \\ = \frac{10+8}{18} \left[6 + 3 \left(\frac{10-1}{9} \right) 9 \right] \\ + \frac{10-1}{18} \left[2(24 + 42 + 36 + 12) \right. \\ \left. + 24 \left(\frac{10-10}{9} \right) 9 \right] \\ = 33 + 114 = 147$$

For $n = 11$, we have $n = 11 = 9.m - 7$ ($m = 2$). Therefore

$$S_{11} = \frac{n+7}{18} \left[2(2a + d_1) + 6 \left(\frac{n-2}{9} \right) (d_1 + d_2 + d_3) \right] \\ + \frac{n-2}{18} \left[2(7a + 20d_1 + 12d_2 + 3d_3) \right. \\ \left. + 21 \left(\frac{n-11}{9} \right) (d_1 + d_2 + d_3) \right]$$

For $n = 12$, we have $n = 12 = 9.m - 6$ ($m = 2$). Therefore

$$S_{12} = \frac{n+6}{18} \left[2(3a + 3d_1) + 9 \left(\frac{n-3}{9} \right) (d_1 + d_2 + d_3) \right] \\ + \frac{n-3}{18} \left[2(6a + 18d_1 + 12d_2 + 3d_3) \right. \\ \left. + 18 \left(\frac{n-12}{9} \right) (d_1 + d_2 + d_3) \right] \\ = \frac{12+6}{18} \left[2(9+6) + 9 \left(\frac{12-3}{9} \right) 9 \right] \\ + \frac{12-3}{18} [2(18+36+36+12) + 0] \\ = 30 + 81 + 102 = 213$$

For $n = 13$, we have $n = 13 = 9.m - 5$ ($m = 2$). Therefore

$$S_{13} = \frac{n+5}{18} \left[2(4a + 6d_1) + 12 \left(\frac{n-4}{9} \right) (d_1 + d_2 + d_3) \right] \\ + \frac{n-4}{18} \left[2(5a + 15d_1 + 12d_2 + 3d_3) \right. \\ \left. + 15 \left(\frac{n-13}{9} \right) (d_1 + d_2 + d_3) \right] \\ = \frac{13+5}{18} \left[2(12+12) + 12 \left(\frac{13-4}{9} \right) 9 \right] \\ + \frac{13-4}{18} [2(15+30+36+12) + 0] \\ = 48 + 108 + 45 + 48 = 249$$

For $n = 14$, we have $n = 14 = 9.m - 4$ ($m = 2$). Therefore

$$S_{14} = \frac{n+4}{18} \left[2(5a + 9d_1 + d_2) \right. \\ \left. + 15 \left(\frac{n-5}{9} \right) (d_1 + d_2 + d_3) \right] \\ + \frac{n-5}{18} \left[2(4a + 12d_1 + 11d_2 + 3d_3) \right. \\ \left. + 12 \left(\frac{n-14}{9} \right) (d_1 + d_2 + d_3) \right] \\ = \frac{14+4}{18} \left[2(15+18+3) + 15 \left(\frac{14-5}{9} \right) 9 \right] \\ + \frac{14-5}{18} [2(12+24+33+12) + 0] \\ = 72 + 135 + 81 = 288$$

For $n = 15$, we have $n = 15 = 9.m - 3$ ($m = 2$). Therefore

$$S_{15} = \frac{n+3}{18} \left[2(6a + 12d_1 + 3d_2) \right. \\ \left. + 18 \left(\frac{n-6}{9} \right) (d_1 + d_2 + d_3) \right] \\ + \frac{n-6}{18} \left[2(3a + 9d_1 + 9d_2 + 3d_3) \right. \\ \left. + 9 \left(\frac{n-15}{9} \right) (d_1 + d_2 + d_3) \right]$$

$$\begin{aligned}
 &= \frac{15+3}{18} \left[2(18+24+9) + 18 \left(\frac{15-6}{9} \right) 9 \right] \\
 &\quad + \frac{15-6}{18} [2(9+18+27+12) + 0] \\
 &= 102 + 162 + 66 = 330
 \end{aligned}$$

For $n = 16$, we have $n = 16 = 9.m - 2$ ($m = 2$). Therefore

$$\begin{aligned}
 S_{16} &= \frac{n+2}{18} \left[2(7a + 15d_1 + 6d_2) \right. \\
 &\quad \left. + 21 \left(\frac{n-7}{9} \right) (d_1 + d_2 + d_3) \right] \\
 &\quad + \frac{n-7}{18} \left[2(2a + 6d_1 + 6d_2 + 3d_3) \right. \\
 &\quad \left. + 6 \left(\frac{n-16}{9} \right) (d_1 + d_2 + d_3) \right] \\
 &= \frac{16+2}{18} \left[2(21 + 30 + 18) + 21 \left(\frac{16-7}{9} \right) 9 \right] \\
 &\quad + \frac{16-7}{18} [2(6 + 12 + 18 + 12) + 0] \\
 &= 138 + 189 + 48 = 375
 \end{aligned}$$

For $n = 17$, we have $n = 17 = 9.m - 1$ ($m = 2$). Therefore

$$\begin{aligned}
 S_{17} &= \frac{n+1}{18} \left[2(8a + 18d_1 + 9d_2 + d_3) \right. \\
 &\quad \left. + 24 \left(\frac{n-8}{9} \right) (d_1 + d_2 + d_3) \right] \\
 &\quad + \frac{n-8}{18} \left[2(a + 3d_1 + 3d_2 + 2d_3) \right. \\
 &\quad \left. + 3 \left(\frac{n-17}{9} \right) (d_1 + d_2 + d_3) \right] \\
 &= \frac{17+1}{18} \left[2(24 + 36 + 27 + 4) + 24 \left(\frac{17-8}{9} \right) 9 \right] \\
 &\quad + \frac{17-8}{18} \left[2(3 + 6 + 9 + 8) \right. \\
 &\quad \left. + 3 \left(\frac{17-17}{9} \right) 9 \right] \\
 &= 182 + 216 + 26 = 424
 \end{aligned}$$

For $n = 18$, we have $n = 18 = 9.m$ ($m = 2$). Therefore

$$\begin{aligned}
 S_{18} &= \frac{n}{18} \left[2(9a + 21d_1 + 12d_2 + 3d_3) \right. \\
 &\quad \left. + 27 \left(\frac{n-9}{9} \right) (d_1 + d_2 + d_3) \right] \\
 &= \frac{18}{18} \left[2(27 + 42 + 36 + 12) + 27 \left(\frac{18-9}{9} \right) 9 \right] \\
 &= 234 + 243 = 477
 \end{aligned}$$

3.4. To Insert n Terms Between First and Last Term of a

3-D.A.P.M.(3): Given the first term a and the last term b of a 3-D.A.P. with multiplicity three, if there exists exactly n terms between them, then to write these terms we proceed as follows:

We have given that a = first term (t_1) and b = $(n+2)^{\text{th}}$ term (t_{n+2}). Let d_1 , d_2 , and d_3 be three successive common differences. Then there arise nine cases:

Case I: When $(n + 2)$ is of the form $(9m - 8; m = 1, 2, \dots)$, then we have

$$\begin{aligned}
 t_{n+2} &= b = a + 3 \left(\frac{n+1}{9} \right) (d_1 + d_2 + d_3) \\
 \Rightarrow (d_1 + d_2 + d_3) &= \frac{3(b-a)}{n+1}
 \end{aligned}$$

From this it is clear that to find d_1 , d_2 , and d_3 , at least two of them must be given. After finding common differences we can write n terms between a and b .

Case II: When $(n + 2)$ is of the form $(9m - 7; m = 1, 2, \dots)$, then we have

$$\begin{aligned}
 t_{n+2} &= b = (a + d_1) + 3 \frac{n}{9} (d_1 + d_2 + d_3) \\
 &= (a + d_1) + \frac{n}{3} (d_1 + d_2 + d_3) \\
 \Rightarrow (d_1 + d_2 + d_3) &= \frac{3(b-a-d_1)}{n}
 \end{aligned}$$

From this it is clear that to find d_1 , d_2 , and d_3 , at least two of them must be given. After finding common differences we can write n terms between a and b .

Case III: When $(n + 2)$ is of the form $(9m - 6; m = 1, 2, \dots)$, then we have

$$\begin{aligned}
 t_{n+2} &= b = (a + 2d_1) + 3 \frac{n-1}{9} (d_1 + d_2 + d_3) \\
 &= (a + 2d_1) + \frac{n-1}{3} (d_1 + d_2 + d_3) \\
 \Rightarrow (d_1 + d_2 + d_3) &= \frac{3(b-a-2d_1)}{n-1}
 \end{aligned}$$

From this it is clear that to find d_1 , d_2 , and d_3 , at least two of them must be given. After finding common differences we can write n terms between a and b .

Case IV: When $(n + 2)$ is of the form $(9m - 5; m = 1, 2, \dots)$, then we have

$$\begin{aligned}
 t_{n+2} &= b = (a + 3d_1) + 3 \frac{n-2}{9} (d_1 + d_2 + d_3) \\
 &= (a + 3d_1) + \frac{n-2}{3} (d_1 + d_2 + d_3) \\
 \Rightarrow (d_1 + d_2 + d_3) &= \frac{3(b-a-3d_1)}{n-2}
 \end{aligned}$$

From this it is clear that to find d_1 , d_2 , and d_3 , at least two of them must be given. After finding common differences we can write n terms between a and b .

Case V: When $(n + 2)$ is of the form $(9m - 4; m = 1, 2, \dots)$, then we have

$$\begin{aligned}
 t_{n+2} &= b = (a + 3d_1 + d_2) + 3 \frac{n-3}{9} (d_1 + d_2 + d_3) \\
 &= (a + 3d_1 + d_2) + \frac{n-3}{3} (d_1 + d_2 + d_3) \\
 \Rightarrow (d_1 + d_2 + d_3) &= \frac{3(b-a-3d_1-d_2)}{n-3}
 \end{aligned}$$

From this it is clear that to find d_1 , d_2 , and d_3 , at least two of them must be given. After finding common differences we can write n terms between a and b .

Case VI: When $(n + 2)$ is of the form $(9m - 3; m = 1, 2, \dots)$, then we have

$$\begin{aligned} t_{n+2} &= b = (a + 3d_1 + 2d_2) + 3\frac{n-4}{9}(d_1 + d_2 + d_3) \\ &= (a + 3d_1 + 2d_2) + \frac{n-4}{3}(d_1 + d_2 + d_3) \\ \Rightarrow (d_1 + d_2 + d_3) &= \frac{3(b-a-3d_1-2d_2)}{n-4} \end{aligned}$$

From this it is clear that to find d_1 , d_2 , and d_3 , at least two of them must be given. After finding common differences we can write n terms between a and b .

Case VII: When $(n + 2)$ is of the form $(9m - 2; m = 1, 2, \dots)$, then we have

$$\begin{aligned} t_{n+2} &= b = (a + 3d_1 + 3d_2) + 3\frac{n-5}{9}(d_1 + d_2 + d_3) \\ &= (a + 3d_1 + 3d_2) + \frac{n-5}{3}(d_1 + d_2 + d_3) \\ \Rightarrow (d_1 + d_2 + d_3) &= \frac{3(b-a-3d_1-3d_2)}{n-5} \end{aligned}$$

From this it is clear that to find d_1 , d_2 , and d_3 , at least two of them must be given. After finding common differences we can write n terms between a and b .

Case VIII: When $(n + 2)$ is of the form $(9m - 1; m = 1, 2, \dots)$, then we have

$$\begin{aligned} t_{n+2} &= b = (a + 3d_1 + 3d_2 + d_3) \\ &\quad + 3\frac{n-6}{9}(d_1 + d_2 + d_3) \\ &= (a + 3d_1 + 3d_2 + d_3) + \frac{n-6}{3}(d_1 + d_2 + d_3) \\ \Rightarrow (d_1 + d_2 + d_3) &= \frac{3(b-a-3d_1-3d_2-d_3)}{n-6} \end{aligned}$$

From this it is clear that to find d_1 , d_2 , and d_3 , at least two of them must be given. After finding common differences we can write n terms between a and b .

Case IX: When $(n + 2)$ is of the form $(9m; m = 1, 2, \dots)$, then we have

$$\begin{aligned} t_{n+2} &= b = (a + 3d_1 + 3d_2 + 2d_3) \\ &\quad + 3\frac{n-7}{9}(d_1 + d_2 + d_3) \\ &= (a + 3d_1 + 3d_2 + 2d_3) + \frac{n-7}{3}(d_1 + d_2 + d_3) \\ \Rightarrow (d_1 + d_2 + d_3) &= \frac{3(b-a-3d_1-3d_2-2d_3)}{n-7} \end{aligned}$$

From this it is clear that to find d_1 , d_2 , and d_3 , at least two of them must be given. After finding common differences we can write n terms between a and b .

Example 3.4: To insert ten terms between the first term 3 and the last term 34 of a 3-D.A.P. with multiplicity three with first c.d. 2 and third c.d. 4.

Here $a = 3$, $b = 34$, $d_1 = 2$, $d_3 = 4$ and $n + 2 = 12 = 9.m - 6$ for $m = 2$ i.e., $n = 10$. Therefore from the relation

$$\begin{aligned} (d_1 + d_2 + d_3) &= \frac{3(b-a-2d_1)}{n-1} \\ \Rightarrow (2 + d_2 + 4) &= \frac{3(34-3-4)}{10-1} = 9 \\ d_2 &= 3 \end{aligned}$$

Therefore the 3-D.G.A.P.M.(3) series is 3, 5, 7, 9, 12, 15, 18, 22, 26, 30, 32, 34.

3.5. A.M. Between Two Consecutive Terms of a 3-D.A.P.M.(3):

Before finding the A.M. we express the number of terms in the form $9m-8$, $9m-7$, $9m-6$, $9m-5$, $9m-4$, $9m-3$, $9m-2$, $9m-1$, $9m$ and then apply the following formulae according to the terms:

(i) Between t_{9m-8} and t_{9m-7}

$$A.M. = a + \frac{d_1}{2} + 3(m-1)(d_1 + d_2 + d_3)$$

(ii) Between t_{9m-7} and t_{9m-6}

$$A.M. = a + \frac{3d_1}{2} + 3(m-1)(d_1 + d_2 + d_3)$$

(iii) Between t_{9m-6} and t_{9m-5}

$$A.M. = a + 5\frac{d_1}{2} + 3(m-1)(d_1 + d_2 + d_3)$$

(iv) Between t_{9m-5} and t_{9m-4}

$$A.M. = a + 3d_1 + \frac{d_2}{2} + 3(m-1)(d_1 + d_2 + d_3)$$

(v) Between t_{9m-4} and t_{9m-3}

$$A.M. = a + 3d_1 + 3\frac{d_2}{2} + 3(m-1)(d_1 + d_2 + d_3)$$

(vi) Between t_{9m-3} and t_{9m-2}

$$A.M. = a + 3d_1 + 5\frac{d_2}{2} + 3(m-1)(d_1 + d_2 + d_3)$$

(vii) Between t_{9m-2} and t_{9m-1}

$$A.M. = a + 3d_1 + 3d_2 + \frac{d_3}{2} + 3(m-1)(d_1 + d_2 + d_3)$$

(viii) Between t_{9m-1} and t_{9m}

$$A.M. = a + 3d_1 + 3d_2 + 3\frac{d_3}{2} + 3(m-1)(d_1 + d_2 + d_3)$$

(ix) Between t_{9m} and t_{9m+1}

$$A.M. = a + 3\frac{d_1}{2} + 3\frac{d_2}{2} + d_3 + 3\frac{(2m-1)}{2}(d_1 + d_2 + d_3)$$

Example 3.5: Find the A.M. between 9th and 10th terms of 3-D.G.A.P. with multiplicity three given by 3, 5, 7, 9, 12, 15, 18, 22, 26, 30, 32, 34, 36, 39, ... with 3 as first term and 2, 3, 4 as successive common differences.

Here $a = 3$, $d_1 = 2$, $d_2 = 3$ and $d_3 = 4$.

Hence A.M. between 9th and 10th terms is = A.M. between t_{9m} and t_{9m+1} for $m = 1$ and is given by the formula

$$\begin{aligned} A.M. &= a + 3\frac{d_1}{2} + 3\frac{d_2}{2} + d_3 + 3\frac{(2m-1)}{2}(d_1 + d_2 + d_3) \\ &= 3 + 3\frac{2}{2} + 3\frac{3}{2} + 4 + 3\frac{1}{2}(2 + 3 + 4) \\ &= 3 + 3 + 4.5 + 4 + 13.5 = 28 \end{aligned}$$

3.6. Forming a 3-D.A.P. with Multiplicity Six: By inserting one A.M. between every two consecutive terms in the 3-D.A.P.M.(3), we get 3-D.A.P. with multiplicity six as shown below:

$$\begin{aligned}
 & a \\
 & a + \frac{d_1}{2} \\
 & a + d_1 \\
 & a + 3\frac{d_1}{2} \\
 & a + 2d_1 \\
 & a + 5\frac{d_1}{2} \\
 & a + 3d_1 \\
 & a + 3d_1 + \frac{d_2}{2} \\
 & a + 3d_1 + d_2 \\
 & a + 3d_1 + 3\frac{d_2}{2} \\
 & a + 3d_1 + 2d_2 \\
 & a + 3d_1 + 5\frac{d_2}{2} \\
 & a + 3d_1 + 3d_2 \\
 & a + 3d_1 + 3d_2 + \frac{d_3}{2} \\
 & a + 3d_1 + 3d_2 + d_3
 \end{aligned}$$

and so on.

3.7. Properties of 3-D.A.P.M.(3): We have the following properties on three dimensional generalized arithmetic progressions:

- (1) If to each terms of a 3-D.A.P.M.(3) a finite number is added or subtracted, the resulting sequence is also a 3-D.A.P.M.(3).
- (2) If each term of a 3-D.A.P.M.(3) is multiplied or divided by a non-zero finite number, the resulting sequence is also a 3-D.A.P.M.(3).
- (3) If corresponding terms of two different 3-D.A.P.M.(3)'s be added or subtracted, the resulting sequence is also a 3-D.A.P.M.(3).
- (4) If corresponding terms of two different 3-D.A.P.M.(3)'s be multiplied or divided, the resulting sequence need not form a 3-D.A.P.M.(3).
- (5) If we take the three c.d.'s equal, we get an A.P. and all the properties of A.P. from the properties of 3-D.A.P.M.(3).

4. CONCLUSION

All the properties discussed for arithmetic progressions of dimension one and multiplicity one have been extended for higher dimension and multiplicities. But the limitation of this article is that we cannot write the general terms, sum of the series and arithmetic means in a single formula. We have to

write them for different cases depending on the dimension and multiplicity.

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