

SOME PROPERTIES AND INEQUALITIES FOR A TWO-PARAMETER EXTENSION OF THE LOWER INCOMPLETE EXPONENTIAL INTEGRAL FUNCTION

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Abstract. In this paper, we introduce an extension of the Incomplete Exponential Integral function. Where $E_{n,p}(x, v)$ and $\epsilon_{n,p}(x, v)$ are the p -analogues of the upper and lower incomplete exponential integral functions respectively. We also establish some properties and inequalities generalizing those satisfied by the p -analogue of the lower Incomplete Exponential Integral function. We employ the Holder's inequality for integrals and the Minkowski's inequality for integral .

Keywords: p -analogue of the lower incomplete exponential integral function, Holder's inequality, Minkowski's inequality.

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1. INTRODUCTION

The exponential integral function is defined by as [1]

$$(1) \quad E_n(x) = \int_1^\infty t^{-n} e^{-tx} dt \quad x > 0, \quad n \in \mathbb{N}.$$

The r -th derivative of (1), r is integer, denoted by $E_n^{(r)}(x)$ is given by [1]

$$(2) \quad E_n^{(r)}(x) = (-1)^{(r)} \int_1^\infty t^{r-n} e^{-tx} dt.$$

Then, the p -analogue of the exponential integral function, $E_{n,p}(x)$ is defined for $x > 0$, $p > 1$ and $n \in \mathbb{N}_0$ by [2]

$$(3) \quad E_{n,p}(x) = \int_1^p t^{-n} A_p^{-xt} dt,$$

and the i -th derivative of (3) is given by [2]

$$(4) \quad E_{n,p}^{(i)}(x) = (\ln A_p^{-1})^i \int_1^p t^{i-n} A_p^{-xt} dt,$$

where, $E_{n,p}(x) \rightarrow E_n(x)$ as $p \rightarrow \infty$, $A_p = (1 + \frac{1}{p})^p$, $E_{n,p}^{(i)}(x) \rightarrow E_n^{(i)}(x)$ as $p \rightarrow \infty$.

Let $x > 0$, $v \geq 1$, $p \in \mathbb{R}^+$, and $n \in \mathbb{N}_0$.

Also, the p -analogue of the upper incomplete exponential integral function is defined as [3]

$$(5) \quad E_{n,p}(x, v) = \int_v^P t^{-n} A_p^{-xt} dt,$$

and the i -th derivative of (5) is given by

$$(6) \quad E_{n,p}^{(i)}(x, v) = (\ln A_p^{-1})^i \int_v^P t^{i-n} A_p^{-xt} dt,$$

where, $E_{n,p}(x, 1) = E_{n,p}(x)$, $E_{n,p}(x, v) \rightarrow E_n(x, v)$ as $p \rightarrow \infty$, $E_{1,p}(1, v) \rightarrow \Gamma(0, v)$ as $p \rightarrow \infty$, $E_{n,p}(x) \rightarrow x^{n-1} \Gamma(1-n, x)$ as $p \rightarrow \infty$ and $E_{n,p}^{(0)}(x, v) \equiv E_{n,p}(x, v)$.

For more properties of this special function, one may refer to [4], [5] and [6]. It is often applied in astrophysics, neutron physics, quantum chemistry as well as other applied sciences. Due to its importance, it has been studied in diverse ways. See for instance [7], [8], [9], [10], [11], [12], [13], [14] and [15]

In the present work, we continue the investigation. Specifically, our objective is to establish the p -analogue of the lower incomplete exponential integral function.

2. PRELIMINARIES

We begin with the following well known results (see for instance [16], [17], [18], [19] or [20]).

Lemma 2.1. (Holder's Inequality) Let $\eta, \mu > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$. If $f(t)$ and $g(t)$ are continuous real-valued functions on $[a, b]$, then inequality

$$(7) \quad \int_a^b |f(t)g(t)| dt \leq \left(\int_a^b |f(t)|^\eta dt \right)^{\frac{1}{\eta}} \left(\int_a^b |g(t)|^\mu dt \right)^{\frac{1}{\mu}},$$

holds. With equality when $|g(t)| = c|f(t)|^{\eta-1}$. If $\eta = \mu = 2$, the inequality becomes Schwarz's Inequality.

Lemma 2.2. (Minkowski's Inequality) Let $\eta > 1$. If $f(t)$ and $g(t)$ are continuous real-valued functions on $[a, b]$, then inequality

$$(8) \quad \left(\int_a^b |f(x) + g(x)|^\eta dx \right)^{\frac{1}{\eta}} \leq \left(\int_a^b |f(x)|^\eta dx \right)^{\frac{1}{\eta}} + \left(\int_a^b |g(x)|^\eta dx \right)^{\frac{1}{\eta}},$$

holds.

Lemma 2.3. (Young's Algebraic Inequality) Let $a, b > 0$, $\eta, \mu > 1$, and $\frac{1}{\eta} + \frac{1}{\mu} = 1$. Then inequality

$$(9) \quad ab \leq \frac{a^\eta}{\eta} + \frac{b^\mu}{\mu},$$

holds.

3. DEFINITION OF THE p -ANALOGUE OF THE LOWER INCOMPLETE EXPONENTIAL INTEGRAL

Definition 3.1. Let $x > 0$, $v \geq 1$, $p \in \mathbb{R}^+$, and $n \in \mathbb{N}_0$. Then, the p -analogue of the lower incomplete exponential integrals is defined as

$$(10) \quad \varepsilon_{n,p}(x, v) = \int_1^v t^{-n} A_p^{-xt} dt,$$

where, $\varepsilon_{n,p}(x, v) \longrightarrow \varepsilon_n(x, v)$ as $p \longrightarrow \infty$, $v \longrightarrow \infty$, $\varepsilon_{0,p}(1, v) = A_p^{-v} - A_p^{-1}$.

3.1. Some Properties and Inequalities of $\varepsilon_{n,p}(x, v)$.

Lemma 3.2. The function $\varepsilon_{n,p}(x, v)$ is decreasing for $x > 0$.

Proof. The following differential equation hold for (10)

$$\begin{aligned} \varepsilon_{n,p}^{(1)}(x, v) &= \ln A_p^{-1} \int_1^v t^{1-n} A_p^{-xt} dt, \\ &= -\ln A_p \int_1^v t^{1-n} A_p^{-xt} dt \leq 0. \end{aligned}$$

Lemma 3.3. The recursive relation

$$(11) \quad \varepsilon_{n,p}(x, v) = \ln A_p^{-x} [A_p^{-x} - v^{-n} A_p^{-vx} - n \varepsilon_{n+1,p}(x, v)],$$

holds for $n \in \mathbb{N}_0$.

Proof. Using (10) and by means of integration by parts

$$\begin{aligned}
 \varepsilon_{n,p}(x, v) &= \int_1^v t^{-n} A_p^{-xt} dt \\
 &= \left[-\frac{t^{-n} A_p^{-xt}}{\ln A_p^x} \right]_1^v - \frac{n}{\ln A_p^x} \int_1^v t^{-(n+1)} A_p^{-xt} dt \\
 &= -\frac{v^{-n} A_p^{-vx}}{\ln A_p^x} + \frac{A_p^{-x}}{\ln A_p^x} - \frac{n}{\ln A_p^x} \varepsilon_{n+1,p}(x, v) \\
 &= \frac{A_p^{-x}}{\ln A_p^x} - \frac{v^{-n} A_p^{-vx}}{\ln A_p^x} - \frac{n}{\ln A_p^x} \varepsilon_{n+1,p}(x, v) \\
 &= \frac{1}{\ln A_p^x} [A_p^{-x} - v^{-n} A_p^{-vx} - n \varepsilon_{n+1,p}(x, v)] \\
 &= \ln A_p^{-x} [A_p^{-x} - v^{-n} A_p^{-vx} - n \varepsilon_{n+1,p}(x, v)],
 \end{aligned}$$

which concludes the proof.

Theorem 3.4. Let $n \in \mathbb{N}_0$, $\eta > 1$, $p \in \mathbb{R}^+$. Then, the inequality

$$(12) \quad \varepsilon_{n,p}\left(\frac{x}{\eta} + \frac{y}{\mu}, v\right) \leq (\varepsilon_{n,p}(x, v))^{\frac{1}{\eta}} (\varepsilon_{n,p}(y, v))^{\frac{1}{\mu}},$$

holds for $x, y > 0$, $v \geq 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$.

Proof. Using (10) and Hölder's inequality for integrals, we have

$$\begin{aligned}
 \varepsilon_{n,p}\left(\frac{x}{\eta} + \frac{y}{\mu}, v\right) &= \int_1^v t^{-n} A_p^{-\left(\frac{x}{\eta} + \frac{y}{\mu}\right)t} dt \\
 &= \int_1^v t^{-n\left(\frac{1}{\eta} + \frac{1}{\mu}\right)} A_p^{-\left(\frac{x}{\eta} + \frac{y}{\mu}\right)t} dt \\
 &= \int_1^v t^{-\frac{n}{\eta}} A_p^{-\frac{xt}{\eta}} t^{-\frac{n}{\mu}} A_p^{-\frac{yt}{\mu}} dt \\
 &\leq \left(\int_1^v \left(t^{-\frac{n}{\eta}} A_p^{-\frac{xt}{\eta}} \right)^{\eta} dt \right)^{\frac{1}{\eta}} \left(\int_1^v \left(t^{-\frac{n}{\mu}} A_p^{-\frac{yt}{\mu}} \right)^{\mu} dt \right)^{\frac{1}{\mu}} \\
 &= \left(\int_1^v t^{-n} A_p^{-xt} dt \right)^{\frac{1}{\eta}} \left(\int_1^v t^{-n} A_p^{-yt} dt \right)^{\frac{1}{\mu}} \\
 &= (\varepsilon_{n,p}(x, v))^{\frac{1}{\eta}} (\varepsilon_{n,p}(y, v))^{\frac{1}{\mu}}.
 \end{aligned}$$

Theorem 3.5. Let $p \in \mathbb{R}^+$ and $m, n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0$.

Then, the inequality

$$(13) \quad \epsilon_{m+n,p} \left(\frac{x}{\eta} + \frac{y}{\mu}, v \right) \leq (\epsilon_{\eta m,p}(x, v))^{\frac{1}{\eta}} (\epsilon_{\mu n,p}(y, v))^{\frac{1}{\mu}},$$

holds for $x, y > 0$, $v \geq 1$, $\eta > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$,

Proof. Using (10) and Hölder's inequality for integrals, we have

$$\begin{aligned} \epsilon_{m+n,p} \left(\frac{x}{\eta} + \frac{y}{\mu}, v \right) &= \int_1^v t^{-(m+n)} A_p^{-\left(\frac{x}{\eta} + \frac{y}{\mu}\right)t} dt \\ &= \int_1^v t^{-m} A_p^{-\frac{xt}{\eta}} t^{-n} A_p^{-\frac{yt}{\mu}} dt \\ &\leq \left(\int_1^v \left(t^{-m} A_p^{-\frac{xt}{\eta}} \right)^{\eta} dt \right)^{\frac{1}{\eta}} \left(\int_1^v \left(t^{-n} A_p^{-\frac{yt}{\mu}} \right)^{\mu} dt \right)^{\frac{1}{\mu}} \\ &= \left(\int_1^v t^{-\eta m} A_p^{-xt} dt \right)^{\frac{1}{\eta}} \left(\int_1^v t^{-\mu n} A_p^{-yt} dt \right)^{\frac{1}{\mu}} \\ &= (\epsilon_{\eta m,p}(x, v))^{\frac{1}{\eta}} (\epsilon_{\mu n,p}(y, v))^{\frac{1}{\mu}}. \end{aligned}$$

Corollary 3.6. Let $m, n \in \mathbb{N}_0$, $p \in \mathbb{R}^+$. Then, the inequality

$$(14) \quad \left(\epsilon_{m+n,p} \left(\frac{x+y}{2}, v \right) \right)^2 \leq \epsilon_{2m,p}(x, v) \epsilon_{2n,p}(y, v),$$

holds for $x, y > 0$ and $v \geq 1$.

Proof. This follows from Theorem 3.5 by letting $\eta = \mu = 2$.

Theorem 3.7. Let $p \in \mathbb{R}^+$, $m, n \in \mathbb{N}_0$ such that $\frac{m}{\eta} + \frac{n}{\mu} \in \mathbb{N}_0$.

Then, the inequality

$$(15) \quad \epsilon_{\frac{m}{\eta} + \frac{n}{\mu}, p} \left(\frac{x}{\eta} + \frac{y}{\mu}, v \right) \leq (\epsilon_{m,p}(x, v))^{\frac{1}{\eta}} (\epsilon_{n,p}(y, v))^{\frac{1}{\mu}},$$

holds for $\eta > 1$, $x, y > 0$, $v \geq 1$, $\frac{1}{\eta} + \frac{1}{\mu} = 1$.

Proof. Using (10) and Hölder's inequality for integrals, we have

$$\begin{aligned}
 \varepsilon_{\frac{m}{\eta} + \frac{n}{\mu}, p} \left(\frac{x}{\eta} + \frac{y}{\mu}, v \right) &= \int_1^v t^{-\left(\frac{m}{\eta} + \frac{n}{\mu}\right)} A_p^{-\left(\frac{x}{\eta} + \frac{y}{\mu}\right)t} dt \\
 &= \int_1^v t^{-\frac{m}{\eta}} A_p^{-\frac{xt}{\eta}} t^{-\frac{n}{\mu}} A_p^{-\frac{yt}{\mu}} dt \\
 &\leq \left(\int_1^v \left(t^{-\frac{m}{\eta}} A_p^{-\frac{xt}{\eta}} \right)^{\eta} dt \right)^{\frac{1}{\eta}} \left(\int_1^v \left(t^{-\frac{n}{\mu}} A_p^{-\frac{yt}{\mu}} \right)^{\mu} dt \right)^{\frac{1}{\mu}} \\
 &= \left(\int_1^v t^{-m} A_p^{-xt} dt \right)^{\frac{1}{\eta}} \left(\int_1^v t^{-n} A_p^{-yt} dt \right)^{\frac{1}{\mu}} \\
 &= (\varepsilon_{m,p}(x, v))^{\frac{1}{\eta}} (\varepsilon_{n,p}(y, v))^{\frac{1}{\mu}}.
 \end{aligned}$$

Corollary 3.8. Let $m, n \in \mathbb{N}_0$, $p \in \mathbb{R}^+$. Then, the inequality

$$(16) \quad \left(\varepsilon_{\frac{m+n}{2}, p} \left(\frac{x+y}{2}, v \right) \right)^2 = \varepsilon_{m,p}(x, v) \varepsilon_{n,p}(y, v),$$

holds for $x, y > 0$ and $v \geq 1$.

Proof. This follows from Theorem 3.7 by letting $\eta = \mu = 2$.

Theorem 3.9. Let $m, n \in \mathbb{N}_0$, $\alpha \in \mathbb{Z}^+$, and $p \in \mathbb{R}^+$.

Then, the inequality

$$(17) \quad [\varepsilon_{m,p}(x, v) + \varepsilon_{n,p}(y, v)]^{\frac{1}{\alpha}} \leq [\varepsilon_{m,p}(x, v)]^{\frac{1}{\alpha}} + [\varepsilon_{n,p}(y, v)]^{\frac{1}{\alpha}},$$

holds for $x, y > 0$ and $v \geq 1$

Proof. Using (10) and the Minkowski's inequality for integrals and letting $a^\alpha + b^\alpha \leq (a+b)^\alpha$, for $a, b \geq 0$ and $\alpha \in \mathbb{Z}^+$. we have

$$\begin{aligned} [\varepsilon_{m,p}(x, v) + \varepsilon_{n,p}(y, v)]^{\frac{1}{\alpha}} &= \left[\int_1^v t^{-m} A_p^{-xt} dt + \int_1^v t^{-n} A_p^{-yt} dt \right]^{\frac{1}{\alpha}} \\ &= \left[\int_1^v \left(\left(t^{-\frac{m}{\alpha}} A_p^{-\frac{xt}{\alpha}} \right)^\alpha + \left(t^{-\frac{n}{\alpha}} A_p^{-\frac{yt}{\alpha}} \right)^\alpha \right) dt \right]^{\frac{1}{\alpha}} \\ &\leq \left[\int_1^v \left(\left(t^{-\frac{m}{\alpha}} A_p^{-\frac{xt}{\alpha}} \right) + \left(t^{-\frac{n}{\alpha}} A_p^{-\frac{yt}{\alpha}} \right) \right)^\alpha dt \right]^{\frac{1}{\alpha}} \\ &\leq \left[\int_1^v \left[t^{-\frac{m}{\alpha}} A_p^{-\frac{xt}{\alpha}} \right]^\alpha dt \right]^{\frac{1}{\alpha}} + \left[\int_1^v \left[t^{-\frac{n}{\alpha}} A_p^{-\frac{yt}{\alpha}} \right]^\alpha dt \right]^{\frac{1}{\alpha}} \\ &= \left[\int_1^v t^{-m} A_p^{-xt} dt \right]^{\frac{1}{\alpha}} + \left[\int_1^v t^{-n} A_p^{-yt} dt \right]^{\frac{1}{\alpha}} \\ &= [\varepsilon_{m,p}(x, v)]^{\frac{1}{\alpha}} + [\varepsilon_{n,p}(y, v)]^{\frac{1}{\alpha}}. \end{aligned}$$

3.2. Some Properties and Inequalities of $\varepsilon_{n,p}^{(i)}(x, v)$.

Lemma 3.10. Let $x > 0$, $p \in \mathbb{R}^+$, $n \in \mathbb{N}_0$, $i \in \mathbb{N}$ such that $i > n$. Then, the i -th derivative of (10) is given by

$$(18) \quad \varepsilon_{n,p}^{(i)}(x, v) = (\ln A_p^{-1})^i \int_1^v t^{i-n} A_p^{-xt} dt,$$

where, $\varepsilon_{n,p}^{(0)}(x, v) \equiv \varepsilon_{n,p}(x, v)$, $\varepsilon_{n,p}^{(i)}(x, v) \longrightarrow E_n^{(i)}(x)$ as $p \longrightarrow \infty$ and $v \longrightarrow \infty$.

Proof. The following differential equations hold for (10)

$$\begin{aligned}\varepsilon_{n,p}^{(1)}(x, v) &= \int_1^v t^{-n} - pt \ln \left(1 + \frac{1}{p} \right) A_p^{-xt} dt \\ &= \ln A_p^{-1} \int_1^v t^{1-n} A_p^{-xt} dt, \\ \varepsilon_{n,p}^{(2)}(x, v) &= \int_1^v t^{1-n} - pt \ln \left(1 + \frac{1}{p} \right) A_p^{-xt} dt \\ &= (\ln A_p^{-1})^2 \int_1^v t^{2-n} A_p^{-xt} dt, \\ \varepsilon_{n,p}^{(3)}(x, v) &= \int_1^v t^{2-n} - pt \ln \left(1 + \frac{1}{p} \right) A_p^{-xt} dt \\ &= (\ln A_p^{-1})^3 \int_1^v t^{3-n} A_p^{-xt} dt, \\ \varepsilon_{n,p}^{(4)}(x, v) &= \int_1^v t^{3-n} - pt \ln \left(1 + \frac{1}{p} \right) A_p^{-xt} dt \\ &= (\ln A_p^{-1})^4 \int_1^v t^{4-n} A_p^{-xt} dt,\end{aligned}$$

and more generally,

$$\varepsilon_{n,p}^{(i)}(x, v) = (\ln A_p^{-1})^i \int_1^v t^{i-n} A_p^{-xt} dt.$$

Theorem 3.11. Let $n \in \mathbb{N}_0$ and $i \in \mathbb{N}$. Then, the inequality,

$$(19) \quad \left| \varepsilon_{n,p}^{(i)}(xy, v) \right| \leq \left| \varepsilon_{n,p}^{(i)}(\eta x, v) \right|^{\frac{1}{\eta}} \left| \varepsilon_{n,p}^{(i)}(\mu y, v) \right|^{\frac{1}{\mu}},$$

holds for $x, y > 0$, $v \geq 1$, $\eta > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$.

Proof. Using (18) and the Holder's inequality for integrals, we have

$$\begin{aligned}
 | \epsilon_{n,p}^{(i)}(xy, v) | &\leq | \epsilon_{n,p}^{(i)}(x+y, v) | \\
 &= (\ln A_p^{-1})^i \int_1^v t^{i-n} A_p^{-(x+y)t} dt \\
 &= (\ln A_p^{-1})^{i\left(\frac{1}{\eta} + \frac{1}{\mu}\right)} \int_1^v t^{i\left(\frac{1}{\eta} + \frac{1}{\mu}\right) - n\left(\frac{1}{\eta} + \frac{1}{\mu}\right)} A_p^{-(x+y)t} dt \\
 &\leq \left[\left((\ln A_p^{-1})^{\frac{i}{\eta}} \int_1^v t^{\frac{i}{\eta} - \frac{n}{\eta}} A_p^{-xt} dt \right)^{\eta} \right]^{\frac{1}{\eta}} \\
 &\quad \times \left[\left((\ln A_p^{-1})^{\frac{i}{\mu}} \int_1^v t^{\frac{i}{\mu} - \frac{n}{\mu}} A_p^{-yt} dt \right)^{\mu} \right]^{\frac{1}{\mu}} \\
 &= \left[(\ln A_p^{-1})^i \int_1^v t^{i-n} A_p^{-\eta xt} dt \right]^{\frac{1}{\eta}} \\
 &\quad \times \left[(\ln A_p^{-1})^i \int_1^v t^{i-n} A_p^{-\mu yt} dt \right]^{\frac{1}{\mu}} \\
 &= | \epsilon_{n,p}^{(i)}(\eta x, v) |^{\frac{1}{\eta}} | \epsilon_{n,p}^{(i)}(\mu y, v) |^{\frac{1}{\mu}} .
 \end{aligned}$$

Theorem 3.12. Let $p \in \mathbb{R}^+$ and $m, n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0$.

Then, the inequality

$$(20) \quad | \epsilon_{m+n,p}^{(i)}\left(\frac{x}{\eta} + \frac{y}{\mu}, v\right) | \leq | \epsilon_{\eta m,p}^{(i)}(x, v) |^{\frac{1}{\eta}} | \epsilon_{\mu n,p}^{(i)}(y, v) |^{\frac{1}{\mu}},$$

holds for $x, y > 0$, $v \geq 1$, $\frac{1}{\eta} + \frac{1}{\mu} = 1$, and $\eta > 1$.

Proof. Using (18) and the Holder's inequality for integrals, we have

$$\begin{aligned}
 & \left| \mathcal{E}_{m+n,p}^{(i)} \left(\frac{x}{\eta} + \frac{y}{\mu}, v \right) \right| \\
 &= (\ln A_p^{-1})^i \int_1^v t^{i-(m+n)} A_p^{-\left(\frac{x}{\eta} + \frac{y}{\mu}\right)t} dt \\
 &= (\ln A_p^{-1})^{i\left(\frac{1}{\eta} + \frac{1}{\mu}\right)} \int_1^v t^{i\left(\frac{1}{\eta} + \frac{1}{\mu}\right) - (m+n)} A_p^{-\left(\frac{x}{\eta} + \frac{y}{\mu}\right)t} dt \\
 &= (\ln A_p^{-1})^{i\left(\frac{1}{\eta} + \frac{1}{\mu}\right)} \int_1^v t^{\frac{i}{\eta} - m} A_p^{-\frac{xt}{\eta}} t^{\frac{i}{\mu} - n} A_p^{-\frac{yt}{\mu}} dt \\
 &\leq \left[\left((\ln A_p^{-1})^{\frac{i}{\eta}} \int_1^v t^{\frac{i}{\eta} - m} A_p^{-\frac{xt}{\eta}} dt \right)^{\eta} \right]^{\frac{1}{\eta}} \\
 &\quad \times \left[\left((\ln A_p^{-1})^{\frac{i}{\mu}} \int_1^v t^{\frac{i}{\mu} - n} A_p^{-\frac{yt}{\mu}} dt \right)^{\mu} \right]^{\frac{1}{\mu}} \\
 &= \left[(\ln A_p^{-1})^i \int_1^v t^{i-\eta m} A_p^{-xt} dt \right]^{\frac{1}{\eta}} \\
 &\quad \times \left[(\ln A_p^{-1})^i \int_1^v t^{i-\mu n} A_p^{-yt} dt \right]^{\frac{1}{\mu}} \\
 &= \left| \mathcal{E}_{\eta m,p}^{(i)}(x, v) \right|^{\frac{1}{\eta}} \left| \mathcal{E}_{\mu n,p}^{(i)}(y, v) \right|^{\frac{1}{\mu}}.
 \end{aligned}$$

Corollary 3.13. Let $m, n \in \mathbb{N}_0$, $p \in \mathbb{R}^+$ and i is an even integer such that $i > m + n$. Then, the inequality

$$(21) \quad \left(\mathcal{E}_{m+n,p}^{(i)} \left(\frac{x+y}{2}, v \right) \right)^2 \leq \mathcal{E}_{2m,p}^{(i)}(x, v) \mathcal{E}_{2n,p}^{(i)}(y, v),$$

holds for $x, y > 0$ and $v \geq 1$.

Proof. This follows from Theorem 3.12 by letting $\eta = \mu = 2$.

Theorem 3.14. Let $p \in \mathbb{R}^+$, $m, n \in \mathbb{N}_0$ such that $\frac{m}{\eta} + \frac{n}{\mu} \in \mathbb{N}_0$.

Then, the inequality

$$(22) \quad \left| \mathcal{E}_{\frac{m}{\eta} + \frac{n}{\mu}, p}^{(i)} \left(\frac{x}{\eta} + \frac{y}{\mu}, v \right) \right| \leq \left(\mathcal{E}_{m,p}^{(i)}(x, v) \right)^{\frac{1}{\eta}} \left| \left(\mathcal{E}_{n,p}^{(i)}(y, v) \right)^{\frac{1}{\mu}} \right|,$$

holds for $\eta > 1$, $x, y > 0$, $v \geq 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$.

Proof. Using (18) and Hölder's inequality for integrals, we have

$$\begin{aligned}
 & \left| \varepsilon_{\frac{m}{\eta} + \frac{n}{\mu}, p}^{(i)} \left(\frac{x}{\eta} + \frac{y}{\mu}, v \right) \right| \\
 &= (\ln A_p^{-1})^i \int_1^v t^{i - \left(\frac{m}{\eta} + \frac{n}{\mu} \right)} A_p^{-\left(\frac{x}{\eta} + \frac{y}{\mu} \right)t} dt \\
 &= (\ln A_p^{-1})^{i \left(\frac{1}{\eta} + \frac{1}{\mu} \right)} \int_1^v t^{i \left(\frac{1}{\eta} + \frac{1}{\mu} \right) - \left(\frac{m}{\eta} + \frac{n}{\mu} \right)} A_p^{-\left(\frac{x}{\eta} + \frac{y}{\mu} \right)t} dt \\
 &= (\ln A_p^{-1})^{\frac{i}{\eta} + \frac{i}{\mu}} \int_1^v t^{\frac{i}{\eta} - \frac{m}{\eta}} A_p^{-\frac{xt}{\eta}} t^{\frac{i}{\mu} - \frac{n}{\mu}} A_p^{-\frac{yt}{\mu}} dt \\
 &\leq \left[\left((\ln A_p^{-1})^{\frac{i}{\eta}} \int_1^v t^{\frac{i}{\eta} - \frac{m}{\eta}} A_p^{-\frac{xt}{\eta}} dt \right)^{\eta} \right]^{\frac{1}{\eta}} \left[\left((\ln A_p^{-1})^{\frac{i}{\mu}} \int_1^v t^{\frac{i}{\mu} - \frac{n}{\mu}} A_p^{-\frac{yt}{\mu}} dt \right)^{\mu} \right]^{\frac{1}{\mu}} \\
 &= \left[(\ln A_p^{-1})^i \int_1^v t^{i-m} A_p^{-xt} dt \right]^{\frac{1}{\eta}} \left[(\ln A_p^{-1})^i \int_1^v t^{i-n} A_p^{-yt} dt \right]^{\frac{1}{\mu}} \\
 &= \left(\varepsilon_{m,p}^{(i)}(x, v) \right)^{\frac{1}{\eta}} \parallel \left(\varepsilon_{n,p}^{(i)}(y, v) \right)^{\frac{1}{\mu}} \mid .
 \end{aligned}$$

Corollary 3.15. Let $m, n \in \mathbb{N}_0$, $p \in \mathbb{R}^+$. Then, the inequality

$$(23) \quad \left(\varepsilon_{\frac{m+n}{2}, p}^{(i)} \left(\frac{x+y}{2}, v \right) \right)^2 = \varepsilon_{m,p}^{(i)}(x, v) \varepsilon_{n,p}^{(i)}(y, v),$$

holds for $x, y > 0$ and $v \geq 1$.

Proof. This follows from Theorem 3.14 by letting $\eta = \mu = 2$.

Theorem 3.16. Let $m, n \in \mathbb{N}_0$, $\alpha \in \mathbb{Z}^+$ and $p \in \mathbb{R}^+$.

Then, the inequality

$$(24) \quad \left(\left| \varepsilon_{m,p}^{(i)}(x, v) \right| + \left| \varepsilon_{n,p}^{(i)}(y, v) \right| \right)^{\frac{1}{\alpha}} \leq \left| \varepsilon_{m,p}^{(i)}(x, v) \right|^{\frac{1}{\alpha}} + \left| \varepsilon_{n,p}^{(i)}(y, v) \right|^{\frac{1}{\alpha}},$$

holds for $x, y > 0$ and $v \geq 1$.

Proof. Using (18) and the Minkowski's inequality for integrals and letting

$a^\alpha + b^\alpha \leq (a+b)^\alpha$, for $a, b \geq 0$, we have

$$\begin{aligned}
 & (|\varepsilon_{m,p}^{(i)}(x, \nu)| + |\varepsilon_{n,p}^{(i)}(y, \nu)|)^{\frac{1}{\alpha}} \\
 &= \left((\ln A_p^{-1})^i \int_1^\nu t^{i-m} A_p^{-xt} dt + (\ln A_p^{-1})^i \int_1^\nu t^{i-n} A_p^{-yt} dt \right)^{\frac{1}{\alpha}} \\
 &= (\ln A_p^{-1})^{\frac{i}{\alpha}} \left(\int_1^\nu \left[\left(t^{\frac{i-m}{\alpha}} A_p^{-\frac{xt}{\alpha}} \right)^\alpha + \left(t^{\frac{i-n}{\alpha}} A_p^{-\frac{yt}{\alpha}} \right)^\alpha \right] dt \right)^{\frac{1}{\alpha}} \\
 &\leq (\ln A_p^{-1})^{\frac{i}{\alpha}} \left(\int_1^\nu \left[\left(t^{\frac{i-m}{\alpha}} A_p^{-\frac{xt}{\alpha}} \right) + \left(t^{\frac{i-n}{\alpha}} A_p^{-\frac{yt}{\alpha}} \right) \right]^\alpha dt \right)^{\frac{1}{\alpha}} \\
 &\leq \left(\left((\ln A_p^{-1})^{\frac{i}{\alpha}} \int_1^\nu t^{\frac{i-m}{\alpha}} A_p^{-\frac{xt}{\alpha}} dt \right)^\alpha \right)^{\frac{1}{\alpha}} + \left(\left((\ln A_p^{-1})^{\frac{i}{\alpha}} \int_1^\nu t^{\frac{i-n}{\alpha}} A_p^{-\frac{yt}{\alpha}} dt \right)^\alpha \right)^{\frac{1}{\alpha}} \\
 &= \left((\ln A_p^{-1})^i \int_1^\nu t^{i-m} A_p^{-xt} dt \right)^{\frac{1}{\alpha}} + \left((\ln A_p^{-1})^i \int_1^\nu t^{i-n} A_p^{-yt} dt \right)^{\frac{1}{\alpha}} \\
 &= |\varepsilon_{m,p}^{(i)}(x, \nu)|^{\frac{1}{\alpha}} + |\varepsilon_{n,p}^{(i)}(y, \nu)|^{\frac{1}{\alpha}}.
 \end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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