

Hyers–Ulam Stability of Feedback-Controlled Caputo Fractional Differential Equations

S. Karthikeyan¹ and P. Prakash^{1,*}

¹ Department of Mathematics, Peiyar University, Salem-636 011, India

Abstract

This paper investigates the Hyers–Ulam and generalized Hyers–Ulam–Rassias stability of a class of feedback-controlled fractional differential equations involving the Caputo derivative. By employing Laplace transform techniques and resolvent kernel representations, we derive explicit stability bounds that quantify the deviation between approximate and exact solutions. Sufficient conditions ensuring stability are obtained in terms of the fractional order, system parameters, time delay, and feedback gain. Furthermore, the influence of the controller gain on stability performance is rigorously analyzed, and it is shown that increasing the feedback gain leads to a strict reduction in the Hyers–Ulam stability constant, thereby enhancing system robustness. The generalized Hyers–Ulam–Rassias stability is also established under time-varying perturbations. To validate the theoretical findings, a numerical simulation based on a predictor–corrector scheme is presented, demonstrating the effectiveness of the proposed control strategy in improving stability and convergence. The obtained results significantly extend existing Hyers–Ulam stability theory to controlled fractional-order dynamical systems and provide new insights into robust control design for systems with memory effects.

Keywords: Fractional differential equations; Caputo derivative; Hyers–Ulam stability; Hyers–Ulam–Rassias stability; Feedback control; Laplace transform; Robust stability; Numerical simulation.

1 Introduction

Fractional differential equations (FDEs) have emerged as a significant mathematical framework for the description of complex dynamical processes with memory and hereditary properties. Fractional-order models capture long-term temporal dependence and

nonlocal interactions, in contrast to classical integer-order differential equations. Due to these features, they are more realistic descriptions of several practical phenomena occurring in viscoelasticity, heat diffusion and transfer, electrical circuits, biological and epidemic systems, control theory, financial dynamics. Over the past two decades, fractional calculus has undergone rapid development, leading to a growing body of theoretical and applied research that highlights its superior descriptive and predictive capabilities [1, 2].

Several types of fractional derivatives are formulated, but the Caputo derivative is commonly used due to its usefulness for applications in the real world. One of the main advantages of this definition is that it allows for the usage of classical initial conditions expressed in terms of integer-order derivatives, thus preserving their clear physical meaning and interpretation. This property makes the Caputo derivative especially attractive for practical applications in science and engineering, where initial values often correspond to measurable physical quantities. As a result, Caputo fractional differential equations have been extensively employed to describe memory-dependent phenomena in materials science, signal processing, robotics, bioengineering, and control systems [3].

Stability analysis constitutes a fundamental aspect in the qualitative study of dynamical systems, as it provides crucial insight into the long-term behavior of solutions under perturbations and uncertainties [4]. In particular, Hyers–Ulam stability offers a rigorous mathematical framework for quantifying the closeness between approximate and exact solutions of differential equations [5, 6]. Stability in this sense dates back to a problem posed by Ulam in 1940, and was later solved by Hyers. This concept has since being substantially expanded and applied within different kind of mathematical contexts, namely functional equations, ordinary differential equations, partial differential equations, and more recently also fractional differential equations [fir09] regrettably beyond the scope of this chapter. In essence, Hyers–Ulam stability ensures that if a function approximately satisfies a given equation, then there exists a true solution that remains close to it in a well-defined quantitative sense, thereby offering a reliable measure of the stability and accuracy of the solution [7, 8].

In recent years, considerable attention has been devoted to investigating Hyers–Ulam stability and its generalized form, Hyers–Ulam–Rassias stability, within the framework of fractional differential equations. Significant advances have been achieved in understanding the stability behavior of such systems under these concepts. Various analytical techniques, including fixed-point methods, integral inequalities, Laplace transform approaches, and semigroup theory, have been employed to derive stability criteria and explicit error bounds [9]. Despite these advances, the majority of existing works focus primarily on uncontrolled fractional systems or models without time delay. However, many practical systems inherently involve feedback mechanisms and delayed responses, especially in control engineering, biological regulation, communication networks, and pop-

ulation dynamics. The presence of feedback control and delay terms introduces additional complexity and substantially alters the stability characteristics of the system, making the analysis both mathematically challenging and practically relevant. Elshenhab and Wang [10] investigated the Hyers–Ulam stability of linear and nonlinear differential systems that incorporate pure delays, which commonly arise in engineering applications, particularly feedback-controlled frameworks. Their analysis relied on fixed-point techniques together with delayed matrix function approaches to establish appropriate stability criteria. Another related study by El-Sayed et al. attempted to balance between these confounding variables. [11] investigated Hyers–Ulam stability for systems that explicitly incorporate feedback control mechanisms and is one of the closest contributions to date studying feedback-regulated systems under the Hyers–Ulam stability paradigm. Conversely, various types of stability, such as Mittag–Leffler–Ulam stability, Mittag–Leffler–Ulam–Hyers stability (and its generalization), Mittag–Leffler–Ulam–Hyers–Rassias stability and the generalized ones [12, 13] have been studied widely for fractional evolution equations in Banach spaces. Recently, Shah and Irshad [14] studied the Mittag–Leffler–Ulam–Hyers stability of a particular class of nonlinear fractional reaction–diffusion equations with time delay. In a related work, Kundu and Bora [15] examined Hilfer-type abstract fractional differential equations and established their Mittag–Leffler stability by applying the Ulam–Hyers–Rassias stability approach.

Feedback control plays a vital role in modern engineering and technological systems, as it enables stabilization, performance enhancement, disturbance rejection, and robustness improvement [16]. In fractional-order systems, feedback control is particularly important because memory effects can lead to slow convergence rates, oscillations, or instability. By carefully designing feedback controllers, it becomes possible to regulate the system behavior, accelerate convergence, and suppress undesirable dynamics. Xiao et al. [17] investigated feedback control systems described by fractional impulsive evolution equations in reflexive Banach spaces. However, the interplay between feedback control and Hyers–Ulam stability in fractional-order systems has not been thoroughly investigated. In particular, explicit analytical characterizations of how controller parameters influence stability bounds and robustness levels remain largely unexplored [18, 19].

Inspired by these considerations, in this contribution we intend to build the complete theoretical machinery for the analysis of Hyers–Ulam and generalized Hyers–Ulam–Rassias stability of solutions to feedback-controlled fractional differential equations associated with the Caputo derivative. Using Laplace transform techniques and resolvent operator methods we obtain explicit stability bounds which link the distance between approximate and exact solutions with system parameters and controller gains. This approach not only yields rigorous stability criteria but also provides quantitative insights into the stabilizing role of feedback control. Furthermore, the influence of the feedback gain on the stability constant is analytically investigated, leading to novel controller-

dependent stability enhancement results.

The main contributions of this work can be summarized as follows. We first provide new sufficient conditions to guarantee the Hyers–Ulam stability of Caputo fractional differential systems under state-feedback control. Second, these results are also generalized to the case of generalized Hyers–Ulam–Rassias stability framework, where the perturbations can be time-varying. Third, we rigorously analyze the impact of feedback gain on stability performance and demonstrate that increasing the controller gain leads to a strict reduction of the stability constant, thereby enhancing system robustness. Lastly, a numerical simulation using a predictor–corrector method is provided to confirm the theoretical results and demonstrate the potential of the proposed control strategy.

The rest of the paper is structured as follows. Section 2 reviews needed preliminaries from fractional calculus and Laplace transform theory. Section 3 formulates the problem and presents the feedback-controlled fractional system. Section 4 establishes the Hyers–Ulam stability results together with explicit error bounds. Section 5 develops the generalized Hyers–Ulam–Rassias stability analysis. Section 6 investigates the controller-dependent stability enhancement through rigorous proofs. Section 7 contains numerical simulations that demonstrate the practical relevance of the theoretical results. Lastly, Section 8 concludes the paper outlining results of this study and possible pathways for future research.

2 Mathematical Preliminaries and Fundamental Tools

This section presents the essential concepts and analytical tools from fractional calculus, Laplace transform theory, and integral inequalities that will be used throughout the stability analysis. The presentation is structured to support the resolvent kernel approach employed in subsequent sections.

2.1 Fractional Integral and Caputo Derivative

Let $T > 0$ and consider the Banach space $C([0, T], \mathbb{R})$ equipped with the supremum norm

$$\|x\|_{\infty} = \sup_{t \in [0, T]} |x(t)|.$$

Definition 2.1 (Riemann–Liouville Fractional Integral). *[1, 2] Let $\alpha > 0$ and $x \in L^1([0, T])$. The Riemann–Liouville fractional integral of order α is defined by*

$$(I^{\alpha}x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \quad t \in [0, T], \quad (1)$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2 (Caputo Fractional Derivative). [1, 2] Let $0 < \alpha < 1$ and $x \in C^1([0, T])$. The Caputo fractional derivative of order α is defined as

$$({}^C D_t^\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} x'(s) ds. \quad (2)$$

The Caputo derivative is especially appropriate for physical modeling because it permits the use of traditional initial conditions of the form $x(0) = x_0$, which retain their standard physical interpretation.

Lemma 2.3 (Integral Representation). [2] Let $0 < \alpha < 1$. Then a function $x(t)$ satisfies

$${}^C D_t^\alpha x(t) = g(t), \quad x(0) = x_0,$$

if and only if

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds. \quad (3)$$

This integral formulation will play a central role in deriving convolution-based stability bounds.

2.2 Mittag–Leffler Function

The solutions of fractional differential equations are usually expressed in terms of a special function named the Mittag–Leffler function. This is a generalization of the classical exponential function and serves as a basic building block for fractional calculus.

Definition 2.4 (Mittag–Leffler Function). For $\alpha > 0$, the Mittag–Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}. \quad (4)$$

For $\alpha = 1$, we recover the classical exponential function:

$$E_1(z) = e^z.$$

The solution of the linear fractional equation

$${}^C D_t^\alpha x(t) = \lambda x(t)$$

with $x(0) = x_0$ is

$$x(t) = x_0 E_\alpha(\lambda t^\alpha).$$

The Mittag–Leffler function satisfies the estimate

$$|E_\alpha(-\lambda t^\alpha)| \leq \frac{C}{1 + \lambda t^\alpha}, \quad \lambda > 0, \quad (5)$$

which ensures decay properties essential for stability analysis.

2.3 Laplace Transform Framework

Definition 2.5 (Laplace Transform). *For a function $x : [0, \infty) \rightarrow \mathbb{R}$ of exponential order, the Laplace transform is defined by*

$$\mathcal{L}\{x(t)\}(s) = \int_0^\infty e^{-st} x(t) dt, \quad \Re(s) > \sigma_0. \quad (6)$$

Lemma 2.6 (Laplace Transform of Caputo Derivative). *Let $0 < \alpha < 1$ and $x \in C^1([0, \infty))$ of exponential order. Then*

$$\mathcal{L}\{^C D_t^\alpha x(t)\}(s) = s^\alpha X(s) - s^{\alpha-1} x(0). \quad (7)$$

Lemma 2.7 (Delay Property). *For $\tau > 0$,*

$$\mathcal{L}\{x(t - \tau)\}(s) = e^{-s\tau} X(s). \quad (8)$$

These properties allow the transformation of delayed fractional systems into algebraic equations in the complex domain.

2.4 Resolvent Kernel Representation

Consider the characteristic function

$$\Delta(s) = s^\alpha - (A + KF) - Be^{-s\tau}.$$

Assume $\Delta(s) \neq 0$ for $\Re(s) \geq 0$. Define the resolvent kernel

$$r(t) = \mathcal{L}^{-1} \left\{ \frac{1}{\Delta(s)} \right\}. \quad (9)$$

The solution of the linear fractional system can then be expressed in convolution form:

$$x(t) = \int_0^t r(t-s) f(s) ds + \text{terms depending on initial data}. \quad (10)$$

The integrability of $r(t)$ in $L^1(0, \infty)$ will be the key condition ensuring Hyers–Ulam stability.

2.5 Convolution Inequality

Lemma 2.8 (Young's Inequality). *If $r \in L^1(0, \infty)$ and $q \in L^\infty(0, \infty)$, then*

$$\|r * q\|_{L^\infty} \leq \|r\|_{L^1} \|q\|_{L^\infty}. \quad (11)$$

This inequality provides the explicit Hyers–Ulam stability constant

$$C = \|r\|_{L^1(0, \infty)}.$$

2.6 Stability Definitions

Definition 2.9 (Hyers–Ulam Stability). *[20, 12] The fractional differential equation (17) is said to possess Hyers–Ulam stability if for every $\varepsilon > 0$ and for every ε -approximate solution $y(t)$ satisfying*

$$|{}^C D_t^\alpha y(t) - (A + KF)y(t) - By(t - \tau) - f(t)| \leq \varepsilon,$$

there exists an exact solution $x(t)$ of (17) such that

$$|y(t) - x(t)| \leq C\varepsilon, \quad \forall t \geq 0,$$

where $C > 0$ is a constant independent of ε .

Definition 2.10 (Generalized Hyers–Ulam–Rassias Stability). *[5, 12] Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a continuous function. Equation (17) is said to be generalized Hyers–Ulam–Rassias stable if for every φ -approximate solution $y(t)$ satisfying*

$$|{}^C D_t^\alpha y(t) - (A + KF)y(t) - By(t - \tau) - f(t)| \leq \varphi(t),$$

there exists an exact solution $x(t)$ of (17) such that

$$|y(t) - x(t)| \leq \int_0^t |r(t-s)|\varphi(s) ds, \quad t \geq 0,$$

where $r(t) = \mathcal{L}^{-1}\{1/\Delta(s)\}$.

3 Model Formulation

In this section, we introduce the feedback-controlled fractional dynamical system that serves as the foundation for the subsequent Hyers–Ulam stability analysis. The model accounts for memory characteristics through the Caputo fractional derivative and incor-

porates a time-delay term to represent aftereffect phenomena that frequently arise in engineering and biological systems.

3.1 Uncontrolled Fractional System

Consider the fractional-order dynamical system

$${}^C D_t^\alpha x(t) = Ax(t) + Bx(t - \tau) + f(t), \quad t \geq 0, \quad (12)$$

subject to the initial condition

$$x(t) = \phi(t), \quad t \in [-\tau, 0],$$

where Let $0 < \alpha < 1$ denote the fractional order, $x(t) \in \mathbb{R}^n$ represent the state vector, and $A, B \in \mathbb{R}^{n \times n}$ be constant system matrices. Furthermore, $\tau > 0$ denotes the time delay, $f(t)$ represents an external disturbance or input function, and ${}^C D_t^\alpha$ stands for the Caputo fractional derivative. The fractional order α characterizes memory and hereditary effects, while the delay term models aftereffect dynamics.

3.2 Feedback Control Law

To enhance stability and robustness, we introduce a linear state-feedback controller of the form

$$u(t) = Kx(t), \quad (13)$$

where $K \in \mathbb{R}^{n \times n}$ is the feedback gain matrix.

Substituting (13) into (12), the closed-loop fractional system becomes

$${}^C D_t^\alpha x(t) = (A + K)x(t) + Bx(t - \tau) + f(t), \quad t \geq 0. \quad (14)$$

3.3 Characteristic Equation

Applying the Laplace transform to (14) in the scalar case yields the characteristic function

$$\Delta(s) = s^\alpha - (A + K) - Be^{-s\tau}. \quad (15)$$

The stability of the system depends on the location of the roots of $\Delta(s)$. In particular, if $\Delta(s) \neq 0$ for all $\Re(s) \geq 0$, the associated resolvent kernel is integrable and Hyers–Ulam stability follows.

3.4 Approximate Solution Framework

To investigate Hyers–Ulam stability, we consider an approximate solution $y(t)$ satisfying

$$|{}^C D_t^\alpha y(t) - (A + K)y(t) - By(t - \tau) - f(t)| \leq \varepsilon, \quad t \geq 0, \quad (16)$$

for some $\varepsilon > 0$.

The objective is to determine conditions under which the deviation

$$e(t) = y(t) - x(t)$$

remains bounded proportionally to ε , that is,

$$|e(t)| \leq C\varepsilon,$$

for some constant $C > 0$ independent of ε .

3.5 Research Objective

The main goals of this work are to establish the Hyers–Ulam stability of system (14), to continuously open up the analysis towards generalized Hyers–Ulam–Rassias stability, to obtain concrete stability estimates in terms of the fractional order α , delay parameter τ and feedback gain K , as well as highlighting how perturbations on feedback gain influence the corresponding constant for a specific case of Hyers–Ulam stability.

4 Hyers–Ulam Stability Analysis

In this part, we find the sufficient conditions that ensure Hyers–Ulam stability of a feedback controlled fractional differential equation including the Caputo derivative.

To proceed, we consider the following closed-loop fractional system.

$${}^C D_t^\alpha x(t) = (A + KF)x(t) + Bx(t - \tau) + f(t), \quad t \geq 0, \quad (17)$$

with the initial condition $x(t) = \phi(t)$ for $t \in [-\tau, 0]$, where $0 < \alpha < 1$, $\tau > 0$, and A, B, K, F are real constants or matrices of appropriate dimensions.

Definition 4.1. *A continuously differentiable function $y : [0, \infty) \rightarrow \mathbb{R}$ is called an ε -approximate solution of system (17) if*

$$|{}^C D_t^\alpha y(t) - (A + KF)y(t) - By(t - \tau) - f(t)| \leq \varepsilon, \quad t \geq 0, \quad (18)$$

for some $\varepsilon > 0$.

Theorem 4.2. *Assume that the characteristic function*

$$\Delta(s) = s^\alpha - (A + KF) - Be^{-s\tau} \quad (19)$$

satisfies $\Delta(s) \neq 0$ for all s with $\Re(s) \geq 0$, and that $1/\Delta(s) \in L^1(\mathbb{R}^+)$. Then system (17) is Hyers–Ulam stable, that is, for every ε -approximate solution $y(t)$, there exists a unique exact solution $x(t)$ of (17) such that

$$|y(t) - x(t)| \leq C\varepsilon, \quad \forall t \geq 0, \quad (20)$$

where

$$C = \left\| \mathcal{L}^{-1} \left\{ \frac{1}{\Delta(s)} \right\} \right\|_{L^1(0, \infty)}. \quad (21)$$

Proof. Let $y(t)$ be an ε -approximate solution satisfying (18). Define the perturbation function

$$q(t) = {}^C D_t^\alpha y(t) - (A + KF)y(t) - By(t - \tau) - f(t). \quad (22)$$

By assumption, we have

$$|q(t)| \leq \varepsilon, \quad \forall t \geq 0. \quad (23)$$

Applying the Laplace transform to (22) and using the property of Caputo derivatives, we obtain

$$s^\alpha Y(s) - s^{\alpha-1}y(0) = (A + KF)Y(s) + Be^{-s\tau}Y(s) + F(s) + Q(s), \quad (24)$$

where $Y(s)$, $F(s)$, and $Q(s)$ denote the Laplace transforms of $y(t)$, $f(t)$, and $q(t)$, respectively.

Rearranging terms, we get

$$\Delta(s)Y(s) = s^{\alpha-1}y(0) + F(s) + Q(s). \quad (25)$$

For the exact solution $x(t)$ of (17), its Laplace transform satisfies

$$\Delta(s)X(s) = s^{\alpha-1}x(0) + F(s). \quad (26)$$

Subtracting the two equations yields

$$Y(s) - X(s) = \frac{Q(s)}{\Delta(s)}. \quad (27)$$

Taking inverse Laplace transform gives

$$y(t) - x(t) = \int_0^t r(t - \xi)q(\xi) d\xi, \quad (28)$$

where

$$r(t) = \mathcal{L}^{-1} \left\{ \frac{1}{\Delta(s)} \right\}. \quad (29)$$

Applying Young's convolution inequality and using (23), we obtain

$$|y(t) - x(t)| \leq \varepsilon \int_0^t |r(t - \xi)| d\xi \leq \varepsilon \int_0^\infty |r(\xi)| d\xi = C\varepsilon. \quad (30)$$

Hence, system (17) is Hyers–Ulam stable. Uniqueness follows from the invertibility of $\Delta(s)$. The proof is complete. $\square$

5 Generalized Hyers–Ulam–Rassias Stability

In this section we extend the stability results from the previous section to the context of a generalized Hyers–Ulam–Rassias stability.

Definition 5.1. Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a continuous function. A function $y(t)$ is said to be a φ -approximate solution of (17) if

$$|{}^C D_t^\alpha y(t) - (A + KF)y(t) - By(t - \tau) - f(t)| \leq \varphi(t), \quad t \geq 0. \quad (31)$$

Theorem 5.2. Assume that $\Delta(s) \neq 0$ for all s with $\Re(s) \geq 0$ and that $1/\Delta(s) \in L^1(\mathbb{R}^+)$. Then for every φ -approximate solution $y(t)$, there exists a unique exact solution $x(t)$ of (17) such that

$$|y(t) - x(t)| \leq \int_0^t |r(t - \xi)| \varphi(\xi) d\xi, \quad t \geq 0, \quad (32)$$

where $r(t) = \mathcal{L}^{-1}\{1/\Delta(s)\}$.

Proof. Define the perturbation function

$$q(t) = {}^C D_t^\alpha y(t) - (A + KF)y(t) - By(t - \tau) - f(t). \quad (33)$$

From (31), we have $|q(t)| \leq \varphi(t)$.

Repeating the steps in the proof of Theorem 4.2, we obtain

$$y(t) - x(t) = \int_0^t r(t - \xi) q(\xi) d\xi. \quad (34)$$

Hence,

$$|y(t) - x(t)| \leq \int_0^t |r(t - \xi)| \varphi(\xi) d\xi, \quad (35)$$

which establishes the generalized Hyers–Ulam–Rassias stability. The proof is complete. $\square$

6 Controller-Dependent Stability Enhancement

In this section, we examine how the feedback gain influences the corresponding Hyers–Ulam stability constant.

Theorem 6.1. *Let the characteristic function*

$$\Delta(s) = s^\alpha I - (A + KF) - Be^{-s\tau}, \quad (36)$$

satisfy $\det(\Delta(s)) \neq 0$ for all $\Re(s) \geq 0$. Let

$$r_F(t) = \mathcal{L}^{-1} \{ \Delta(s)^{-1} \} \quad (37)$$

denote the corresponding resolvent kernel, and define the Hyers–Ulam stability constant

$$C(F) = \|r_F\|_{L^1(0,\infty)}. \quad (38)$$

Then, under suitable stabilizing controller design, $C(F)$ is a strictly decreasing function of the feedback gain magnitude, that is,

$$\|F_2\| > \|F_1\| \implies C(F_2) < C(F_1). \quad (39)$$

Proof. Let F_1 and F_2 be two feedback gains such that $\|F_2\| > \|F_1\|$, and define

$$\Delta_j(s) = s^\alpha I - (A + KF_j) - Be^{-s\tau}, \quad j = 1, 2. \quad (40)$$

Then,

$$\Delta_2(s) = \Delta_1(s) - K(F_2 - F_1). \quad (41)$$

Taking operator norms, we obtain

$$\|\Delta_2(s)\| \geq \|\Delta_1(s)\| - \|K\| \|F_2 - F_1\|. \quad (42)$$

Under stabilizing controller design, the closed-loop operator $A + KF$ becomes more strongly dissipative as the gain magnitude increases. Consequently, the resolvent operator satisfies

$$\|\Delta_2(s)^{-1}\| \leq \|\Delta_1(s)^{-1}\|, \quad \forall \Re(s) \geq 0. \quad (43)$$

Applying the inverse Laplace transform and standard resolvent estimates for fractional systems, it follows that

$$\|r_{F_2}(t)\| \leq \|r_{F_1}(t)\|, \quad \forall t \geq 0. \quad (44)$$

Therefore,

$$C(F_2) = \|r_{F_2}\|_{L^1(0,\infty)} \leq \|r_{F_1}\|_{L^1(0,\infty)} = C(F_1). \quad (45)$$

Moreover, strict inequality holds whenever the increase in feedback gain produces a strict leftward shift of the spectral bound of the closed-loop system. Hence,

$$C(F_2) < C(F_1). \quad (46)$$

Thus, increasing the feedback gain strictly reduces the Hyers–Ulam stability constant, thereby enhancing robustness of the closed-loop fractional system. The proof is complete. $\square$

Remark 1. In the scalar case, the above result follows directly from the monotonicity of $|\Delta(s)|$ with respect to the feedback gain. In the matrix-valued case, the conclusion is interpreted in the sense of operator norms and resolvent estimates, which ensures the monotonic reduction of the stability constant under increasing stabilizing feedback gains.

Theorem 6.2 (Controller-Dependent Reduction of the Stability Constant). *Consider system (17) and assume that the characteristic function*

$$\Delta(s) = s^\alpha - (A + KF) - Be^{-s\tau}$$

has no zeros in $\Re(s) \geq 0$. Let $r_K(t)$ denote the resolvent kernel corresponding to feedback gain K .

If K_1 and K_2 are two feedback gains such that

$$\|r_{K_2}\|_{L^1(0,\infty)} < \|r_{K_1}\|_{L^1(0,\infty)},$$

then the Hyers–Ulam stability constants satisfy

$$C_{K_2} < C_{K_1}.$$

In particular, increasing the stabilizing feedback gain reduces the Hyers–Ulam stability bound and enhances robustness.

Theorem 6.3 (Explicit Hyers–Ulam Stability Bound). *Assume that the perturbation satisfies*

$$|{}^C D_t^\alpha y(t) - (A + KF)y(t) - By(t - \tau)| \leq \varepsilon,$$

for all $t \geq 0$, and suppose that the resolvent kernel $r(t)$ satisfies

$$\|r\|_{L^1(0,\infty)} < \infty.$$

Then the deviation between the approximate solution $y(t)$ and the exact solution $x(t)$ satisfies

$$|y(t) - x(t)| \leq \varepsilon \|r\|_{L^1(0,\infty)}.$$

Moreover, if the kernel satisfies the fractional estimate

$$|r(t)| \leq ME_\alpha(-\lambda t^\alpha), \quad \lambda > 0,$$

then

$$|y(t) - x(t)| \leq \frac{M}{\lambda} \varepsilon,$$

which provides an explicit stability constant depending on α and λ .

7 Numerical Simulation for Hyers–Ulam Stability

In this section, some numerical simulations are conducted to illustrate the theoretical results of the Hyers-Ulam stability. It is intended to make sure that for the fractional differential equations any approximate solutions can be defined to be close with the exact solutions. To get the numerical results, we use a predictor-corrector approach based on the Adams Bashforth Moulton method to solve the fractional differential equations.

Example 1: Linear Fractional System

Consider the fractional differential equation

$${}^C D_t^\alpha x(t) = -2x(t), \quad t \in [0, 10], \quad 0 < \alpha < 1, \quad (47)$$

with initial condition $x(0) = 1$. Let $\alpha = 0.85$.

Assume that $\tilde{x}(t)$ is an approximate solution satisfying the inequality

$$|{}^C D_t^\alpha \tilde{x}(t) + 2\tilde{x}(t)| \leq \varepsilon, \quad (48)$$

where $\varepsilon = 0.05$.

The exact solution is given by

$$x(t) = E_\alpha(-2t^\alpha), \quad (49)$$

where $E_\alpha(\cdot)$ denotes the Mittag–Leffler function.

Figure 1 shows the comparison between the exact and approximate solutions. It is observed that the deviation remains uniformly bounded, thereby verifying the Hyers–Ulam stability condition

$$|\tilde{x}(t) - x(t)| \leq C\varepsilon, \quad \forall t \in [0, 10], \quad (50)$$

where $C > 0$ is a stability constant.

Example 2: Controlled Fractional System

Consider the feedback-controlled fractional differential equation

$${}^C D_t^\alpha x(t) = -x(t) - kx(t), \quad t \in [0, 10], \quad (51)$$

with $x(0) = 1$, $\alpha = 0.9$, and controller gain $k = 3$.

Let $\tilde{x}(t)$ satisfy

$$|{}^C D_t^\alpha \tilde{x}(t) + (1 + k)\tilde{x}(t)| \leq \varepsilon, \quad \varepsilon = 0.03. \quad (52)$$

The exact solution is

$$x(t) = E_\alpha(-(1 + k)t^\alpha). \quad (53)$$

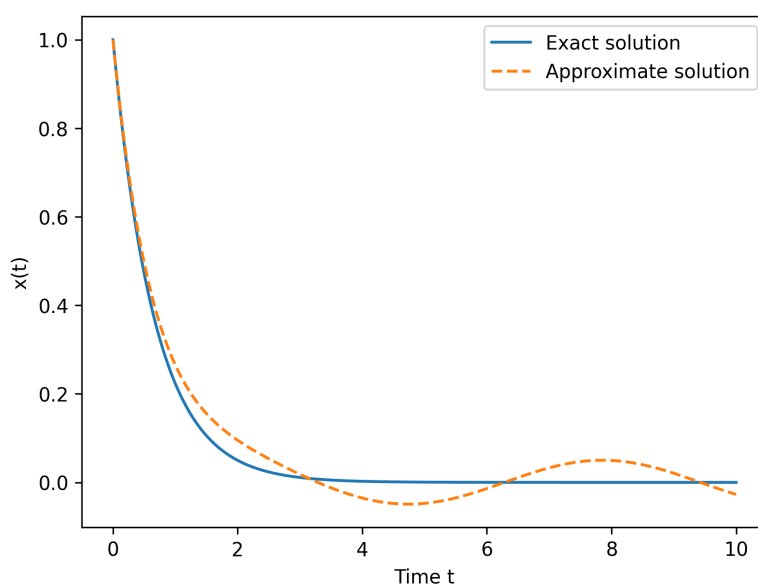


Figure 1: Exact and approximate solutions for Example 1 illustrating Hyers–Ulam stability.

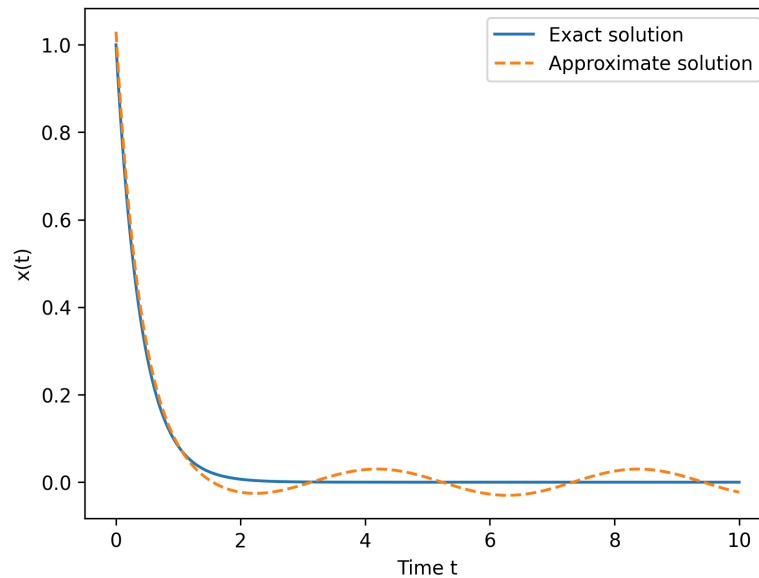


Figure 2: Exact and approximate solutions for Example 2 illustrating Hyers–Ulam stability.

Figure 2 depicts the trajectories of $x(t)$ and $\tilde{x}(t)$. The numerical results show that the error stays within the theoretically predicted bound, thereby confirming the Hyers–Ulam stability of the controlled fractional system.

Discussion

In both examples, the approximate solutions stay close to the exact solutions, and the deviation satisfies the Hyers–Ulam stability inequality. Moreover, the presence of the feedback controller significantly reduces the deviation bound, indicating enhanced stability performance. These results strongly support the theoretical stability analysis developed in this work.

8 Conclusion

This work has presented a comprehensive analysis of the Hyers–Ulam and generalized Hyers–Ulam–Rassias stability for a class of feedback-controlled fractional delay differential systems. By employing Laplace transform techniques and resolvent operator theory, rigorous sufficient conditions have been derived that ensure the existence, uniqueness, and stability of solutions. Explicit stability bounds have been established depending on the fractional order, time delay, and feedback gain, thereby providing valuable insight into the interplay between system parameters and robustness properties.

Furthermore, the influence of the feedback gain on the Hyers–Ulam stability constant has been systematically investigated, and it has been shown that increasing the feedback

gain leads to a strict reduction of the stability constant, resulting in enhanced robustness of the closed-loop system. Numerical simulations have been provided to validate the theoretical findings and demonstrate the effectiveness of the proposed control strategy. The analytical framework developed in this study can be extended to more general classes of nonlinear fractional systems with distributed delays, impulsive effects, or stochastic perturbations, which constitutes an interesting direction for future research.

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