



An Information Theoretic Approach to Prevalence Estimation and Data Gap in Rough Fuzzy Setting

Rohit Kumar Verma

Associate Professor, Department of Mathematics, Bharti Vishwavidyalaya, Durg, Chhattisgarh, India.

ARTICLE INFO	ABSTRACT
Published Online: 09 March 2026 Corresponding Author: Rohit Kumar Verma	For the given known data $\mu_{\bar{A}}(x_1), \dots, \mu_{\bar{A}}(x_n), \mu_{\underline{A}}(x_1), \dots, \mu_{\underline{A}}(x_n)$ and given that $\mu_{\bar{A}}(x), \mu_{\underline{A}}(x)$ belongs to the same set as $\mu_{\bar{A}}(x_1), \dots, \mu_{\bar{A}}(x_n), \mu_{\underline{A}}(x_1), \dots, \mu_{\underline{A}}(x_n)$ and $a \leq \mu_{\bar{A}}(x) \leq b$, $a \leq \mu_{\underline{A}}(x) \leq b$, distinct maximum information estimates of the data gap $\mu_{\bar{A}}(x), \mu_{\underline{A}}(x)$ are obtained using different information measures and different methods. The first approach uses either a or b or $\mu_{\bar{A}}(x), \mu_{\underline{A}}(x)$ as the estimate, where $\mu_{\bar{A}}(x), \mu_{\underline{A}}(x)$ is an information mean of $\mu_{\bar{A}}(x_1), \dots, \mu_{\bar{A}}(x_n), \mu_{\underline{A}}(x_1), \dots, \mu_{\underline{A}}(x_n)$. The second technique relies on a, b and $\sum_{i=1}^n \mu_{\bar{A}}(x_i), \sum_{i=1}^n \mu_{\underline{A}}(x_i)$. In the third approach, all of the data $\mu_{\bar{A}}(x_1), \dots, \mu_{\bar{A}}(x_n), \mu_{\underline{A}}(x_1), \dots, \mu_{\underline{A}}(x_n)$ are dependent on it.
KEYWORDS: Data gap, Maximum information estimates, Fuzzy rough set, Equivalence relation etc	

1. INTRODUCTION

The estimation of the data gap is very important area in many fields of research like agriculture, economics, and science laboratory and data management. When we observed that some data are found missing due to nature climates or any other reasons and are very difficult to repeat the experiments for example the field trials of agriculture crops or disrupted due to nature climates, can the technique to predict or estimate this value are need to be developed. Various scientist and technocrats have suggested different method valuably the max entropy method described by Kapur [7] and that have play an better role then Yate [15].

In the case of an information system (U, A) , where A is a non-empty finite collection of characteristics and U is a non-empty set of finite objects as the universe of discourse, such that for every a in A , $a: U \rightarrow V_a$. The range of possible values for attribute an is known as the set V_a . Decision systems, often known as decision tables, are information systems with a decision attribute that includes classes or decision values for every item. Formally speaking, it is a pair $(U, A \cup \{d\})$, where the decision attribute is d , which is not a member of A . We refer to the components of A as

conditional attributes. In contrast to RST, FRST allows for multiple expressions of indiscernibility.

Any fuzzy connection that establishes the degree of indiscernibility between two items is referred to as a fuzzy indiscernibility relation (FIR). We look at a few FIR unique

instances.

(i) The fuzzy tolerance relation: this relation possesses symmetrical and reflexive features, with the reflexive being $A(x, x) = 1$. Synchronous: $A(x, y) = A(y, x)$

(ii) In addition to being reflexive and symmetric, the similarity relation (also known as the fuzzy equivalence relation) is transitive. It is defined as $\min(A(x, y), A(y, z)) \leq A(x, z)$.

(iii) T-similarity relation (also known as the fuzzy T-equivalence relation): this relation is a fuzzy tolerance relation that is also T-transitive. For a given triangular norm T , $T(A(x, y), A(y, z)) \leq A(x, z)$.

Jensen and Shen presented the following equations as the tolerance relations on a quantitative attribute a, V_a .

$$(i) V_a(x, y) = 1 - \frac{|a(x) - a(y)|}{|a_{max} - a_{min}|}$$

$$(ii) V_a(x, y) = e^{-\frac{(a(x) - a(y))^2}{2\sigma_a^2}}$$

$$(iii) V_a(x, y) = \max\left(\min\left(\frac{a(y) - a(x) + \sigma_a}{\sigma_a}, \frac{a(x) - a(y) + \sigma_a}{\sigma_a}\right), 0\right)$$

where a_{min} and a_{max} are the user-supplied minimum and maximum values of the data, and σ_a^2 is the variance of feature a .

The traditional method of object discrimination is applied to a qualitative (*i.e.*, nominal) property a , meaning that $V_a(x, y) = 1$ if $a(x) = a(y)$ and $V_a(x, y) = 0$ otherwise. The fuzzy B -indiscernibility relation can thus be defined for any

subset B of A using the formula $V_B(x, y) = T(V_A(x, y))$, where T is a t-norm operator, such as minimum, product. In general, any aggregation operator, such as the average, can be used in place of Radzikowska and Kerre [7] state that an impicator I and a t-norm T are used to generalise the lower and higher approximations in the context of FRST. The fuzzy B -upper and B -lower approximations of a fuzzy set set A in U are as follows.

Definition 1.1 Assume that A is an equivalence relation and (U, A) is a finite set. This is known as the Pawlak [6] approximation space. The lower and higher approximations of x_i for each $X \subseteq U$ are provided by $\mu_{\underline{A}}(x_i) = (x_i \in X: [x]_A \subseteq X)$ and $\mu_{\overline{A}}(x_i) = (x_i \in U: [x]_A \cap X \neq \emptyset)$ signify an equivalence class in A that contains an element $x_i \in U$. Pawlak [6] states that if $\mu_{\underline{A}}(x_i) \neq \mu_{\overline{A}}(x_i)$, then x_i is referred to be a rough set.

Definition 1.2 Assume that (U, A) is a fuzzy approximation space, A is a fuzzy relation on $U \times U$, and U is the universal set. The fuzzy set of U is the lower and upper approximation of X , i.e., $\mu_{\overline{A}}(x_i)$ and $\mu_{\underline{A}}(x_i)$, with respect to approximation space (U, A) , for any set $X \in F(U)$. Its membership functions are defined as follows:

$$\mu_{\underline{A}}(x_i) = \bigvee_{y \in U} (\mu_A(x, y) \wedge A(y)) \text{ and } \mu_{\overline{A}}(x_i) = \bigwedge_{x \in U} (\mu_A(x, y) \vee A(y))$$

The pair $(\mu_{\underline{A}}(x_i), \mu_{\overline{A}}(x_i))$ is hence referred to as a fuzzy rough set. Infimum and supremum values are represented by the symbols $\wedge$ and $\vee$. Now, the hybrid Shannon [8] information measure, or Modified Verma [10]

$$V_{\alpha}(A) = \sum_{i=1}^n \ln(1 + \alpha \mu_{\underline{A}}(x_i)) - \sum_{i=1}^n \mu_{\underline{A}}(x_i) \ln \mu_{\underline{A}}(x_i) - \sum_{i=1}^n \ln(1 + \alpha) \mu_{\underline{A}}(x_i) + \sum_{i=1}^n \ln(1 + \alpha \mu_{\overline{A}}(x_i)) - \sum_{i=1}^n \mu_{\overline{A}}(x_i) \ln \mu_{\overline{A}}(x_i) - \sum_{i=1}^n \ln(1 + \alpha) \mu_{\overline{A}}(x_i); \alpha > 0.$$

as well as Verma [9], a hybrid Burg information measure

$$V_{\alpha}(A) = - \sum_{i=1}^n \ln(1 + \alpha \mu_{\underline{A}}(x_i)) + \sum_{i=1}^n \ln \mu_{\underline{A}}(x_i) + \sum_{i=1}^n \ln(1 + \alpha) \mu_{\underline{A}}(x_i) - \sum_{i=1}^n \ln(1 + \alpha \mu_{\overline{A}}(x_i)) + \sum_{i=1}^n \ln \mu_{\overline{A}}(x_i) + \sum_{i=1}^n \ln(1 + \alpha) \mu_{\overline{A}}(x_i); \alpha > 0.$$

Shannon's measure of information or equality provides one such metric for the probability or proportion distribution.

$$\frac{\mu_{\overline{A}}(x_1)}{\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x_1)} + \frac{\mu_{\underline{A}}(x_1)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x_1)}, \frac{\mu_{\overline{A}}(x_2)}{\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x_2)} + \frac{\mu_{\underline{A}}(x_2)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x_2)}, \dots, \frac{\mu_{\overline{A}}(x_n)}{\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x_n)} + \frac{\mu_{\underline{A}}(x_n)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x_n)}, \dots, \frac{\mu_{\overline{A}}(x_n)}{\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x_n)} + \frac{\mu_{\underline{A}}(x_n)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x_n)} \quad (1.1)$$

is given by

$$S(\mu_{\overline{A}}(x)) = - \sum_{i=1}^n \frac{\mu_{\overline{A}}(x_i)}{\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x_i)} \ln \frac{\mu_{\overline{A}}(x_i)}{\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x_i)}$$

$$- \frac{\mu_{\overline{A}}(x)}{\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x)} \ln \frac{\mu_{\overline{A}}(x)}{\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x)} \quad (1.2)$$

and

$$S(\mu_{\underline{A}}(x)) = - \sum_{i=1}^n \frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x_i)} \ln \frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x_i)}$$

$$- \frac{\mu_{\underline{A}}(x)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} \ln \frac{\mu_{\underline{A}}(x)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} \quad (1.3)$$

Through maximize $S(\mu_{\overline{A}}(x))$ and $S(\mu_{\underline{A}}(x))$ we obtained the estimates

$$S(\mu_{\overline{A}}(x)) = (\mu_{\overline{A}}(x_1)^{\mu_{\overline{A}}(x_1)} \mu_{\overline{A}}(x_2)^{\mu_{\overline{A}}(x_2)} \dots \mu_{\overline{A}}(x_n)^{\mu_{\overline{A}}(x_n)})^{\frac{1}{\sum_{i=1}^n \mu_{\overline{A}}(x_i)}} \quad (1.4-a)$$

And

$$S(\mu_{\underline{A}}(x)) = (\mu_{\underline{A}}(x_1)^{\mu_{\underline{A}}(x_1)} \mu_{\underline{A}}(x_2)^{\mu_{\underline{A}}(x_2)} \dots \mu_{\underline{A}}(x_n)^{\mu_{\underline{A}}(x_n)})^{\frac{1}{\sum_{i=1}^n \mu_{\underline{A}}(x_i)}} \quad (1.4-b)$$

which is the entropic mean of $\mu_{\overline{A}}(x_1), \mu_{\overline{A}}(x_2), \dots, \mu_{\overline{A}}(x_n)$ and $\mu_{\underline{A}}(x_1), \mu_{\underline{A}}(x_2), \dots, \mu_{\underline{A}}(x_n)$. Using Burg's [1] measure of entropy,

$$B(\mu_{\overline{A}}(x)) = \sum_{i=1}^n \ln \frac{\mu_{\overline{A}}(x)}{\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x)} + \sum_{i=1}^n \ln \frac{\mu_{\overline{A}}(x_i)}{\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x)} \quad (1.5-a)$$

and

$$B(\mu_{\underline{A}}(x)) = \sum_{i=1}^n \ln \frac{\mu_{\underline{A}}(x)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} + \sum_{i=1}^n \ln \frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} \quad (1.5-b)$$

after maximization we obtain the estimates

$$B(\mu_{\overline{A}}(x)) = \frac{\mu_{\overline{A}}(x_1) + \mu_{\overline{A}}(x_2) + \dots + \mu_{\overline{A}}(x_n)}{n} \quad (1.6-a)$$

and

$$B(\mu_{\underline{A}}(x)) = \frac{\mu_{\underline{A}}(x_1) + \mu_{\underline{A}}(x_2) + \dots + \mu_{\underline{A}}(x_n)}{n} \quad (1.6-b)$$

Lastly, we employ the Havrda-Charvat [2] entropy of second order

$$H(\mu_{\overline{A}}(x)) = 1 - \sum_{i=1}^n \left(\frac{\mu_{\overline{A}}(x_i)}{\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x)} \right)^2 - \frac{(\mu_{\overline{A}}(x))^2}{(\sum_{i=1}^n \mu_{\overline{A}}(x_i) + \mu_{\overline{A}}(x))^2} \quad (1.7-a)$$

and

$$H(\mu_{\underline{A}}(x)) = 1 - \sum_{i=1}^n \left(\frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} \right)^2 - \frac{(\mu_{\underline{A}}(x))^2}{(\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x))^2} \quad (1.7-b)$$

b)

We receive the estimate,

$$H(\mu_{\bar{A}}(x)) = \frac{\sum_{i=1}^n (\mu_{\bar{A}}(x_i))^2}{\sum_{i=1}^n \mu_{\bar{A}}(x_i)} \quad \text{and} \quad H(\mu_{\underline{A}}(x)) = \frac{\sum_{i=1}^n (\mu_{\underline{A}}(x_i))^2}{\sum_{i=1}^n \mu_{\underline{A}}(x_i)} \quad (1.8)$$

2. NEW RESULT

2.1 Utilising modified Verma [10] i. e. hybrid Shannon [8] information measure when $\alpha = 0$

In this instance, we must maximize (1.2) and (1.3) while keeping in mind that $a \leq \mu_{\bar{A}}(x) \leq b$ and $a \leq \mu_{\underline{A}}(x) \leq b$

Now from (1.2) and (1.3)

$$\frac{dS}{d\mu_{\bar{A}}(x)} = \frac{1}{(\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x))^2} (\sum_{i=1}^n \mu_{\bar{A}}(x_i) \ln \mu_{\bar{A}}(x_i) - \sum_{i=1}^n \mu_{\bar{A}}(x_i) \ln \mu_{\bar{A}}(x)) \quad (2.1.1)$$

$$\text{and} \quad \frac{dS}{d\mu_{\underline{A}}(x)} = \frac{1}{(\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x))^2} (\sum_{i=1}^n \mu_{\underline{A}}(x_i) \ln \mu_{\underline{A}}(x_i) - \sum_{i=1}^n \mu_{\underline{A}}(x_i) \ln \mu_{\underline{A}}(x)) \quad (2.1.2)$$

$$\text{Also,} \quad \frac{d^2 S}{d(\mu_{\bar{A}}(x))^2} = \frac{1}{(\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x))^3} (2(\sum_{i=1}^n \mu_{\bar{A}}(x_i) \ln \mu_{\bar{A}}(x) - \sum_{i=1}^n \mu_{\bar{A}}(x_i) \ln \mu_{\bar{A}}(x_i)) - \sum_{i=1}^n \mu_{\bar{A}}(x_i) \frac{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)}{\mu_{\bar{A}}(x)}) \quad (2.1.3)$$

$$\text{and} \quad \frac{d^2 S}{d(\mu_{\underline{A}}(x))^2} = \frac{1}{(\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x))^3} (2(\sum_{i=1}^n \mu_{\underline{A}}(x_i) \ln \mu_{\underline{A}}(x) - \sum_{i=1}^n \mu_{\underline{A}}(x_i) \ln \mu_{\underline{A}}(x_i)) - \sum_{i=1}^n \mu_{\underline{A}}(x_i) \frac{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)}{\mu_{\underline{A}}(x)}) \quad (2.1.4)$$

When $\mu_{\bar{A}}(x) = 0$ and $\mu_{\underline{A}}(x) = 0$ then $S = 0$,

$$\text{Similarly,} \quad \frac{dS}{d\mu_{\bar{A}}(x)} = \infty \quad \text{and} \quad \frac{dS}{d\mu_{\underline{A}}(x)} = \infty, \quad (2.1.5)$$

$$\text{Also,} \quad \frac{d^2 S}{d(\mu_{\bar{A}}(x))^2} < 0 \quad \text{and} \quad \frac{d^2 S}{d(\mu_{\underline{A}}(x))^2} < 0 \quad (2.1.6)$$

2.2 Verma [9] i. e. hybrid Burg information measure when $\alpha = 0$

Use of hybrid Burg [9] information measure when $\alpha = 0$

$$\frac{dB}{d\mu_{\bar{A}}(x)} = -\frac{n+1}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} + \frac{1}{\mu_{\bar{A}}(x)} \quad (2.2.1)$$

$$\text{and} \quad \frac{dB}{d\mu_{\underline{A}}(x)} = -\frac{n+1}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} + \frac{1}{\mu_{\underline{A}}(x)} \quad (2.2.2)$$

$$\frac{d^2 B}{d(\mu_{\bar{A}}(x))^2} = \frac{1}{(\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x))^2} - \frac{1}{(\mu_{\bar{A}}(x))^2} \quad (2.2.3)$$

$$\text{and} \quad \frac{d^2 B}{d(\mu_{\underline{A}}(x))^2} = \frac{1}{(\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x))^2} - \frac{1}{(\mu_{\underline{A}}(x))^2} \quad (2.2.4)$$

2.3 Applying an alternative measure of information

$$\begin{aligned} K_1(x) = & -\sum_{i=1}^n \left(\frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + a} - \frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} \right) \ln \left(\frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + a} - \frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} \right) \\ & - \sum_{i=1}^n \left(\frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + a} - \frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} \right) \ln \left(\frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + a} - \frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} \right) \\ & - \sum_{i=1}^n \left(\frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} - \frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + b} \right) \ln \left(\frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} - \frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + b} \right) \\ & - \sum_{i=1}^n \left(\frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} - \frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + b} \right) \ln \left(\frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} - \frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + b} \right) \\ & - \left(\frac{\mu_{\bar{A}}(x)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} - \frac{a}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + a} \right) \ln \left(\frac{\mu_{\bar{A}}(x)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} - \frac{a}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + a} \right) \\ & - \left(\frac{\mu_{\underline{A}}(x)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} - \frac{a}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + a} \right) \ln \left(\frac{\mu_{\underline{A}}(x)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} - \frac{a}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + a} \right) \\ & - \left(\frac{b}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + b} - \frac{\mu_{\bar{A}}(x)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} \right) \ln \left(\frac{b}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + b} - \frac{\mu_{\bar{A}}(x)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} \right) \\ & - \left(\frac{b}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + b} - \frac{\mu_{\underline{A}}(x)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} \right) \ln \left(\frac{b}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + b} - \frac{\mu_{\underline{A}}(x)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} \right) \end{aligned} \quad (2.3.1)$$

By default, its maximization will guarantee that $a < \mu_{\bar{A}}(x_i) < b$ and $a < \mu_{\underline{A}}(x) < b$

$$\frac{b - \mu_{\bar{A}}(x)}{\mu_{\bar{A}}(x) - a} = \frac{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + b}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + a} \quad \text{and} \quad \frac{b - \mu_{\underline{A}}(x)}{\mu_{\underline{A}}(x) - a} = \frac{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + b}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + a} \quad (2.3.2)$$

$$\text{or } K_1(\mu_{\bar{A}}(x)) = \frac{b \sum_{i=1}^n \mu_{\bar{A}}(x_i) + a \sum_{i=1}^n \mu_{\bar{A}}(x_i) + 2ab}{2 \sum_{i=1}^n \mu_{\bar{A}}(x_i) + a + b} \text{ and } K_1(\mu_{\underline{A}}(x)) = \frac{b \sum_{i=1}^n \mu_{\underline{A}}(x_i) + a \sum_{i=1}^n \mu_{\underline{A}}(x_i) + 2ab}{2 \sum_{i=1}^n \mu_{\underline{A}}(x_i) + a + b} \quad (2.3.3)$$

Similarly, if we maximize the measure of information corresponding to Burg's information measure

$$\begin{aligned} B_1(\mu_{\bar{A}}(x)) &= \sum_{i=1}^n \ln \left(\frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + a} - \frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} \right) \\ &+ \sum_{i=1}^n \ln \left(\frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} - \frac{\mu_{\bar{A}}(x_i)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + b} \right) \\ &+ \sum_{i=1}^n \ln \left(\frac{\mu_{\bar{A}}(x)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} - \frac{a}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + a} \right) \\ &+ \sum_{i=1}^n \ln \left(\frac{b}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + b} - \frac{\mu_{\bar{A}}(x)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} \right) \end{aligned} \quad (2.3.4)$$

At the same time

$$\begin{aligned} B_1(\mu_{\underline{A}}(x)) &= \sum_{i=1}^n \ln \left(\frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + a} - \frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} \right) \\ &+ \sum_{i=1}^n \ln \left(\frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} - \frac{\mu_{\underline{A}}(x_i)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + b} \right) \\ &+ \sum_{i=1}^n \ln \left(\frac{\mu_{\underline{A}}(x)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} - \frac{a}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + a} \right) \\ &+ \sum_{i=1}^n \ln \left(\frac{b}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + b} - \frac{\mu_{\underline{A}}(x)}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} \right) \end{aligned} \quad (2.3.5)$$

Then, maximization gives

$$\frac{b+a-2\mu_{\bar{A}}(x)}{(\mu_{\bar{A}}(x)-a)(b-\mu_{\bar{A}}(x))} = \frac{1}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} \quad \text{and} \quad \frac{b+a-2\mu_{\underline{A}}(x)}{(\mu_{\underline{A}}(x)-a)(b-\mu_{\underline{A}}(x))} = \frac{1}{\sum_{i=1}^n \mu_{\underline{A}}(x_i) + \mu_{\underline{A}}(x)} \quad (2.3.6)$$

So that, $a < B_1(\mu_{\bar{A}}(x)) < \frac{a+b}{2} < b$ at the same time $a < B_1(\mu_{\underline{A}}(x)) < \frac{a+b}{2} < b$.

2.4 Use of another principle

Consider the three-probability distribution,

$$\begin{aligned} A: & \frac{\mu_{\bar{A}}(x_1)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x_1)} + \frac{\mu_{\bar{A}}(x_2)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x_2)} + \frac{\mu_{\bar{A}}(x_2)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x_2)} + \frac{\mu_{\bar{A}}(x_2)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x_2)}, \dots \\ & \dots, \frac{\mu_{\bar{A}}(x_n)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x_n)} + \frac{\mu_{\bar{A}}(x_n)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x_n)}, \frac{\mu_{\bar{A}}(x)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} + \frac{\mu_{\bar{A}}(x)}{\sum_{i=1}^n \mu_{\bar{A}}(x_i) + \mu_{\bar{A}}(x)} \\ B: & \frac{\mu_{\bar{B}}(x_1)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x_1)} + \frac{\mu_{\bar{B}}(x_1)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x_1)}, \frac{\mu_{\bar{B}}(x_2)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x_2)} + \frac{\mu_{\bar{B}}(x_2)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x_2)}, \dots \\ & \dots, \frac{\mu_{\bar{B}}(x_n)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x_n)} + \frac{\mu_{\bar{B}}(x_n)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x_n)}, \frac{b}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} + \frac{b}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} \\ C: & \frac{\mu_{\bar{C}}(x_1)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x_1)} + \frac{\mu_{\bar{C}}(x_1)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x_1)}, \frac{\mu_{\bar{C}}(x_2)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x_2)} + \frac{\mu_{\bar{C}}(x_2)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x_2)}, \dots \\ & \dots, \frac{\mu_{\bar{C}}(x_n)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x_n)} + \frac{\mu_{\bar{C}}(x_n)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x_n)}, \frac{c}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c} + \frac{c}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c} \end{aligned}$$

We have no reason to think that the directed-divergence of B from A should differ from the directed-divergence of A from C in the absence of any further information, so we take

$$D(B:A) = D(A:C),$$

Naturally, following the use of the Kullback-Leibler [5] measure of divergence

$$\begin{aligned} & \sum_{i=1}^n \frac{\mu_{\bar{B}}(x_i)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} \ln \frac{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} + \sum_{i=1}^n \frac{\mu_{\bar{B}}(x_i)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} \ln \frac{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} \\ & + \frac{b}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} \ln \frac{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} + \frac{b}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} \ln \frac{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} = \\ & \sum_{i=1}^n \frac{\mu_{\bar{C}}(x_i)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} \ln \frac{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} + \sum_{i=1}^n \frac{\mu_{\bar{C}}(x_i)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} \ln \frac{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} \\ & + \frac{\mu_{\bar{C}}(x)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} \ln \frac{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} + \frac{\mu_{\bar{C}}(x)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} \ln \frac{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} \end{aligned} \quad (2.4.1)$$

In other words,

$$\ln \frac{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} + \ln \frac{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} = \ln \frac{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} + \ln \frac{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} \quad (2.4.2)$$

The above means that

$$\frac{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} + \frac{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} = \frac{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} + \frac{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} \quad (2.4.3)$$

$$\text{Or } \sqrt{(\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b)(\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c)} < \sum_{i=1}^n \mu_{\bar{B}}(x_i) + \frac{b}{2} + \sum_{i=1}^n \mu_{\bar{C}}(x_i) + \frac{c}{2}$$

$$\text{and } \sqrt{(\sum_{i=1}^n \mu_{\underline{B}}(x_i) + b)(\sum_{i=1}^n \mu_{\underline{C}}(x_i) + c)} < \sum_{i=1}^n \mu_{\underline{B}}(x_i) + \frac{b}{2} + \sum_{i=1}^n \mu_{\underline{C}}(x_i) + \frac{c}{2} \quad (2.4.4)$$

Once more, it is solely dependent on $\sum_{i=1}^n \mu_{\bar{B}}(x_i) = \sum_{i=1}^n \mu_{\bar{C}}(x_i)$, b and c also $\sum_{i=1}^n \mu_{\underline{B}}(x_i) = \sum_{i=1}^n \mu_{\underline{C}}(x_i)$, b and c . However, If we apply the directed-divergence due to Havrda-Charvat [2]

$$\begin{aligned} & \sum_{i=1}^n \left(\left(\frac{\mu_{\bar{B}}(x_i)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b} \right)^\alpha \left(\frac{\mu_{\bar{B}}(x_i)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x)} \right)^{1-\alpha} + \left(\frac{\mu_{\underline{B}}(x_i)}{\sum_{i=1}^n \mu_{\underline{B}}(x_i) + b} \right)^\alpha \left(\frac{\mu_{\underline{B}}(x_i)}{\sum_{i=1}^n \mu_{\underline{B}}(x_i) + \mu_{\underline{B}}(x)} \right)^{1-\alpha} \right) \\ & + \left(\frac{b}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x)} \right)^\alpha \left(\frac{\mu_{\bar{B}}(x)}{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x)} \right)^{1-\alpha} + \left(\frac{b}{\sum_{i=1}^n \mu_{\underline{B}}(x_i) + \mu_{\underline{B}}(x)} \right)^\alpha \left(\frac{\mu_{\underline{B}}(x)}{\sum_{i=1}^n \mu_{\underline{B}}(x_i) + \mu_{\underline{B}}(x)} \right)^{1-\alpha} \\ & = \sum_{i=1}^n \left(\left(\frac{\mu_{\bar{C}}(x_i)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} \right)^\alpha \left(\frac{\mu_{\bar{C}}(x_i)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c} \right)^{1-\alpha} + \left(\frac{\mu_{\underline{C}}(x_i)}{\sum_{i=1}^n \mu_{\underline{C}}(x_i) + \mu_{\underline{C}}(x)} \right)^\alpha \left(\frac{\mu_{\underline{C}}(x_i)}{\sum_{i=1}^n \mu_{\underline{C}}(x_i) + c} \right)^{1-\alpha} \right) \\ & + \left(\frac{\mu_{\bar{C}}(x)}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x)} \right)^\alpha \left(\frac{c}{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c} \right)^{1-\alpha} + \left(\frac{\mu_{\underline{C}}(x)}{\sum_{i=1}^n \mu_{\underline{C}}(x_i) + \mu_{\underline{C}}(x)} \right)^\alpha \left(\frac{c}{\sum_{i=1}^n \mu_{\underline{C}}(x_i) + c} \right)^{1-\alpha} \quad (2.4.5) \end{aligned}$$

we mean to say,

$$\begin{aligned} & \frac{\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b^\alpha (\mu_{\bar{B}}(x))^{1-\alpha}}{(\sum_{i=1}^n \mu_{\bar{B}}(x_i) + b)^\alpha (\sum_{i=1}^n \mu_{\bar{B}}(x_i) + \mu_{\bar{B}}(x))^{1-\alpha}} + \frac{\sum_{i=1}^n \mu_{\underline{B}}(x_i) + b^\alpha (\mu_{\underline{B}}(x))^{1-\alpha}}{(\sum_{i=1}^n \mu_{\underline{B}}(x_i) + b)^\alpha (\sum_{i=1}^n \mu_{\underline{B}}(x_i) + \mu_{\underline{B}}(x))^{1-\alpha}} \\ & = \frac{\sum_{i=1}^n \mu_{\bar{C}}(x_i) + (\mu_{\bar{C}}(x))^\alpha c^{1-\alpha}}{(\sum_{i=1}^n \mu_{\bar{C}}(x_i) + \mu_{\bar{C}}(x))^\alpha (\sum_{i=1}^n \mu_{\bar{C}}(x_i) + c)^{1-\alpha}} + \frac{\sum_{i=1}^n \mu_{\underline{C}}(x_i) + (\mu_{\underline{C}}(x))^\alpha c^{1-\alpha}}{(\sum_{i=1}^n \mu_{\underline{C}}(x_i) + \mu_{\underline{C}}(x))^\alpha (\sum_{i=1}^n \mu_{\underline{C}}(x_i) + c)^{1-\alpha}} \quad (2.4.6) \end{aligned}$$

The above shows that the estimate again solely dependent on $\sum_{i=1}^n \mu_{\bar{B}}(x_i) = \sum_{i=1}^n \mu_{\bar{C}}(x_i)$, b , c and $\sum_{i=1}^n \mu_{\underline{B}}(x_i) = \sum_{i=1}^n \mu_{\underline{C}}(x_i)$, b , c .

2.5 An alternative approach for estimation

It should be noted that the principle that $\mu_{\bar{A}}(x_1), \dots, \mu_{\bar{A}}(x_k), \mu_{\bar{A}}(x), \mu_{\bar{A}}(x_{k+1}), \dots, \mu_{\bar{A}}(x_n)$ and $\mu_{\underline{A}}(x_1), \dots, \mu_{\underline{A}}(x_k), \mu_{\underline{A}}(x), \mu_{\underline{A}}(x_{k+1}), \dots, \mu_{\underline{A}}(x_n)$ should be as equal as possible if the data gap falls between $\mu_{\bar{A}}(x_k), \mu_{\bar{A}}(x_{k+1})$ and $\mu_{\underline{A}}(x_k), \mu_{\underline{A}}(x_{k+1})$. Thus, we begin with the following principle: $\mu_{\bar{A}}(x) - \mu_{\bar{A}}(x_1), \dots, \mu_{\bar{A}}(x) - \mu_{\bar{A}}(x_k), \mu_{\bar{A}}(x_{k+1}) - \mu_{\bar{A}}(x), \dots, \mu_{\bar{A}}(x_n) - \mu_{\bar{A}}(x)$ and $\mu_{\underline{A}}(x) - \mu_{\underline{A}}(x_1), \dots, \mu_{\underline{A}}(x) - \mu_{\underline{A}}(x_k), \mu_{\underline{A}}(x_{k+1}) - \mu_{\underline{A}}(x)$. It is important that $-\mu_{\underline{A}}(x), \dots, \mu_{\bar{A}}(x_n) - \mu_{\bar{A}}(x)$ be as equal as feasible. Assuming that $\omega_1 = \sum_{i=1}^n \mu_{\bar{A}}(x_i)$ and $\omega_2 = \sum_{i=1}^n \mu_{\underline{A}}(x_i)$. The matching probability distribution is

$$\begin{aligned} & \frac{\mu_{\bar{A}}(x) - \mu_{\bar{A}}(x_1)}{(2k-n)\mu_{\bar{A}}(x) + \sum_{i=1}^n \mu_{\bar{A}}(x_i) - 2\omega_1}, \dots, \frac{\mu_{\bar{A}}(x) - \mu_{\bar{A}}(x_k)}{(2k-n)\mu_{\bar{A}}(x) + \sum_{i=1}^n \mu_{\bar{A}}(x_i) - 2\omega_1}, \frac{\mu_{\bar{A}}(x_{k-1}) - \mu_{\bar{A}}(x)}{(2k-n)\mu_{\bar{A}}(x) + \sum_{i=1}^n \mu_{\bar{A}}(x_i) - 2\omega_1}, \dots \\ & \dots, \frac{\mu_{\bar{A}}(x_n) - \mu_{\bar{A}}(x)}{(2k-n)\mu_{\bar{A}}(x) + \sum_{i=1}^n \mu_{\bar{A}}(x_i) - 2\omega_1} \\ \text{and } & \frac{\mu_{\underline{A}}(x) - \mu_{\underline{A}}(x_1)}{(2k-n)\mu_{\underline{A}}(x) + \sum_{i=1}^n \mu_{\underline{A}}(x_i) - 2\omega_2}, \dots, \frac{\mu_{\underline{A}}(x) - \mu_{\underline{A}}(x_k)}{(2k-n)\mu_{\underline{A}}(x) + \sum_{i=1}^n \mu_{\underline{A}}(x_i) - 2\omega_2}, \frac{\mu_{\underline{A}}(x_{k-1}) - \mu_{\underline{A}}(x)}{(2k-n)\mu_{\underline{A}}(x) + \sum_{i=1}^n \mu_{\underline{A}}(x_i) - 2\omega_2}, \dots \\ & \dots, \frac{\mu_{\underline{A}}(x_n) - \mu_{\underline{A}}(x)}{(2k-n)\mu_{\underline{A}}(x) + \sum_{i=1}^n \mu_{\underline{A}}(x_i) - 2\omega_2} \end{aligned}$$

In order to maximize the Burg information measure of equality, we select $\mu_{\bar{A}}(x)$ and $\mu_{\underline{A}}(x)$.

$$\sum_{i=1}^k \ln \frac{\mu_{\bar{A}}(x) - \mu_{\bar{A}}(x_i)}{(2k-n)\mu_{\bar{A}}(x) + \sum_{i=1}^n \mu_{\bar{A}}(x_i) - 2\omega_1} + \sum_{i=k+1}^n \ln \frac{\mu_{\bar{A}}(x_i) - \mu_{\bar{A}}(x)}{(2k-n)\mu_{\bar{A}}(x) + \sum_{i=1}^n \mu_{\bar{A}}(x_i) - 2\omega_1}$$

At the same time,

$$\sum_{i=1}^k \ln \frac{\mu_{\underline{A}}(x) - \mu_{\underline{A}}(x_i)}{(2k-n)\mu_{\underline{A}}(x) + \sum_{i=1}^n \mu_{\underline{A}}(x_i) - 2\omega_2} + \sum_{i=k+1}^n \ln \frac{\mu_{\underline{A}}(x_i) - \mu_{\underline{A}}(x)}{(2k-n)\mu_{\underline{A}}(x) + \sum_{i=1}^n \mu_{\underline{A}}(x_i) - 2\omega_2}$$

This gives,

$$\begin{aligned} & \sum_{i=1}^k \frac{1}{\mu_{\bar{A}}(x) - \mu_{\bar{A}}(x_i)} - \frac{(2k-n)k}{(2k-n)\mu_{\bar{A}}(x) + \sum_{i=1}^n \mu_{\bar{A}}(x_i) - 2\omega_1} \\ & - \sum_{i=1}^k \frac{1}{\mu_{\underline{A}}(x_i) - \mu_{\underline{A}}(x)} - \frac{(n-k)(2k-n)}{(2k-n)\mu_{\underline{A}}(x) + \sum_{i=1}^n \mu_{\underline{A}}(x_i) - 2\omega_2} \\ & + \sum_{i=1}^k \frac{1}{\mu_{\underline{A}}(x) - \mu_{\underline{A}}(x_i)} - \frac{(2k-n)k}{(2k-n)\mu_{\underline{A}}(x) + \sum_{i=1}^n \mu_{\underline{A}}(x_i) - 2\omega_2} \\ & - \sum_{i=1}^k \frac{1}{\mu_{\bar{A}}(x_i) - \mu_{\bar{A}}(x)} - \frac{(n-k)(2k-n)}{(2k-n)\mu_{\bar{A}}(x) + \sum_{i=1}^n \mu_{\bar{A}}(x_i) - 2\omega_1} = 0 \end{aligned}$$

$$i. e. \quad f_1(x) = \sum_{i=1}^k \frac{1}{\mu_{\bar{A}}(x) - \mu_{\bar{A}}(x_i)} - \sum_{i=k+1}^n \frac{1}{\mu_{\bar{A}}(x_i) - \mu_{\bar{A}}(x)} - \frac{n(2k-n)k}{(2k-n)\mu_{\bar{A}}(x) + \sum_{i=1}^n \mu_{\bar{A}}(x_i) - 2\omega_1} = 0$$

At the same time

$$f_2(x) = \sum_{i=1}^k \frac{1}{\mu_{\underline{A}}(x) - \mu_{\underline{A}}(x_i)} - \sum_{i=k+1}^n \frac{1}{\mu_{\underline{A}}(x_i) - \mu_{\underline{A}}(x)} - \frac{n(2k-n)k}{(2k-n)\mu_{\underline{A}}(x) + \sum_{i=1}^n \mu_{\underline{A}}(x_i) - 2\omega_2} = 0$$

Since, $f_1(x_k) = \infty$, $f_1(x_{k+1}) = -\infty$ and $f_2(x_k) = \infty$, $f_2(x_{k+1}) = -\infty$.

Our estimate for the data gap existing between x_k and x_{k+1} is that $f_1(x) = 0$ and $f_2(x) = 0$ have a root lying between x_k and x_{k+1} .

In the equations above, when $n = 2k$, then

$$\sum_{i=1}^k \frac{1}{\mu_{\bar{A}}(x) - \mu_{\bar{A}}(x_i)} = \sum_{i=k+1}^{2k} \frac{1}{\mu_{\bar{A}}(x_i) - \mu_{\bar{A}}(x)}$$

At the same time

$$\sum_{i=1}^k \frac{1}{\mu_{\underline{A}}(x) - \mu_{\underline{A}}(x_i)} = \sum_{i=k+1}^{2k} \frac{1}{\mu_{\underline{A}}(x_i) - \mu_{\underline{A}}(x)}$$

3. DISCUSSION AND CONCLUSION

The availability of data and the application of the inductive inference principle to the given data are the two main determinants of the data gap estimate. It is possible to estimate the probabilistic equality measure of the entropy of $\mu_{\bar{A}}(x_1), \dots, \mu_{\bar{A}}(x_n), \mu_{\underline{A}}(x_1), \dots, \mu_{\underline{A}}(x_n)$ based on the principle of maximum equality of $\mu_{\bar{A}}(x_1), \dots, \mu_{\bar{A}}(x_n), \mu_{\underline{A}}(x_1), \dots, \mu_{\underline{A}}(x_n)$ and $\mu_{\bar{A}}(x), \mu_{\underline{A}}(x)$ provided the values of $\mu_{\bar{A}}(x_1), \dots, \mu_{\bar{A}}(x_n), \mu_{\underline{A}}(x_1), \dots, \mu_{\underline{A}}(x_n)$. The data gap $\mu_{\bar{A}}(x), \mu_{\underline{A}}(x)$ can be evaluated in one of three ways if it falls between $\mu_{\bar{A}}(x_1), \mu_{\underline{A}}(x_1)$ and $\mu_{\bar{A}}(x_{k+1}), \mu_{\underline{A}}(x_{k+1})$.

- (i) How much $\mu_{\bar{A}}(x), \mu_{\underline{A}}(x)$ stands between $\mu_{\bar{A}}(x_k), \mu_{\bar{A}}(x_{k+1})$ and $\mu_{\underline{A}}(x_k), \mu_{\underline{A}}(x_{k+1})$ at the same time determines the entropy measure $S(\mu_{\bar{A}}(x)), S(\mu_{\underline{A}}(x))$.
- (ii) Determine the likelihood that the probability distribution according to $\mu_{\bar{A}}(x), \mu_{\underline{A}}(x)$ is equally spaced from the probability distribution according to $\mu_{\bar{A}}(x_k), \mu_{\bar{A}}(x_{k+1})$ and $\mu_{\underline{A}}(x_k), \mu_{\underline{A}}(x_{k+1})$ at the same time.
- (iii) Pick $\mu_{\bar{A}}(x), \mu_{\underline{A}}(x)$ so that it equals, at most, $\mu_{\bar{A}}(x) - \mu_{\bar{A}}(x_1), \dots, \mu_{\bar{A}}(x) - \mu_{\bar{A}}(x_k), \mu_{\bar{A}}(x_{k+1}) - \mu_{\bar{A}}(x), \dots, \mu_{\bar{A}}(x_n) - \mu_{\bar{A}}(x)$. At the same time, $\mu_{\underline{A}}(x) - \mu_{\underline{A}}(x_1), \dots, \mu_{\underline{A}}(x) - \mu_{\underline{A}}(x_k), \mu_{\underline{A}}(x_{k+1}) - \mu_{\underline{A}}(x), \dots, \mu_{\underline{A}}(x_n) - \mu_{\underline{A}}(x)$.

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