

Extension of the Sets of F-Statistical Convergence and Strong F-Cesaro Convergence by Moduli and Double Set Sequences

Alauddin Dafadar

Assistant Professor, Department of Mathematics, Bhatler College, Dantan (Autonomous), Paschim Medinipur-721426, West Bengal, India

ARTICLE INFO	ABSTRACT
Published Online: 28 November 2025	In this paper, we extend the concept of f-density to f-double natural density of a subset of $N \times N$ where f is an unbounded modulus function. We explore the notion of f-statistical convergence, and strongly f-Cesaro summability defined by modulus f by f-double natural density. In this article, we also give some inclusion relations between f-statistical convergence and strongly f-Cesaro summability with respect to modulus f for double sequence of sets in the Wijsman sense.
Corresponding Author: Alauddin Dafadar	
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I. INTRODUCTION

Statistical convergence depends on the natural density of subsets of the set $N = \{1, 2, 3, \dots\}$. The natural density $\delta(K)$ ([16]) of a set $K \subseteq N$ is defined by

$$\delta(K) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in K\}|$$

Where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n . Note that $\delta(N) = 1$ and $\delta(K) = 0$ if and only if K is finite.

The strong Cesaro convergence, and the statistical convergence, two notions introduced at different times and different contexts, are related by J. Connor's result ([19]), which was sharpened by Khan and Orhan ([20]). Among other results, Khan and Orhan ([20]) show that a sequence is strongly Cesaro convergent if and only if it is statistically convergent and uniformly integrable. Recently in ([6]), the authors obtained Connor-Khan-Orhan's ([19], [20]) result for different kinds of statistical convergence defined by moduli. In ([21]), Mursaleen and Edely obtained Connor's result for double sequences.

The notion of a modulus function was introduced by Nakano ([13]). Ruckle ([36]) and Maddox ([15]) introduced and discussed some properties of sequence spaces defined by using a modulus function. In the year 2014, Aizpuru et al. ([1]) defined a new concept of density with the help of an unbounded modulus function and, as a consequence, they obtained a new concept of non-matrix convergence, namely, f-statistical convergence, which is intermediate between the ordinary convergence and the statistical convergence and agrees with the statistical convergence when the modulus function is the identity mapping. Quite recently, Bhardwaj and

Dhawan ([30]), and Bhardwaj et al. ([31]), have introduced and studied the concepts of f-statistical convergence of order α and f-statistical boundedness, respectively, by using the approach of Aizpuru et al. ([1]) (see also [28], [32], [33], [34]). The concept of Wijsman statistical convergence is implementation of the concept of statistical convergence to sequences of sets presented by Nuray and Rhoades ([7]) in 2012. For further results one may refer to ([5], [8], [23], [24], [25], [26], [33]). The notion of Wijsman statistical convergence has been extended by Bhardwaj et al. ([34]) to that of f-Wijsman statistical convergence, where f is an unbounded modulus. Recently, Colak and Altin ([27]) introduced statistical convergence of order α for double sequences and examined some inclusion relations. Ulusu and Nuray et al. ([29]) proposed the idea of Wijsman strongly lacunary summability for set sequences and discussed its relation with Wijsman strongly Cesaro summability. Nuray et al. ([9]) introduced the notion of statistical convergence and I-statistical convergence in the Wijsman sense for double sequences of sets. Sengul et al. ([14]) introduced the concepts of f-lacunary statistical convergence of order α and strong f-lacunary summability of order α for double sequences and give some inclusion relations between these concepts. In ([17]) I. S. Ibrahim and R. Colak established relations between the sets of strongly Cesaro summable sequences of complex numbers for modulus functions f and g satisfying various conditions.

Let us first introduce some preliminaries which are essential for the background of the paper.

II. WIJSMAN CONVERGENCE AND CESARO SUMMABILITY

We recall some basic definitions and concepts ([3], [10], [11]). Let $(X; \rho)$ be a metric space. For any point $x \in X$ and for any non-empty subset U of X , we define the distance from x to U by $d(x, U) = \inf_{u \in U} \rho(x, u)$.

Let X be any non-empty set. The function $\phi : \mathbb{N} \rightarrow P(X)$ is defined by $\phi(n) = U_n (\in P(X))$ for each $n \in \mathbb{N}$. Where $P(X)$ is the power set of X . The sequence $\{U_n\} = \{U_1, U_2, U_3, \dots\}$, which is the ranges element of ϕ , is said to be a set sequences or a sequence of sets.

There are different types of convergence notions for the sequences of sets. In ([3], [25], [26]) the concepts of different types invariant convergence and statistical convergence by double sequence of sets was discussed in the sense of Wijsman. Nuray et al. ([10]) introduced the concepts of Wijsman convergence and Wijsman statistically convergence for a double sequence of sets, as follows:

Definition II.1. Let $(X; \rho)$ be a metric space and U be any non-empty closed subset of X . A double sequence of closed sets $\{U_{ij}\} (\subseteq X)$ is said to be Wijsman convergent to U if for each $x \in X$,

$$\lim_{i,j \rightarrow \infty} \{d(x, U_{ij}) - d(x, U)\} = 0.$$

It is denoted by $[W_2] - \lim_{i,j \rightarrow \infty} U_{ij} = U$.

Example 1. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$\{U_{ij}\} = \{(x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = \frac{1}{ij}\}.$$

The double sequence of sets is Wijsman convergent to $U = \{(0,1)\}$.

Definition II.2. Let $(X; \rho)$ be a metric space and U be any non-empty closed subset of X . A double sequence of closed sets $\{U_{ij}\} (\subseteq X)$ is said to be Wijsman statistically convergent to U if for each $\epsilon > 0$ and each $x \in X$,

$$\lim_{m,n \rightarrow \infty} |\{(i,j) : i \leq m, j \leq n : |d(x, U_{ij}) - d(x, U)| \geq \epsilon\}| = 0.$$

It is denoted by $[WS] - \lim_{i,j \rightarrow \infty} U_{ij} = U$.

Definition II.3. A double sequence of closed sets is said to be bounded if for each $x \in X$,

$$\sup_{i,j} \{d(x, U_{ij})\} < \infty.$$

The set of all bounded sequence of sets is denoted by L_∞ .

Example 2. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$U_{ij} = \begin{cases} (x, y) \in \mathbb{R}^2, & (x-1)^2 + y^2 = \frac{1}{ij} \\ \text{if } i, j \text{ are square integer;} \\ \{(0,0)\}, & \text{otherwise;} \end{cases}$$

Then the double sequence $\{U_{ij}\}$ is bounded and Wijsman statistically convergent to the set $U = \{(0,0)\}$ but it is not Wijsman convergent.

Example 3. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$U_{ij} = \begin{cases} (x, y) \in \mathbb{R}^2, & (x-i)^2 + (y-j)^2 = \frac{1}{ij} \\ \text{if } i, j \text{ are square integer;} \\ \{(0,0)\}, & \text{otherwise;} \end{cases}$$

Then the double sequence $\{U_{ij}\}$ is Wijsman statistically convergent to the set $U = \{(0,0)\}$ but it is not Wijsman convergent.

If a double sequence of sets $\{U_{ij}\}$ is Wijsman statistically convergent to the set U , then $\{U_{ij}\}$ need not be Wijsman convergent. Also, it is not necessary be bounded.

Example 4. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$U_{ij} = \begin{cases} (x, y) \in \mathbb{R}^2, & (x-1)^2 + (y-1)^2 = ij \\ \text{if } i, j \text{ are square integer;} \\ \{(2,2)\}, & \text{otherwise;} \end{cases}$$

Then the double sequence $\{U_{ij}\}$ is Wijsman statistically convergent to the set $U = \{(2,2)\}$ but it is neither Wijsman convergent nor bounded.

Definition II.4. Let $(X; \rho)$ be a metric space and U be any non-empty closed subset of X . A double sequence of closed sets $\{U_{ij}\} (\subseteq X)$ is said to be Wijsman Cesaro summable to U if for each $x \in X$,

$$\lim_{p,q \rightarrow \infty} \frac{1}{pq} \sum_{i=1}^p \sum_{j=1}^q d(x, U_{ij}) = d(x, U)$$

Definition II.5. Let $(X; \rho)$ be a metric space and U be any non-empty closed subset of X . A double sequence of closed sets $\{U_{ij}\} (\subseteq X)$ is said to be Wijsman strongly Cesaro summable to U if for each $x \in X$,

$$\lim_{p,q \rightarrow \infty} \frac{1}{pq} \sum_{i=1}^p \sum_{j=1}^q |d(x, U_{ij}) - d(x, U)| = 0.$$

Definition II.6. Let $(X; \rho)$ be a metric space and U be any non-empty closed subset of X . A double sequence of closed sets $\{U_{ij}\} (\subseteq X)$ is said to be Wijsman strongly α -Cesaro summable to U if for each $x \in X$,

$$\lim_{p,q \rightarrow \infty} \frac{1}{pq} \sum_{i=1}^p \sum_{j=1}^q |d(x, U_{ij}) - d(x, U)|^\alpha = 0.$$

Example 5. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$U_{ij} = \begin{cases} (x, y) \in \mathbb{R}^2, & x^2 + (y-1)^2 = \frac{1}{ij} \\ & \text{if } i, j \text{ are square integer;} \\ & \{(1, 0)\}, \text{ otherwise;} \end{cases}$$

Then the double sequence $\{U_{ij}\}$ is Wijsman strongly Cesaro summable to the set $U = \{(1, 0)\}$.

If $\{U_{ij}\}$ is convergent but unbounded then it is Wijsman statistically convergent but it neither Wijsman Cesaro summable nor Wijsman strongly Cesaro summable.

Example 6. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$U_{ij} = \begin{cases} (x, y) \in \mathbb{R}^2, & (x-1)^2 + (y-1)^2 = i, \text{ if } j = 1, \text{ for all } i; \\ (x, y) \in \mathbb{R}^2, & (x-1)^2 + (y-1)^2 = j, \text{ if } i = 1, \text{ for all } j; \\ & \{(0, 0)\}, \text{ otherwise;} \end{cases}$$

Then the double sequence of sets $\{U_{ij}\}$ is Wijsman convergent to the set $U = \{(0, 0)\}$.

but the limit

$\lim_{p, q \rightarrow \infty} \frac{1}{pq} \sum_{i=1}^p \sum_{j=1}^q d(x, U_{ij})$ does not tend to a definite

limit. Therefore $\{U_{ij}\}$ is not Wijsman Cesaro-summable and hence is not Wijsman strongly Cesaro-summable although

$$\begin{aligned} & \lim_{p, q \rightarrow \infty} \frac{1}{pq} |\{i \leq p, j \leq q : |d(x, U_{ij}) - d(x, \{(0, 0)\})| \geq \epsilon\}| \\ &= \lim_{p, q \rightarrow \infty} \frac{p+q-1}{pq} = 0 \end{aligned}$$

so $\{U_{ij}\}$ is Wijsman statistically convergent to $\{(0, 0)\}$.

The notion of a modulus function was given by Nakano ([12]). Following Maddox ([15]), we recall that a modulus function f is a function from $[0, \infty]$ to $[0, \infty]$ such that

- (i) $f(x) = 0$ if and only if $x = 0$;
- (ii) $f(x+y) \leq f(x) + f(y)$;
- (iii) f is increasing;
- (iv) f is continuous from the right at 0.

A modulus function may be bounded or unbounded.

For example $f(x) = \frac{x}{x+1}$ is a bounded modulus function

where as $f(x) = x^p$ ($0 < p \leq 1$) is unbounded. From (ii) we easily get the inequality $f(nx) \leq n f(x)$ and so that $f(n) \leq n f(1)$ for every positive integer n and real $x \geq 0$.

Let $K (\subseteq \mathbb{N} \times \mathbb{N})$ be a two-dimensional set of positive integers, and let $K(m, n) = \{(j, k) : j \leq m, k \leq n\}$. Then, the two-dimensional analogue of natural density ([18], [22]) can be defined as follows: The lower asymptotic density of a set $K (\subseteq \mathbb{N} \times \mathbb{N})$ is defined as

$$\underline{\delta}_2(K) = \liminf_{m, n \rightarrow \infty} \frac{K(m, n)}{mn}. \text{ In this case, if the sequence } \left(\frac{K(m, n)}{mn}\right) \text{ has a limit in Pringsheim's sense, then we say that } K \text{ has a double natural density and is defined as } \lim_{m, n \rightarrow \infty} \frac{K(m, n)}{mn} = \delta_2(K).$$

Example 7: Let $K = \{(j^2 k^2 : j, k \in \mathbb{N})\}$. Then

$$\delta_2(K) = \lim_{m, n \rightarrow \infty} \frac{K(m, n)}{mn} \leq \lim_{m, n \rightarrow \infty} \frac{\sqrt{m}\sqrt{n}}{mn} = 0. \text{ Then } K \text{ has double natural density zero, while the set } \{(j, 2k) : j, k \in \mathbb{N}\} \text{ has double natural density } \frac{1}{2}.$$

Note: For any modulus f , $\lim_{h \rightarrow 0} \frac{f(h)}{h}$ exists and $\lim_{h \rightarrow 0} \frac{f(h)}{h} = \inf_{h \in (0, \infty)} \frac{f(h)}{h}$.

Definition II.7 For any unbounded modulus function f , the f -density of a subset U of $\mathbb{N} \times \mathbb{N}$ is defined by

$$\delta_f(U) = \lim_{m, n \rightarrow \infty} \frac{1}{f(m, n)} f(|\{(u_1, u_2) : u_1 \leq m, u_2 \leq n; (u_1, u_2) \in U\}|) \text{ if the limit exists.}$$

If we take $f(h) = h$, then the f -density becomes the natural density.

Remark II.8. In the case of natural density, it is obvious that for any $U \subseteq \mathbb{N} \times \mathbb{N}$, we have $\delta(U) + \delta(\mathbb{N} \times \mathbb{N} - U) = 1$.

But this case is different for f -density, i.e.,

$$\delta_f(U) + \delta_f(\mathbb{N} \times \mathbb{N} - U) = 1 \text{ does not hold good, in general.}$$

To verify this situation, we may take $U = \{(1, 2), (3, 4), (5, 6), \dots\}$ and the modulus function $f(h) = \log(h+1)$, then we have $\delta_f(\mathbb{N} \times \mathbb{N} - U) = 1 = \delta_f(U)$. But $\delta_f(U) = 0$ holds only for unbounded modulus function ([35]).

If we take a finite $U \subseteq \mathbb{N} \times \mathbb{N}$, then f -density and natural density coincides, that is, $\delta_f(U) = 0$ and so that $\delta_f(U) + \delta_f(\mathbb{N} \times \mathbb{N} - U) = 1$.

Again for any unbounded modulus function f and for any $U \subseteq \mathbb{N} \times \mathbb{N}$, $\delta_f(U) = 0$ implies $\delta(U) = 0$ ([1]). But the converse does not hold good. For example if we take $f(h) = \log(h+1)$ and the set $U = \{(u_1^2, u_2^2) : (u_1, u_2) \in \mathbb{N} \times \mathbb{N}\}$. Then $\delta(U) = 0$ but $\delta_f(U) = \frac{1}{2}$. Moreover, if $U \subseteq \mathbb{N} \times \mathbb{N}$ is finite and $\delta(U) = 0$, then $\delta_f(U) = 0$.

Our intention of this paper is to extend the notion of f -statistical convergence and strong f -Cesaro convergence from single sequences to double sequence of sets and discussed on different types of f -statistical convergence, which are defined by a density in $\mathbb{N} \times \mathbb{N}$ with the aid of modulus functions f and obtained some inclusion relations related to this concepts.

III. MAIN RESULTS

Every double limit we use will be considered in Pringsheim's sense ([2]) unless otherwise stated. The Wijsman f -strong Cesaro convergence for double sequences is defined as follows:

Given a double sequence of sets $\{U_{ij}\}$ and U , we denote

$$U_\epsilon(m, n) = \{(u_1, u_2) : u_1 \leq m, u_2 \leq n; |d(x, U_{ij}) - d(x, U)| \geq \epsilon\}.$$

Based on the above notion of density, we could define the Wijsman f -statistically convergence as follows:

Definition III.1. Let f be an unbounded modulus function. A double sequence of sets $\{U_{ij}\}$ of complex numbers is said to be Wijsman f statistically convergent to U (WS_f -convergent to U) if for any $\epsilon > 0$,

$$\lim_{m,n \rightarrow \infty} \frac{1}{f(mn)} f(|U_\epsilon(m, n)|) = 0,$$

Where the the vertical bar denotes the cardinality of the enclosed set.

In this case we write $WS_f - \lim_{i,j \rightarrow \infty} U_{ij} = U$ or $U_{ij} \rightarrow U(WS_f)$.

The class of all WS_f -convergent sequences of sets will be denoted by WS_f I.e.,

$$WS_f = \left\{ U = (U_{ij}) : \lim_{m,n \rightarrow \infty} \frac{1}{f(mn)} f(|U_\epsilon(m, n)|) = 0 \right\}$$

for all $\epsilon > 0$ and for some $U \in \mathbb{C}$.

Definition III.2 For a modulus function f , a double sequence of sets $\{U_{ij}\}$ of complex numbers is said to be Wijsman f -strongly Cesaro summable to $U \in \mathbb{C}$ (WC_f -summable to U) if,

$$\lim_{m,n \rightarrow \infty} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f(|d(U_{ij}, x) - d(U, x)|) = 0.$$

In this case we write $WC_f - \lim_{i,j \rightarrow \infty} U_{ij} = U$ or

$U_{ij} \rightarrow U(WC_f)$. The class of all WC_f -summable sequences of sets will be denoted by WC_f I.e.,

$$WC_f = \left\{ U = (U_{ij}) : \lim_{m,n \rightarrow \infty} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f(|d(U_{ij}, x) - d(U, x)|) = 0 \right\}$$

for some $U \in \mathbb{C}$. The set of all Wijsman strongly Cesaro summable sequences of sets is denoted by WC .

Note 1: This denition does not require the modulus function f to be unbounded.

Note 2: WS_f -convergence reduces to WS -convergence if $f(h)=h$ and Wijsman f -strongly Cesaro summability reduces to Wijsman strongly Cesaro summability if $f(h)=h$.

First we state two lemmas as follows:

Lemma III.3 For any modulus function f , $\lim_{h \rightarrow \infty} \frac{f(h)}{h}$ exist, and

$$\lim_{h \rightarrow \infty} \frac{f(h)}{h} = \inf_{h \in (0, \infty)} \frac{f(h)}{h}.$$

Lemma III.4 If the double sequence of sets $(U)_{ij} \in WS_f$, then its WS_f -limit is unique.

Theorem III.5 For any modulus function f and g , if

$$\sup_{h \in (0, \infty)} \frac{f(h)}{g(h)} < \infty, \text{ then } WC_g \subset WC_f.$$

Proof: Suppose $S = \sup_{h \in (0, \infty)} \frac{f(h)}{g(h)} < \infty$. Then we can write,

$\frac{f(h)}{g(h)} \leq S$ for every $h \in [0, \infty)$. It is apparent that $S > 0$ and if the double sequence of sets $(U)_{ij}$ is g -strongly Cesaro summable to U , we can write

$$\begin{aligned} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f(|d(U_{ij}, x) - d(U, x)|) \\ \leq \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n S \cdot g(|d(U_{ij}, x) - d(U, x)|) \end{aligned}$$

Taking limit as $m, n \rightarrow \infty$, we find that $(U)_{ij} \in WC_g$ implies that $(U)_{ij} \in WC_f$.

Remark III.6 The converse of the above theorem, does not have to be correct for every modulus functions f and g with

$$\sup_{h \in (0, \infty)} \frac{f(h)}{g(h)} < \infty, \text{ in general.}$$

The following example shows that there exist at least for certain specific modulus function for which the inclusion $WC_g \subset WC_f$ can be strict.

Example 8. Consider the double sequence of sets $\{U_{ij}\}$ be defined by

$$U_{ij} = \begin{cases} \{ij\}, & \text{if } i = m^3, j = n^3; \\ \{(0,1)\}, & \text{if } i \neq m^3, j \neq n^3; \end{cases}$$

where $m, n \in \mathbb{N}$ and take the modulus function $f(h) = \frac{h}{h+1}$ and $g(h) = h$.

In that case $\sup_{h \in (0, \infty)} \frac{f(h)}{g(h)} = 1 < \infty$ so that $WC_g \subset WC_f$.

Using the equality $f(0) = 0$, we have

$$\begin{aligned} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f(|d(U_{ij}, x)|) \\ = \frac{1}{mn} \sum_{i=1, i=m^3}^m \sum_{j=1, j=n^3}^n f(ij) \\ + \frac{1}{mn} \sum_{i=1, i \neq m^3}^m \sum_{j=1, j \neq n^3}^n f((0,0)) \\ = \frac{1}{mn} \sum_{i=1, i=m^3}^m \sum_{j=1, j=n^3}^n \frac{ij}{1+ij} \\ < \frac{1}{mn} \sum_{i=1, i=m^3}^m \sum_{j=1, j=n^3}^n 1 \\ \leq \frac{\sqrt[3]{mn}}{mn} \end{aligned}$$

Since $\frac{\sqrt[3]{mn}}{mn} \rightarrow 0$ as $m, n \rightarrow \infty$, we can write $\{U_{ij}\} \in WC_f$. However,

$$\begin{aligned}
 & \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n g(|d(U_{ij}, x)|) \\
 &= \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n g(d(U_{ij}, x)) \\
 &= \frac{1}{mn} \sum_{i=1, i \neq m^3}^m \sum_{j=1, j \neq n^3}^n g(ij) \\
 & \quad + \frac{1}{mn} \sum_{i=1, i \neq m^3}^m \sum_{j=1, j \neq n^3}^n g((0,0)) \\
 &> \frac{1}{mn} (1^3 + 4^3 + 9^3 + \dots + k^3); \max_{i,j \in \mathbb{N}} \leq mn.
 \end{aligned}$$

$= \frac{1}{mn} \sum_{k=1}^t k^6$ and this series is diverges to ∞ as $m, n \rightarrow \infty$ so that $\{U_{ij}\} \in WC_g$.

This implies that $\{U_{ij}\} \in WC_f - WC_g$ and the inclusion $WC_g \subset WC_f$ is strict.

Theorem III.7 For any two modulus function f and g , if $\inf_{h \in (0, \infty)} \frac{f(h)}{g(h)} > 0$, then $WC_f \subset WC_g$.

Proof: Suppose $I = \inf_{h \in (0, \infty)} \frac{f(h)}{g(h)} > 0$. Then, we can write $\frac{f(h)}{g(h)} \geq I$ for every $h \in [0, \infty)$. If the double sequence of sets (U_{ij}) is f -strongly Cesaro summable to U , we can write

$$\begin{aligned}
 & \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n g(|d(U_{ij}, x) - d(U, x)|) \\
 & \leq \frac{1}{I} \cdot \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n g(|d(U_{ij}, x) - d(U, x)|)
 \end{aligned}$$

Taking limits on both sides as $m, n \rightarrow \infty$, we find that $(U_{ij}) \in WC_f$ implies that $(U_{ij}) \in WC_g$.

Remark III.8 The converse of the above theorem does not have to be correct for every modulus functions f and g with $\inf_{h \in (0, \infty)} \frac{f(h)}{g(h)} > 0$, in general. To demonstrate this, consider the double sequence of sets in previous example and take the modulus function $g(h) = \frac{h}{h+1}$ and $f(h) = h$. Then $\inf_{h \in (0, \infty)} \frac{f(h)}{g(h)} > 0$ and hence $x \in WC_g$ but $x \notin WC_f$. This establish that at least for certain specific modulus function, the inclusion $WC_f \subset WC_g$ can be strict.

Combining the above two theorems we obtain a corollary as: Corollary III.9 For any two modulus function f and g ,

$$WC_f = WC_g \text{ holds if } \inf_{h \in (0, \infty)} \frac{f(h)}{g(h)} \leq \sup_{h \in (0, \infty)} \frac{f(h)}{g(h)} < \infty.$$

Relation between Wijsman strong Cesaro summability and Wijsman statistical convergence with respect modulus functions:

Theorem III.10 For any two modulus function f and g , if $\inf_{h \in (0, \infty)} \frac{f(h)}{g(h)} > 0$ and $\lim_{h \rightarrow \infty} \frac{g(h)}{h} > 0$, then $WC_f \subset WC_g$ I.e.,

every Wijsman f -strongly Cesaro summable double sequences of sets is Wijsman g -statistically convergent.

Proof: Suppose $I = \inf_{h \in (0, \infty)} \frac{f(h)}{g(h)} > 0$. Then, we can write

$\frac{f(h)}{g(h)} \geq I$ for every $h \in [0, \infty)$. If the double sequence of sets (U_{ij}) is f -strongly Cesaro summable to U , we can write

$$\begin{aligned}
 & \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f(|d(U_{ij}, x) - d(U, x)|) \\
 & \geq I \cdot \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n g(|d(U_{ij}, x) - d(U, x)|) \\
 & \geq I \cdot \frac{1}{mn} \sum_{i=1}^m \sum_{j=1, |d(U_{ij}, x) - d(U, x)| \geq \epsilon}^n g(|d(U_{ij}, x) - d(U, x)|) \\
 & \geq I \cdot \frac{1}{mn} \{i \leq m, j \leq n : |d(U_{ij}, x) - d(U, x)| \geq \epsilon\} \cdot g(\epsilon) \\
 & \geq I \cdot \frac{1}{mn} g(\{i \leq m, j \leq n : |d(U_{ij}, x) - d(U, x)| \geq \epsilon\}) \cdot \frac{g(\epsilon)}{g(1)} \\
 & = I \cdot \frac{g(\{i \leq m, j \leq n : |d(U_{ij}, x) - d(U, x)| \geq \epsilon\})}{\frac{g(mn)}{mn} \cdot \frac{g(\epsilon)}{g(1)}}
 \end{aligned}$$

Taking limits on both sides as $m, n \rightarrow \infty$, we find that $(U_{ij}) \in WC_f$ implies that $(U_{ij}) \in WS_g$ using the hypothesis

$$\lim_{h \rightarrow \infty} \frac{g(h)}{h} > 0.$$

Remark III.11 The converse of the above theorem does not have to be correct for every unbounded modulus functions f and g with $\inf_{h \in (0, \infty)} \frac{f(h)}{g(h)} > 0$ and $\lim_{h \rightarrow \infty} \frac{g(h)}{h} > 0$, in general. The following example can establishes that at least for certain specific unbounded modulus function, the inclusion $WC_f \subset WS_g$ can be strict.

Example 9. Consider the double sequence of sets in previous example and take the modulus functions $f(h) = g(h) = h$. Then, $\inf_{h \in (0, \infty)} \frac{f(h)}{g(h)} > 0$ and $\lim_{h \rightarrow \infty} \frac{g(h)}{h} > 0$.

Also we have,

$$\begin{aligned}
 & \frac{1}{g(mn)} g(\{i \leq m, j \leq n : |d(U_{ij}, x) - d(0, x)|\}) \\
 & \leq \frac{g(\sqrt[3]{mn})}{g(mn)}
 \end{aligned}$$

Taking limits on both sides as $m, n \rightarrow \infty$, we find that

$(U_{ij}) \in WS_g$. However it has shown in example. 8 that $x \notin WC_f$.

If we assume $f(h) = g(h)$ in Theorem III.10, then we have the outcome as a corollary:

Corollary III.12 For an unbounded modulus function f , if $\inf_{h \in (0, \infty)} \frac{f(h)}{g(h)} > 0$, then every Wijsman f -strongly Cesaro summable double sequence of sets is Wijsman f -statistically convergent.

If we assume $g(h) = h$ in Theorem III.10, then as well we have the outcome as a corollary:

Corollary III.13 For an unbounded modulus function f , if $\inf_{h \in (0, \infty)} \frac{f(h)}{g(h)} > 0$, then every Wijsman f -strongly Cesaro summable double sequence of sets is Wijsman statistically convergent.

If we assume $f(h) = h$ in Corollary III.13, then we have the similar outcome as a corollary:

Corollary 3.14. A Wijsman strongly Cesaro summable double sequence of sets is Wijsman statistically convergent.

Data availability: The data that support the findings of this study are available on request from the corresponding author. Conflicts of interest: The authors declare that they have no conflicts of interest.

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