



Quantification of Theoretic Assertions Pertaining to Information Entropy Grouping Divergence Under Fuzzy Context

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ARTICLE INFO	ABSTRACT
Published Online: 29 January 2026	Strong groupings and their connection to the diverging principle for the information measure are asserted in a fuzzy ambience in the current communication. It is demonstrated that for every
Corresponding Author: Rohit Kumar Verma	feasible value of n and $n(>k)$, H_n may be stated in terms of $H_{(n-k+1)}$ and H_k . The diverging property results from its attempts at groupings as well.
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1. INTRODUCTION

A measure of entropy $H_n(p_1, p_2, \dots, p_n)$ will be said to have the grouping property with respect to grouping function in fuzzy ambience $g(p_1, p_2)$ if

$$H_n(p_1, p_2, \dots, p_n) = H_{n-1}(p_1 + p_2, p_3, \dots, p_n) + g(p_1, p_2) H_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right)$$

holds for all $n \geq 3$. Here $g(p_1, p_2)$ is a known continuous function. The property is called grouping property because if we know $H_2(p_1, p_2)$ we can find $H_3(p_1, p_2, p_3)$ knowing which we can find $H_4(p_1, p_2, p_3, p_4)$ knowing which we can find $H_5(p_1, p_2, p_3, p_4, p_5)$ and continuing in this way we can find $H_n(p_1, p_2, \dots, p_n)$ for all values of n .

If we consider entropy measures of the sum form, i.e. if we consider entropy measures of the form

$$H_n(p_1, p_2, \dots, p_n) = \phi(p_1) + \phi(p_2) + \dots + \phi(p_n)$$

Since knowing the function $\phi(p)$, we can write down the expression for $H_n(p_1, p_2, \dots, p_n)$, at once. However the above two equations gives

$$\phi(p_1) + \phi(p_2) = \phi(p_1 + p_2) + g(p_1, p_1 + p_2) \left[\phi\left(\frac{p_1}{p_1+p_2}\right) + \phi\left(\frac{p_2}{p_1+p_2}\right) \right]$$

In almost all the books on the subject (Aczel and Daroczy[1], Kapur [3, 4, 5], Mathai and Rathie [6]) these are mentioned as two separate properties of Shannon's measure of entropy. On the other hand, a key feature of Shannon entropy is the Diverging Principle, also known in information theory as the Diverging Axiom or Additivity on Independent Events. It states that the total uncertainty of a complex system can be divided (or "diverged") into the uncertainty of each of its parts. However, this is difficult in the case of Verma measure [8, 9, 10, 11]. Therefore, we can say that the intuition that

information for independent events is total is formalized by the Diverging Principle: the sum of the information obtained from each independent process is the total information obtained by observing the outcomes of two independent processes. Mathematically, for a system described by a set of probabilities, the diverging principle holds.

$$H_{m_1+m_2+\dots+m_n}(p_{11}, p_{12}, \dots, p_{1m_1}; p_{21}, p_{22}, \dots, p_{2m_2}; \dots; p_{n1}, p_{n2}, \dots, p_{nm_n}) = H_n(p_1, p_2, \dots, p_n) +$$

$$\sum_{i=1}^n p_i H_{m_i}\left(\frac{p_{i1}}{p_i}, \frac{p_{i2}}{p_i}, \dots, \frac{p_{im_i}}{p_i}\right),$$

$$\text{where, } p_i = \sum_{j=1}^{m_i} p_{ij}, \quad i = 1, 2, \dots, n.$$

It is easy to modify entropic divergence (and strong divergence) to conduct more than one-step lookahead. Smaller search trees would often result from this, but each node would take longer. (For strong divergence, two-step lookahead has been demonstrated to reduce overall run-time on some really difficult problems (when combined with implication information from the two-step lookahead). Conducting deeper lookaheads exclusively on candidates who appear promising based on shallower lookaheads is one strategy to mitigate this tradeoff. Before performing any lookahead, the candidate pool could potentially be reduced using heuristics.

For example, algorithms for decision tree induction and our entropic lookahead method for question selection at search nodes have an intriguing (albeit not perfect) parallel. A question is added at the leaf that yields the largest information gain in the majority of grouping decision tree induction techniques. Similar to this, in search, we can create children that, in a way, likewise yield the highest information gain by selecting queries whose offspring have the lowest entropy.

Zadeh [12], In contrast to full members (1) or non-members (0), fuzzy sets, a generalization of classical set theory, permit elements to have partial membership, denoted by a degree of membership between 0 and 1. Fuzzy sets are helpful for expressing ambiguous and imprecise thoughts since they are defined by a membership function that gives each element a particular grade of membership.

2. OUR CLAIMS

CLAIM 2.1 For every, ambiguity $\mu_A(x_i)$, $i = 1, \dots, n$. Show that the equality

$$\begin{aligned} & H_n(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n)) + H_n(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n)) = \\ & = H_{n-k+1}(\mu_A(x_1) + \mu_A(x_2) + \dots + \mu_A(x_k), \mu_A(x_{k+1}), \dots, \mu_A(x_n)) + \\ & \quad (\mu_A(x_1) + \mu_A(x_2) + \dots + \mu_A(x_k))^\alpha \\ & \quad H_k\left(\frac{\mu_A(x_1)}{\sum_{j=1}^k \mu_A(x_j)}, \dots, \frac{\mu_A(x_k)}{\sum_{j=1}^k \mu_A(x_j)}\right) + \\ & \quad H_{n-k+1}(k - \mu_A(x_1) - \mu_A(x_2) - \dots - \mu_A(x_k), 1 - \mu_A(x_{k+1}), \dots, 1 - \mu_A(x_n)) + \\ & \quad (k - \mu_A(x_1) - \mu_A(x_2) - \dots - \mu_A(x_k))^\alpha \\ & \quad H_k\left(\frac{1 - \mu_A(x_1)}{\sum_{j=1}^k (1 - \mu_A(x_j))}, \dots, \frac{1 - \mu_A(x_k)}{\sum_{j=1}^k (1 - \mu_A(x_j))}\right) \end{aligned}$$

Proof: Since

$$\begin{aligned} & H_n(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n)) + H_n(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n)) = \\ & H_{n-1}(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3), \dots, \mu_A(x_n)) \\ & + H_{n-1}(2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3), \dots, 1 - \mu_A(x_n)) \\ & + g(\mu_A(x_1), \mu_A(x_2)) H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2)}\right) \\ & + g(1 - \mu_A(x_1), 1 - \mu_A(x_2)) H_2\left(\frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)}, \frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)}\right) \end{aligned} \quad (2.1.1)$$

$$\begin{aligned} & = H_{n-2}(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3), \mu_A(x_4), \dots, \mu_A(x_n)) \\ & + H_{n-2}(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3), 1 - \mu_A(x_4), \dots, 1 - \mu_A(x_n)) \\ & + g(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3)) H_2\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) \\ & + g(2 - \mu_A(x_1) - \mu_A(x_2), \mu_A(x_3)) H_2\left(\frac{2 - \mu_A(x_1) - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_3)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) \\ & + g(\mu_A(x_1), \mu_A(x_2)) H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2)}\right) \\ & + g(1 - \mu_A(x_1), 1 - \mu_A(x_2)) H_2\left(\frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)}, \frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)}\right) \end{aligned} \quad (2.1.2)$$

$$\begin{aligned} & = H_{n-3}(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3) + \mu_A(x_4), \mu_A(x_5), \dots, \mu_A(x_n)) \\ & + H_{n-3}(4 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3) - \mu_A(x_4), 1 - \mu_A(x_5), \dots, 1 - \mu_A(x_n)) \\ & + g(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3), \mu_A(x_4)) \\ & \quad H_2\left(\frac{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3) + \mu_A(x_4)}, \frac{\mu_A(x_4)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3) + \mu_A(x_4)}\right) \\ & + g(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3), 1 - \mu_A(x_4)) \\ & \quad H_2\left(\frac{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}{4 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3) - \mu_A(x_4)}, \frac{\mu_A(x_4)}{4 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3) - \mu_A(x_4)}\right) \\ & + g(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3)) H_2\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) \\ & + g(2 - \mu_A(x_1) - \mu_A(x_2), \mu_A(x_3)) H_2\left(\frac{2 - \mu_A(x_1) - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_3)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) \\ & + g(\mu_A(x_1), \mu_A(x_2)) H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2)}\right) \\ & + g(1 - \mu_A(x_1), 1 - \mu_A(x_2)) H_2\left(\frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)}, \frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)}\right) \end{aligned} \quad (2.1.3)$$

More generally

$$\begin{aligned} & H_n(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n)) + H_n(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n)) = \\ & H_{n-k}(\mu_A(x_1) + \mu_A(x_2) + \dots + \mu_A(x_k), \mu_A(x_{k+1}), \dots, \mu_A(x_n)) \end{aligned}$$

Let X be a space, and let x be its generic element, so that $X = \{x\}$. Then, the following equation

$$A = \{(x, \mu_A(x)) : x \in X\}$$

defines a fuzzy set. Where $\mu_A: X \rightarrow [0,1]$ is the fuzzy set A 's membership function and $\mu_A(x) \in [0,1]$ is the element x 's degree of membership to the set A .

$$\begin{aligned}
 &+ H_{n-1}(k - \mu_A(x_1) - \mu_A(x_2) - \dots - \mu_A(x_k), 1 - \mu_A(x_{k+1}), \dots, 1 - \mu_A(x_n)) \\
 &+ \sum_{i=1}^k g(\sum_{j=1}^i \mu_A(x_j), \mu_A(x_{i+1})) H_2\left(\frac{\sum_{j=1}^i \mu_A(x_j)}{\sum_{j=1}^{i+1} \mu_A(x_j)}, \frac{\mu_A(x_{i+1})}{\sum_{j=1}^{i+1} \mu_A(x_j)}\right) \\
 &+ \sum_{i=1}^k g(\sum_{j=1}^i 1 - \mu_A(x_j), 1 - \mu_A(x_{i+1})) H_2\left(\frac{\sum_{j=1}^i 1 - \mu_A(x_j)}{\sum_{j=1}^{i+1} 1 - \mu_A(x_j)}, \frac{1 - \mu_A(x_{i+1})}{\sum_{j=1}^{i+1} 1 - \mu_A(x_j)}\right)
 \end{aligned} \tag{2.1.4}$$

$$\begin{aligned}
 &H_3\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) + \\
 &H_3\left(\frac{1 - \mu_A(x_1)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_3)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) = \\
 &H_2\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) \\
 &+ H_2\left(\frac{2 - \mu_A(x_1) - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_3)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) \\
 &+ g\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) H_2\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) \\
 &+ g\left(\frac{2 - \mu_A(x_1) - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_3)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) H_2\left(\frac{2 - \mu_A(x_1) - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_3)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right)
 \end{aligned} \tag{2.1.5}$$

By (2.1.2) and (2.1.5)

$$\begin{aligned}
 &H_n(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n)) + H_n(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n)) \\
 &= H_{n-2}(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3), \mu_A(x_4), \dots, \mu_A(x_n)) + g(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3)) \\
 &\left[H_3\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) \right. \\
 &- g\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) \\
 &+ g(\mu_A(x_1), \mu_A(x_2)) H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2)}\right) \\
 &+ H_{n-2}(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3), 1 - \mu_A(x_4), \dots, 1 - \mu_A(x_n)) \\
 &+ g(2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3)) \\
 &\left. \left[H_3\left(\frac{1 - \mu_A(x_1)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_3)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) \right. \right. \\
 &- g\left(\frac{1 - \mu_A(x_1)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) H_2\left(\frac{1 - \mu_A(x_1)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) \\
 &\left. \left. + g(1 - \mu_A(x_1), 1 - \mu_A(x_2)) H_2\left(\frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)}, \frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)}\right) \right] \right]
 \end{aligned}$$

This will be expressed in terms of H_{n-2} and H_3 only if

$$\begin{aligned}
 &g(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3)) g\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}, \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) \\
 &+ g(2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3)) g\left(\frac{1 - \mu_A(x_1)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}, \frac{1 - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) \\
 &= g(\mu_A(x_1), \mu_A(x_2)) + g(1 - \mu_A(x_1), 1 - \mu_A(x_2))
 \end{aligned} \tag{2.1.6}$$

$$\begin{aligned}
 &\text{This will be satisfied if } g(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3)) + g(2 - \mu_A(x_1) - \mu_A(x_2), \mu_A(x_3)) \\
 &= f(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)) + f(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)) \\
 &\text{i.e. } g(\mu_A(x_1), \mu_A(x_2)) + g(1 - \mu_A(x_1), 1 - \mu_A(x_2)) \\
 &= f(\mu_A(x_1) + \mu_A(x_2)) + f(2 - \mu_A(x_1) - \mu_A(x_2))
 \end{aligned} \tag{2.1.7}$$

Equations (2.1.6) and (2.1.7) gives

$$\begin{aligned}
 &[f(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)) + f(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3))] \\
 &\left[f\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) + f\left(\frac{2 - \mu_A(x_1) - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) \right] \\
 &= [f(\mu_A(x_1) + \mu_A(x_2)) + f(2 - \mu_A(x_1) - \mu_A(x_2))] \\
 &\text{i.e. } \left[f\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) + f\left(\frac{2 - \mu_A(x_1) - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) \right] \\
 &= \frac{[f(\mu_A(x_1) + \mu_A(x_2)) + f(2 - \mu_A(x_1) - \mu_A(x_2))]}{[f(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)) + f(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3))]}
 \end{aligned} \tag{2.1.8}$$

The functional equation (2.1.8) has the solution

$$\begin{aligned}
 & \left[f\left(\frac{\mu_A(x_1)+\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right) + f\left(\frac{2-\mu_A(x_1)-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right) \right] \\
 &= \left[\left(\frac{\mu_A(x_1)+\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right) + \left(\frac{2-\mu_A(x_1)-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right) \right]^\alpha \\
 H_n(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n)) &+ H_n(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n)) \\
 &= H_{n-1}(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3), \dots, \mu_A(x_n)) + \\
 &\quad (\mu_A(x_1) + \mu_A(x_2))^\alpha H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right) + \\
 &\quad H_{n-1}(2 - \mu_A(x_1) - \mu_A(x_2), \mu_A(x_3), \dots, \mu_A(x_n)) + \\
 &\quad (2 - \mu_A(x_1) - \mu_A(x_2))^\alpha H_2\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}, \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right) \\
 &= H_{n-2}(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3), \mu_A(x_4), \dots, \mu_A(x_n)) + \\
 &\quad (\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3))^\alpha \\
 &\quad H_3\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right) + \\
 &\quad H_{n-2}(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3), \mu_A(x_4), \dots, \mu_A(x_n)) + \\
 &\quad (3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3))^\alpha \\
 &\quad H_3\left(\frac{1-\mu_A(x_1)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}, \frac{1-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}, \frac{1-\mu_A(x_3)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right) \\
 &= H_{n-k+1}(\mu_A(x_1) + \mu_A(x_2) + \dots + \mu_A(x_k), \mu_A(x_{k+1}), \dots, \mu_A(x_n)) + \\
 &\quad (\mu_A(x_1) + \mu_A(x_2) + \dots + \mu_A(x_k))^\alpha \\
 &\quad H_k\left(\frac{\mu_A(x_1)}{\sum_{j=1}^k \mu_A(x_j)}, \dots, \frac{\mu_A(x_k)}{\sum_{j=1}^k \mu_A(x_j)}\right) + \\
 &\quad H_{n-k+1}(k - \mu_A(x_1) - \mu_A(x_2) - \dots - \mu_A(x_k), 1 - \mu_A(x_{k+1}), \dots, 1 - \mu_A(x_n)) + \\
 &\quad (k - \mu_A(x_1) - \mu_A(x_2) - \dots - \mu_A(x_k))^\alpha \\
 &\quad H_k\left(\frac{1-\mu_A(x_1)}{\sum_{j=1}^k (1-\mu_A(x_j))}, \dots, \frac{1-\mu_A(x_k)}{\sum_{j=1}^k (1-\mu_A(x_j))}\right).
 \end{aligned}$$

Thus, the required result is obtained.

CLAIM 2.2 For every, vagueness $\mu_A(x_i)$, $i = 1, \dots, n$ we claim the following equality

$$\begin{aligned}
 & H_{m_1+m_2+\dots+m_k}(\mu_A(x_{11}), \dots, \mu_A(x_{1m_1}), \mu_A(x_{21}), \dots, \mu_A(x_{2m_2}), \mu_A(x_{n1}), \mu_A(x_{n1}), \dots, \mu_A(x_{nm_k})) \\
 & \quad H_{m_1+m_2+\dots+m_k}(1 - \mu_A(x_1), \dots, 1 - \mu_A(x_{1m_1}), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_{2m_2}), \\
 & \quad \quad 1 - \mu_A(x_{n1}), 1 - \mu_A(x_{n1}), \dots, 1 - \mu_A(x_{nm_k})) \\
 &= H_n(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n)) + H_n(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n)) \\
 & \quad + \sum_{i=1}^n \mu_A(x_i) H_{m_i} \left\{ \frac{\mu_A(x_{i1}), \mu_A(x_{i2}), \dots, \mu_A(x_{im_i})}{\mu_A(x_i)} \right\} \\
 & \quad + \sum_{i=1}^n (1 - \mu_A(x_i)) H_{m_i} \left\{ \frac{1-\mu_A(x_{i1}), 1-\mu_A(x_{i2}), \dots, 1-\mu_A(x_{im_i})}{1-\mu_A(x_i)} \right\}.
 \end{aligned}$$

Proof: Since the grouping principle stated as follows

$$\begin{aligned}
 & H_n(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n)) + H_n(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n)) = \\
 & \quad H_{n-1}(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3), \dots, \mu_A(x_n)) \\
 & \quad + H_{n-1}(2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3), \dots, 1 - \mu_A(x_n)) \\
 & \quad + (\mu_A(x_1) + \mu_A(x_2)) H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right) \\
 & \quad + (2 - \mu_A(x_1) - \mu_A(x_2)) H_2\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}, \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right) \\
 & \quad \quad \quad n = 3, 4, 5, \dots \quad (2.2.1)
 \end{aligned}$$

$$\begin{aligned}
 & H_2(\mu_A(x_1), \mu_A(x_2)) + H_n(1 - \mu_A(x_1), 1 - \mu_A(x_2)) = \\
 & \quad -\mu_A(x_1) \ln \mu_A(x_1) - (1 - \mu_A(x_1)) \ln(1 - \mu_A(x_1)) \\
 & \quad -\mu_A(x_2) \ln \mu_A(x_2) - (1 - \mu_A(x_2)) \ln(1 - \mu_A(x_2)) \quad (2.2.2)
 \end{aligned}$$

$$\begin{aligned}
 & H_3(\mu_A(x_1), \mu_A(x_2), \mu_A(x_3)) + H_3(1 - \mu_A(x_1), 1 - \mu_A(x_2), 1 - \mu_A(x_3)) = \\
 & \quad H_2(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3)) + H_2(2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3))
 \end{aligned}$$

$$\begin{aligned}
 & +(\mu_A(x_1) + \mu_A(x_2))H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right) + \\
 & + (2 - \mu_A(x_1) - \mu_A(x_2))H_2\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}, \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right) \\
 = & -(\mu_A(x_1) + \mu_A(x_2))\ln(\mu_A(x_1) + \mu_A(x_2)) \\
 & - (2 - \mu_A(x_1) - \mu_A(x_2))\ln(2 - \mu_A(x_1) - \mu_A(x_2)) \\
 & - \mu_A(x_3)\ln\mu_A(x_3) - (1 - \mu_A(x_3))\ln(1 - \mu_A(x_3)) \\
 & + (\mu_A(x_1) + \mu_A(x_2))\left\{-\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}\ln\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)} - \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\ln\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right\} \\
 & + (2 - \mu_A(x_1) - \mu_A(x_2))\left\{-\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}\ln\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)} \right. \\
 & \quad \left. - \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\ln\frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right\} \\
 = & -\mu_A(x_1)\ln\mu_A(x_1) - (1 - \mu_A(x_1))\ln(1 - \mu_A(x_1)) \\
 & - \mu_A(x_2)\ln\mu_A(x_2) - (1 - \mu_A(x_2))\ln(1 - \mu_A(x_2)) \\
 & - \mu_A(x_3)\ln\mu_A(x_3) - (1 - \mu_A(x_3))\ln(1 - \mu_A(x_3)) \tag{2.2.3}
 \end{aligned}$$

$$\begin{aligned}
 \text{Letting } H_n(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n)) + H_n(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n)) \\
 = -\sum_{i=1}^n \mu_A(x_i)\ln\mu_A(x_i) - \sum_{i=1}^n (1 - \mu_A(x_i))\ln(1 - \mu_A(x_i)) \tag{2.2.4}
 \end{aligned}$$

Then from (2.2.1) we have

$$\begin{aligned}
 & H_{n+1}(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_{n+1})) + H_{n+1}(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_{n+1})) = \\
 & H_n(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3), \dots, \mu_A(x_{n+1})) \\
 & + H_n(2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3), \dots, 1 - \mu_A(x_{n+1})) \\
 & + (\mu_A(x_1) + \mu_A(x_2))H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right) \\
 & + (2 - \mu_A(x_1) - \mu_A(x_2))H_2\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}, \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right) \\
 = & -(\mu_A(x_1) + \mu_A(x_2))\ln(\mu_A(x_1) + \mu_A(x_2)) \\
 & - (2 - \mu_A(x_1) - \mu_A(x_2))\ln(2 - \mu_A(x_1) - \mu_A(x_2)) \\
 & - \sum_{i=3}^{n+1} \mu_A(x_i)\ln\mu_A(x_i) - \sum_{i=3}^{n+1} (1 - \mu_A(x_i))\ln(1 - \mu_A(x_i)) \\
 & + (\mu_A(x_1) + \mu_A(x_2))\left\{-\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}\ln\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)} - \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\ln\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right\} \\
 & + (2 - \mu_A(x_1) - \mu_A(x_2))\left\{-\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}\ln\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)} \right. \\
 & \quad \left. - \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\ln\frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right\} \\
 = & -\sum_{i=1}^{n+1} \mu_A(x_i)\ln\mu_A(x_i) - \sum_{i=1}^{n+1} (1 - \mu_A(x_i))\ln(1 - \mu_A(x_i)) \tag{2.2.5}
 \end{aligned}$$

From (2.2.1)

$$\begin{aligned}
 & H_3(\mu_A(x_1), \mu_A(x_2), \mu_A(x_3)) + H_3(1 - \mu_A(x_1), 1 - \mu_A(x_2), 1 - \mu_A(x_3)) = \\
 & H_2(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3)) + H_2(2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3)) \\
 & + (\mu_A(x_1) + \mu_A(x_2))H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right) + \\
 & + (2 - \mu_A(x_1) - \mu_A(x_2))H_2\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}, \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right) \tag{2.2.6}
 \end{aligned}$$

Again from (2.2.1)

$$\begin{aligned}
 & H_4(\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \mu_A(x_4)) + H_4(1 - \mu_A(x_1), 1 - \mu_A(x_2), 1 - \mu_A(x_3), 1 - \mu_A(x_4)) = \\
 & H_3(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3), \mu_A(x_4)) + H_3(2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3), 1 - \mu_A(x_4)) \\
 & + (\mu_A(x_1) + \mu_A(x_2))H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right) + \\
 & + (2 - \mu_A(x_1) - \mu_A(x_2))H_2\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}, \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right) \tag{2.2.7}
 \end{aligned}$$

On using (2.2.1) again

$$\begin{aligned}
 & H_4(\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \mu_A(x_4)) + H_4(1 - \mu_A(x_1), 1 - \mu_A(x_2), 1 - \mu_A(x_3), 1 - \mu_A(x_4)) = \\
 & H_2(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3), \mu_A(x_4)) + H_2(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3), 1 - \mu_A(x_4)) \\
 & + (\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3))H_2\left(\frac{\mu_A(x_1)+\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right) \\
 & + (3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3))H_2\left(\frac{2-\mu_A(x_1)-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}, \frac{\mu_A(x_3)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)
 \end{aligned}$$

$$\begin{aligned}
 & +(\mu_A(x_1) + \mu_A(x_2))H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right) + \\
 & + (2 - \mu_A(x_1) - \mu_A(x_2))H_2\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}, \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right) \\
 = & H_2(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3), \mu_A(x_4)) + H_2(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3), 1 - \mu_A(x_4)) \\
 & + (\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)) \left\{ H_2\left(\frac{\mu_A(x_1)+\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right) \right. \\
 & + \frac{\mu_A(x_1)+\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)} H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right) \Big\} \\
 & + (3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)) \left\{ H_2\left(\frac{2-\mu_A(x_1)-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}, \frac{1-\mu_A(x_3)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right) \right. \\
 & + \frac{2-\mu_A(x_1)-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)} H_2\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}, \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right) \Big\} \quad (2.2.8)
 \end{aligned}$$

Using (2.2.6) for the fuzzy probability distribution $\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}, \frac{1-\mu_A(x_1)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}, \frac{1-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}, \frac{1-\mu_A(x_3)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}$ the expression within curly bracket comes out to be

$$\begin{aligned}
 & H_3\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right) \\
 & + H_3\left(\frac{1-\mu_A(x_1)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}, \frac{1-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}, \frac{1-\mu_A(x_3)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right) \\
 \text{Hence} \\
 & H_4(\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \mu_A(x_4)) + H_4(1 - \mu_A(x_1), 1 - \mu_A(x_2), 1 - \mu_A(x_3), 1 - \mu_A(x_4)) = \\
 & H_2(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3), \mu_A(x_4)) + H_2(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3), 1 - \mu_A(x_4)) \\
 & + (\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)) \\
 & H_3\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}, \frac{\mu_A(x_3)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right) \\
 & + (3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)) \\
 & + H_3\left(\frac{1-\mu_A(x_1)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}, \frac{1-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}, \frac{1-\mu_A(x_3)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right) \quad (2.2.9)
 \end{aligned}$$

Now, sub outcomes by using mathematical induction

$$\begin{aligned}
 & H_{n+k}(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n), \mu_A(x_{n+1}), \mu_A(x_{n+2}), \dots, \mu_A(x_{n+k})) \\
 & + H_{n+k}(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n), 1 - \mu_A(x_{n+1}), 1 - \mu_A(x_{n+2}), \dots, 1 - \mu_A(x_{n+k})) \\
 = & H_{k+1}(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n), \mu_A(x_{n+1}), \mu_A(x_{n+2}), \dots, \mu_A(x_{n+k})) \\
 & + H_{k+1}(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n), 1 - \mu_A(x_{n+1}), 1 - \mu_A(x_{n+2}), \dots, 1 - \mu_A(x_{n+k})) \\
 & + (\mu_A(x_1) + \dots + \mu_A(x_n))H_n\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\dots+\mu_A(x_n)}, \dots, \frac{\mu_A(x_n)}{\mu_A(x_1)+\dots+\mu_A(x_n)}\right) \\
 & + (n - \mu_A(x_1) + \dots + \mu_A(x_n))H_n\left(\frac{1-\mu_A(x_1)}{n-\mu_A(x_1)-\dots-\mu_A(x_n)}, \dots, \frac{1-\mu_A(x_n)}{n-\mu_A(x_1)-\dots-\mu_A(x_n)}\right) \quad (2.2.10)
 \end{aligned}$$

The first term on the R.H.S. is decompose into two terms

$$\begin{aligned}
 & H_k((n+1) + \mu_A(x_1) + \dots + \mu_A(x_n) + \mu_A(x_{n+1}), \mu_A(x_{n+2}), \dots, \mu_A(x_{n+k})) \\
 & + H_k((n+1) - \mu_A(x_1) - \mu_A(x_2) - \dots - \mu_A(x_n) - \mu_A(x_{n+1}), 1 - \mu_A(x_{n+2}), \dots, 1 - \mu_A(x_{n+k})) \\
 & + (\mu_A(x_1) + \dots + \mu_A(x_{n+1}))H_2\left(\frac{\mu_A(x_1)+\dots+\mu_A(x_n)}{\mu_A(x_1)+\dots+\mu_A(x_{n+1})}, \dots, \frac{\mu_A(x_{n+1})}{\mu_A(x_1)+\dots+\mu_A(x_{n+1})}\right) + \\
 & ((n+1) - \mu_A(x_1) - \dots - \mu_A(x_{n+1}))H_2\left(\frac{n-\mu_A(x_1)-\dots-\mu_A(x_n)}{(n+1)-\mu_A(x_1)-\dots-\mu_A(x_{n+1})}, \dots, \frac{1-\mu_A(x_{n+1})}{(n+1)-\mu_A(x_1)-\dots-\mu_A(x_{n+1})}\right) \quad (2.2.11)
 \end{aligned}$$

Combining the second terms on the R.H.S. of (2.2.10) and (2.2.11) we get

$$\begin{aligned}
 & (\mu_A(x_1) + \dots + \mu_A(x_{n+1})) \left\{ H_2\left(\frac{\mu_A(x_1)+\dots+\mu_A(x_n)}{\mu_A(x_1)+\dots+\mu_A(x_{n+1})}, \dots, \frac{\mu_A(x_{n+1})}{\mu_A(x_1)+\dots+\mu_A(x_{n+1})}\right) \right. \\
 & \quad \left. + H_n\left(\frac{\mu_A(x_1)}{n-\mu_A(x_1)+\dots+\mu_A(x_n)}, \dots, \frac{\mu_A(x_n)}{n-\mu_A(x_1)+\dots+\mu_A(x_n)}\right) \right\} + \\
 & ((n+1) - \mu_A(x_1) - \dots - \mu_A(x_{n+1})) \\
 & \quad \left\{ H_2\left(\frac{n-\mu_A(x_1)-\dots-\mu_A(x_n)}{(n+1)-\mu_A(x_1)-\dots-\mu_A(x_{n+1})}, \dots, \frac{1-\mu_A(x_{n+1})}{(n+1)-\mu_A(x_1)-\dots-\mu_A(x_{n+1})}\right) \right. \\
 & \quad \left. + H_n\left(\frac{1-\mu_A(x_1)}{n-\mu_A(x_1)+\dots+\mu_A(x_n)}, \dots, \frac{1-\mu_A(x_n)}{n-\mu_A(x_1)+\dots+\mu_A(x_n)}\right) \right\} \quad (2.2.12)
 \end{aligned}$$

Now, using (2.2.7) for $k = 1$, for the fuzzy distribution

$$\frac{\mu_A(x_1)}{\mu_A(x_1)+\dots+\mu_A(x_{n+1})}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\dots+\mu_A(x_{n+1})}, \dots, \frac{\mu_A(x_{n+1})}{\mu_A(x_1)+\dots+\mu_A(x_{n+1})}$$

We get the expression within square branches of (2.2.12) is

$$H_{n+1} \left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\dots+\mu_A(x_{n+1})}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\dots+\mu_A(x_{n+1})}, \dots, \frac{\mu_A(x_{n+1})}{\mu_A(x_1)+\dots+\mu_A(x_{n+1})} \right) \\ H_{n+1} \left(\frac{1-\mu_A(x_1)}{(n+1)-\mu_A(x_1)-\dots-\mu_A(x_{n+1})}, \frac{1-\mu_A(x_2)}{(n+1)-\mu_A(x_1)-\dots-\mu_A(x_{n+1})}, \dots, \frac{1-\mu_A(x_{n+1})}{(n+1)-\mu_A(x_1)-\dots-\mu_A(x_{n+1})} \right) \quad (2.2.13)$$

From (2.2.7), (2.2.8), (2.2.9) and (2.2.10) we found that this also holds good for $(n + 1)$ outcomes and hence

$$H_{n+k}(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n), \mu_A(x_{n+1}), \dots, \mu_A(x_{n+k})) \\ + H_{n+k}(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n), 1 - \mu_A(x_{n+1}), \dots, 1 - \mu_A(x_{n+k})) \\ = H_{k+1}(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n), \mu_A(x_{n+1}), \dots, \mu_A(x_{n+k})) \\ + H_{k+1}(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n), 1 - \mu_A(x_{n+1}), \dots, 1 - \mu_A(x_{n+k})) \\ + (\mu_A(x_1) + \dots + \mu_A(x_n))H_n \left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\dots+\mu_A(x_n)}, \dots, \frac{\mu_A(x_n)}{\mu_A(x_1)+\dots+\mu_A(x_n)} \right) \\ + (n - \mu_A(x_1) + \dots + \mu_A(x_n))H_n \left(\frac{1-\mu_A(x_1)}{n-\mu_A(x_1)-\dots-\mu_A(x_n)}, \dots, \frac{1-\mu_A(x_n)}{n-\mu_A(x_1)-\dots-\mu_A(x_n)} \right) \quad (2.2.14)$$

Valid for all n, k . Ofcourse, we can combine another set of outcomes and hence the branching property,

$$H_{m_1+m_2+\dots+m_k}(\mu_A(x_{11}), \dots, \mu_A(x_{1m_1}), \mu_A(x_{21}), \dots, \mu_A(x_{2m_2}), \mu_A(x_{n1}), \mu_A(x_{n1}), \dots, \mu_A(x_{nm_k})) \\ H_{m_1+m_2+\dots+m_k}(1 - \mu_A(x_1), \dots, 1 - \mu_A(x_{1m_1}), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_{2m_2}), \\ 1 - \mu_A(x_{n1}), 1 - \mu_A(x_{n1}), \dots, 1 - \mu_A(x_{nm_k})) \\ = H_n(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n)) + H_n(1 - \mu_A(x_1), 1 - \mu_A(x_2), \dots, 1 - \mu_A(x_n)) \\ + \sum_{i=1}^n \mu_A(x_i)H_{m_i} \left\{ \frac{\mu_A(x_{i1}), \mu_A(x_{i2}), \dots, \mu_A(x_{im_i})}{\mu_A(x_i)} \right\} \\ + \sum_{i=1}^n (1 - \mu_A(x_i))H_{m_i} \left\{ \frac{1-\mu_A(x_{i1}), 1-\mu_A(x_{i2}), \dots, 1-\mu_A(x_{im_i})}{1-\mu_A(x_i)} \right\} \quad (2.2.15)$$

Which completes the proof. Ofcourse, we can say the grouping leads to diverging property. Thus Havrda-Charvat [2] and Shannon [2, 7] measures are the only measures of entropy for which the grouping principle applies.

3. CONCLUDING REMARKS

We have demonstrated that,

(i) For any $n(> k)$, the stronger aggregation allows us to express H_n in terms of two entropies, one of type H_{n-k+1} and one of type H_k .

(ii) According to recursion, the entropy of a joint distribution can be written as the sum of two entropies: the conditional distribution of the other distribution and the entropy of a marginal distribution.

(iii) The fact that the total entropy can be split into entropy within and between classes is also emphasised.

REFERENCES

1. Aczel, J. and Daroczy, Z. (1975): Measures of Information and their Applications, Academic Press, New York.
2. Havrda, J. H. and Charvat, F. (1967): Quantification of Methods of Classifications Processes: Concept of structural a entropy. *Kybernetika* 3.30-35.
3. Kapur, J. N. (1989): Maximum Entropy Models in Science and Engineering, Wiley Eastern, New Delhi and John Wiley, New York.
4. Kapur, J. N. and Kesavan, H. K. (1987): Generalised Maximum Entropy Principle (with Applications) Sand Ford Educational Press, Waterloo University, Canada.
5. Kapur, J. N. and Kesavan, H. K. (1992): Entropy Optimization Principles and their Applica-tions Academic press, New York.
6. Mathai, A. and Rathie, P. N. (1975): Basic Concepts of Information Theory and Statistics, John Wiley, New York.
7. Shannon, C. E. (1948): Mathematical Theory of Communication, Bell System Tech. Jour. 27.379-423, 623-659.
8. Verma, R. K. (2023): Information radius via Verma information measure in intuitionistic fuzzy environment. *Int J Math Res.*;15(1):1-8.
9. Verma R. K. (2023): Modified version of Verma measures of information and their kinship with past information measures. *Int J Pure Appl. Math. Sci.*;16(1):17-24.
10. Verma, R. K. (2023): Obvious disparities on optimal guesswork wiretapper moments under mismatch related to non-Shannon cipher system. *Asian J Prob Stat.*; 24(4):17-22. DOI:10.9734/AJPAS/2023/v24i4531.
11. Verma R. K. (2023): On optimal channel capacity theorems via Verma information measure with two-sided input in noisy state. *Asian J Prob Stat.*; 22(2):1-7. DOI:10.9734/AJPAS/2023/v22i2478.
12. Zadeh, L. A. (1965): Fuzzy Sets, Information and Control, Vol. 8, pp. 94-102.