



Relations Among Basic Hypergeometric Series Via Transformation Formulas

Dr. Jayprakash Yadav

Department of Mathematics & Statistics, Prahladrai Dalmia College of Commerce and Economics, Sundar Nagar, Malad (W), Mumbai-400064, INDIA.

ARTICLE INFO	ABSTRACT
Published Online: 16 April 2026	Basic hypergeometric series, or q -series, play a fundamental role in many areas of mathematics, including combinatorics, number theory, and mathematical physics. The study of their transformation formulas provides deep insights into the interrelations among various classes of these series. This paper investigates several transformation formulas that establish connections between different forms of basic hypergeometric series. By employing classical results on transformations, new relations and identities among terminating and non-terminating q -series are derived. The approach unifies several known results and extends them to more generalized parameter domains. The obtained transformations not only simplify the evaluation of certain q -series but also reveal structural symmetries underlying their behavior. These findings contribute to the broader theory of basic hypergeometric functions and open avenues for applications in partition theory, special functions, and other analytical studies involving q -analogues.
Corresponding Author: Dr. Jayprakash Yadav	
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1. INTRODUCTION

The theory of **basic hypergeometric series**, also known as q -hypergeometric series, forms a significant branch of special function theory and q -analysis. These series arise naturally as q -analogues of the classical hypergeometric series and play a vital role in various areas of mathematics, including combinatorics, number theory, orthogonal polynomials, and mathematical physics. The fundamental idea behind these series is the replacement of ordinary ratios of successive terms by q -ratios, leading to a rich structure governed by the properties of q -shifted factorials.

Transformation formulas are central to the study of hypergeometric and basic hypergeometric series. They provide relationships between series of different parameters and arguments, thereby simplifying expressions and uncovering hidden connections among various series. Classical results such as **Heine's transformation**, **Watson's transformation**, **Bailey's transformations**, and **Sears' transformation** have long served as essential tools in deriving identities and establishing equivalences between multiple q -series. These transformations often unify apparently distinct forms of series under a common

framework and reveal deep algebraic and analytical properties.

The exploration of relations among basic hypergeometric series through transformation formulas not only consolidates known results but also leads to new identities and generalizations. Such relations are of particular importance in the theory of special functions and their applications to mathematical physics, partition theory, and combinatorial analysis. Moreover, these transformations frequently serve as generating mechanisms for summation theorems and bilateral series identities.

In this study, attention is focused on establishing and analysing **relations among various forms of basic hypergeometric series using transformation formulas**. By systematically applying classical and generalized transformations, we aim to demonstrate how diverse series are interconnected and how these relationships can simplify computations. The results presented herein contribute to the ongoing development of q -series theory and provide a foundation for further investigations into their applications in both pure and applied mathematics.

The generalized basic hyper geometric series is defined by

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q, Z \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1, q)_n (a_2, q)_n \dots (a_r, q)_n}{(b_1, q)_n (b_2, q)_n \dots (b_s, q)_n} [(-1)^n q^{n(n-1)/2}]^{1+s-r} Z^n \quad (1)$$

where r and s are positive integers, $q \neq 0$ when $r > s + 1$, the numerator parameters $a_1; \dots; a_r$ and the denominator parameters $b_1; \dots; b_s$ being complex quantities provided that $b_j \neq q^{-m}; m = 0; 1; \dots; j = 1; 2; \dots; s$

For real or complex $a, q < 1$, the q -shifted factorial is defined by

$$(a, q)_n = \begin{cases} 1 & \text{if } n = 0; \\ (1-a)(1-aq)(1-aq^2) \dots (1-aq^{n-1}) & \text{if } n \in \mathbb{N}. \end{cases} \quad (2)$$

In 1971 Verma established the following q -series identity:

$$\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{(q, \gamma q^n; q)_n} \sum_{k=0}^{\infty} \frac{(\alpha, \beta; q)_{n+k}}{(q, \gamma q^{2n+1}; q)_k} B_{n+k} z^k \sum_{j=0}^{\infty} \frac{(q^{-n}, \gamma q^n; q)_j}{(q, \alpha, \beta; q)_j} A_j (wq)^j = \sum_{n=0}^{\infty} A_n B_n \frac{(zw)^n}{(q; q)_n} \quad (3)$$

The multi-basic hypergeometric series with bases $q_1, q_2, \dots, q_m$ is defined as

$$\begin{aligned} & \phi \left[\begin{matrix} a_1, a_2, \dots, a_r; c_{1,1}, \dots, c_{1,r_1}; \dots; c_{m,1}, \dots, c_{1,r_m}; q; q_1, q_2, \dots, q_m; Z \end{matrix} \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1, q)_n (a_2, q)_n \dots (a_r, q)_n}{(b_1, q)_n (b_2, q)_n \dots (b_s, q)_n} Z^n \prod_{j=0}^m \frac{(c_{j,1}, c_{j,2}, \dots, c_{j,r_j}; q_j)_j}{(d_{j,1}, d_{j,2}, \dots, c_{j,s_j}; q_j)_j} \end{aligned} \quad (4)$$

where $(a_1, a_2, \dots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_r; q)_n$.

We shall use the following summation formulae due to Verma and Jain [8]

$${}_4\phi_3 \left[\begin{matrix} b^2 x^4 q^{2+2n}, x^2, x^2 q, q^{-2n}, q^2; q^2 \end{matrix} \right] = \frac{x^{2n} (-q, bq; q)_n}{(-x^2 q, bx^2 q; q)_n} \quad (5)$$

$${}_5\phi_4 \left[\begin{matrix} a, aq, aq^2, a^3 q^{3+3n}, q^{-3n}, q^3; q^3 \end{matrix} \right] = \frac{a^n (q^3, q^3)_n (aq; q)_n}{(a^3 q^3; q^3)_n (q; q)_n} \quad (6)$$

$${}_6\phi_5 \left[\begin{matrix} a^{1/3}, a^{1/3} q, a^{1/3} q^2, q\sqrt{a}, aq^{n+1}, q^{-n}, q; q \end{matrix} \right] = \frac{a^{(n-m)/2} (q, \sqrt{a}; q)_n (aq^3, q^6 \sqrt{a}; q^3)_m}{(aq, q^2 \sqrt{a}; q^3)_n (q^3, \sqrt{a}; q^3)_m} \quad (7)$$

2 Main Results

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\alpha, \beta, q^{3/2} \sqrt{a}, -q^{3/2} \sqrt{a}; q)_n (a^{1/3} q, a^{1/3} q w^2; q)_m (-z)^n (\sqrt{a})^{n-m} q^{n(n-1)/2}}{(\sqrt{aq}, -\sqrt{aq}, \frac{aq^2}{\alpha}, \frac{aq^2}{\beta}; q)_n (q, qw, qw^2; q)_m} \\ &= \prod \left[\begin{matrix} aq^2, \frac{aq^2}{\alpha\beta}, q \\ \frac{aq^2}{\alpha}, \frac{aq^2}{\beta} \end{matrix} \right] {}_5\phi_4 \left[\begin{matrix} \alpha, \beta, a^{1/3}, a^{1/3} w, a^{1/3} w^2; q; \frac{aq^2}{\alpha\beta} \\ q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix} \right] \end{aligned} \quad (1)$$

$$\sum_{n=0}^{\infty} \frac{(\alpha, \beta, q^{3/2}\sqrt{a}, -q^{3/2}\sqrt{a}, \sqrt{a}; q)_n (a^{1/3}q, a^{1/3}qw, a^{1/3}qw^2, a^{1/6}q^2w, a^{1/6}q^2w^2; q)_m}{(\sqrt{aq}, -\sqrt{aq}, \frac{aq^2}{\alpha}, \frac{aq^2}{\beta}, q^2\sqrt{a}; q)_n (q, qw, qw^2, a^{1/6}q^{2/3}, a^{1/6}q^{2/3}w, a^{1/6}q^{2/3}w^2; q)_m} (-z)^n (\sqrt{a})^{n-m} q^{n(n-1)/2}$$

$$= \prod \left[\begin{matrix} aq^2, \frac{aq^2}{\alpha\beta}, q \\ \frac{aq^2}{\alpha}, \frac{aq^2}{\beta} \end{matrix} \right] {}_6\phi_5 \left[\begin{matrix} \alpha, \beta, a^{1/3}, a^{1/3}w, a^{1/3}w^2, q\sqrt{a}; q; \frac{aq^2}{\alpha\beta} \\ q^2\sqrt{a}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix} \right] \quad (2)$$

Proof of (1):

Choosing $A_j = \frac{(\alpha, \beta, a^{1/3}, a^{1/3}w, a^{1/3}w^2; q)_j}{(q\sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_j}$, $B_n = 1$, $w = 1$, $z = \frac{aq^2}{\alpha\beta}$ and $\gamma = aq$ in equation (3) and making use of equation (5) and Gauss Summation Formula [Slater [7]; App.iv (iv.2)] in order to sum the inner series on the left-hand side of (3), we get (1).

Proof of (2):

Choosing $A_j = \frac{(\alpha, \beta, a^{1/3}, a^{1/3}w, a^{1/3}w^2, q\sqrt{a}; q)_j}{(q^2\sqrt{a}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_j}$, $B_n = 1$, $w = 1$, $z = \frac{aq^2}{\alpha\beta}$ and $\gamma = aq$ in equation (3) and making use of equation (5) and Gauss Summation Formula [Slater [7]; App.iv (iv.2)] in order to sum the inner series on the left hand side of (3), we get (2).

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