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On 3-Prime $\Gamma-$ Near Rings with Generalized Derivations

L.Madhuchelvi

Abstract

Let N be a 3-prime $\Gamma-$ near ring.The purpose of this paper is to extend the results of Ashraf[1] and Ashraf and Rehman[2] in the setting of a semigroup ideal of N admitting a generalized derivation.

Keywords: 3-Prime$\Gamma-$ near ring, Semigroup ideal,Generalized derivation.

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1 Introduction

The notion of a $\Gamma-$ ring, a concept more general than a ring was defined by Nobusawa[9].As a generalization of near-rings, $\Gamma-$near rings were introduced by Satyanarayana[11].Recently Booth and Groenewald [4,5],Satyanarayana[12,13],Selvaraj and George[14,16],Selvaraj and Madhuchelvi[15,17] studied several aspects in $\Gamma-$ near rings.In the case of rings, generalized derivations have received significant attention in recent years.The derivations in $\Gamma-$ near rings have been introduced by Bell and Mason[3].They studied basic properties of derivations in $\Gamma-$ near rings.In [1], Ashraf et.al. investigates the commutativity of a prime ring admitting a generalized derivation with associated derivation satisfying certain properties. Daif and Bell[6] proved that a semiprime ring R must be commutative if it admits a derivation d such that $d([x,y]) = [x,y]$ or $d([x,y]) + [x,y] = 0$. Further Ashraf and Rehman[2] extended the mention result for Lie ideals of R.

Motivated by the above,in this paper, we extend M.Ashraf's[1] results to a semigroup ideal of a 3-prime$\Gamma-$ near ring and generalize the results of Ashraf and Rehman[2] for generalized derivation and semigroup ideals of a 3-prime $\Gamma-$ near ring.

2 Preliminaries

Throughout this paper N stands for a zero symmetric right $\Gamma-$ near ring. In this section we collect all basic concepts and results in $\Gamma-$ near rings mostly from Booth[4] and Satyanarayana[11] which are required for our study.

Definition 2.1 *A Γ - **near ring** is a triple $(N, +, \Gamma)$, where*

(i) $(N, +)$ is a (not necessarily abelian) group;

(ii) Γ *is a non-empty set of binary operations on* N *such that for each* $\gamma \in \Gamma$, $(N, +, \gamma)$ *is a right near -ring and;*

(iii) $(x\gamma y)\,\mu z = x\gamma\,(y\mu z)$ *for all* $x, y, z \in N$ *and* $\gamma, \mu \in \Gamma$.

Γ-near rings generalize near-rings in the sense that every near-ring N is a Γ-near ring with $\Gamma = \{\cdot\}$ where $\cdot$ is the multiplication defined on N.

Example 2.2 *Let* $N = \mathbb{Z}_6$ *with* $\Gamma = \{\gamma_1, \gamma_2\}$ *where* γ_1, γ_2 *are given by the Schemes 1:* $(0, 1, 0, 0, 0, 0)$ *and 2:* $(0, 0, 1, 0, 0, 0)$ *(see p.409, Pilz[10])*

γ_1	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	0	0	0
2	0	2	0	0	0	0
3	0	3	0	0	0	0
4	0	4	0	0	0	0
5	0	5	0	0	0	0

γ_2	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	1	0	0	0
2	0	0	2	0	0	0
3	0	0	3	0	0	0
4	0	0	4	0	0	0
5	0	0	5	0	0	0

Then N *is a* Γ*-near ring.*

Definition 2.3 *Let* N *be a zero symmetric right* $\Gamma-$*near ring.An additive mapping* $d : N \to N$ *is said to be a* ***derivation*** *on* N *if* $d(x\alpha y) = x\alpha d(y) + d(x)\alpha y$ *for all* $x, y \in N, \alpha \in \Gamma$.

Definition 2.4 *For all* $x, y \in N$ *and* $\alpha \in \Gamma$
$[x, y]_\alpha = x\alpha y - y\alpha x$
$(x \circ y)_\alpha = x\alpha y + y\alpha x$
$(x, y) = x + y - x - y$

Definition 2.5 *Let* N *be a zero symmetric right* $\Gamma-$*near ring.An additive mapping* $f : N \to N$ *is said to be a* ***right generalized derivation (resp.left generalized derivation)*** *on* N *if there exists a derivation* d *of* N *such that* $f(x\alpha y) = f(x)\alpha y + x\alpha d(y)(f(x\alpha y) = x\alpha f(y) + d(x)\alpha y)$ *for all* $x, y \in N, \alpha \in \Gamma$. *An additive mapping* $f : N \to N$ *is said to be a* ***generalized derivation*** *if it is both right and left generalized derivation.*

Definition 2.6 *A nonempty subset* U *of* N *is called a* ***semigroup right ideal(resp.semigroup left ideal)*** *if* $U\Gamma N \subset U(N\Gamma U \subset U)$. U *is called* ***semigroup ideal*** *if it is both right and left semigroup ideal.*

Definition 2.7 *A* $\Gamma-$ *near ring* N *is called* ***3-prime*** *if* $a\Gamma N\Gamma b = 0 \Rightarrow a = 0$ *or* $b = 0$ *for all* $a, b \in N$.

Definition 2.8 *A* $\Gamma-$ *near ring* N *is called* ***2-torsion-free*** *if* $(N, +)$ *has no elements of order 2.*

Definition 2.9 *An element* x *of a* $\Gamma-$ *near ring* N *is called* ***distributive*** *if* $x\alpha(a+b) = x\alpha a + x\alpha b$ *for all* $a, b \in N$ *and* $\alpha \in \Gamma$*.If all the elements of a* $\Gamma-$ *near ring* N *are distributive, then* N *is said to be a* ***distributive*** $\Gamma-$ ***near ring***.

Definition 2.10 *For all* $x, y, z \in N$ *and* $\alpha, \beta \in \Gamma$,
$[x\beta y, z]_\alpha = x\beta [y, z]_\alpha + [x, z]_\alpha \beta y + x\beta z\alpha y - x\alpha z\beta y$,
$[x, y\beta z]_\alpha = y\beta [x, z]_\alpha + [x, y]_\alpha \beta z + y\alpha x\beta z - y\beta x\alpha z$,
$(x\beta y \circ z)_\alpha = x\beta (y \circ z)_\alpha - [x, z]_\alpha \beta y + x\beta z\alpha y - x\alpha z\beta y = (x \circ z)_\alpha \beta y + x\beta [y, z]_\alpha + x\alpha z\beta y - x\beta z\alpha y$ *and*
$(x \circ y\beta z)_\alpha = (x \circ y)_\alpha \beta z - y\beta [x, z]_\alpha + y\alpha x\beta z - y\beta x\alpha z = y\beta (x \circ z)_\alpha + [x, y]_\alpha \beta z + y\beta x\alpha z - y\alpha x\beta z$.

3 Generalized Derivations of 3-Prime $\Gamma-$ Near Rings

Throughout this section, a $\Gamma-$ near ring N is satisfying the assumption (*):$a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in N, \alpha, \beta \in \Gamma$.

Lemma 3.1 *If* $d \neq 0$ *is a derivation on a 2-torsion free* $\Gamma-$ *near ring* N *and* U *a semigroup ideal of* N *such that* $U + U \subset U$ *and* $d(U) = 0$*, then* $U \subset Z$.

Proof. Let $u \in U, x \in N$. By the hypothesis and by[7,Theorem] , $d([u, x]_\alpha) = 0 \Rightarrow u \in Z$ .Thus $U \subset Z$. ■

Theorem 3.2 *Let* N *be a 2-torsion free 3-prime distributive* $\Gamma-$ *near ring satisfying (*),* U *a semigroup ideal such that* $U+U \subset U$ *and* f *a generalized derivation associated with* $d \neq 0$. *If* $(f(y) \circ d(x))_\alpha = 0$ *for all* $x, y \in U$ *and* $\alpha \in \Gamma$*, then* $U \subseteq Z$.

Proof. Given that $(f(y) \circ d(x))_\alpha = 0$ for all $x, y \in U$ and $\alpha \in \Gamma$
Replace y by $y\beta z$ and using (*), we get

$$-[y, d(x)]_\alpha \beta f(z) + d(y)\beta (z \circ d(x))_\alpha - [d(y), d(x)]_\alpha \beta z = 0 \text{ for } x, y \in U, z \in N, \alpha, \beta \in \Gamma.$$

Replace y by $d(x)$ for any $x \in V$ where $V = \{u \in U | d(u) \in U\}$, we get

$$d^2(x)\beta(z \circ d(x))_\alpha - \left[d^2(x), d(x)\right]_\alpha \beta z = 0 \quad (1)$$

Replace z by $z\gamma y$ in (1) and using (1), we get

$$d^2(x)\beta z\gamma [y, d(x)]_\alpha = 0 \text{ for } x, y \in U, z \in N, \alpha, \beta, \gamma \in \Gamma$$

Hence $d^2(x)\beta N\gamma [y, d(x)]_\alpha = 0$. This implies that either $[y, d(x)]_\alpha = 0$ or $d^2(x) = 0$.If $[y, d(x)]_\alpha = 0$ then $d(x)\alpha y = y\alpha d(x)$ for $x, y \in U$. This implies that $U \subseteq Z$. If $d^2(x) = 0$ then $d(d(U)) = 0 \Rightarrow d(U) = 0$ since $d \neq 0$.By Lemma 3.1, $U \subseteq Z$. ■

Theorem 3.3 *Let N be a 2-torsion free 3-prime distributive $\Gamma-$ near ring satisfying (*), U a semigroup ideal such that $U+U \subset U$ and f a generalized derivation associated with $d \neq 0$. If $[d(x), f(y)]_\alpha = 0$ for all $x, y \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z$.*

Proof. By hypothesis we have $[d(x), f(y)]_\alpha = 0$ for all $x, y \in U$ and $\alpha \in \Gamma$. Replace y by $y\beta z$ and using (*), we get

$$f(y)\beta[d(x), z]_\alpha + y\beta[d(x), d(z)]_\alpha + [d(x), y]_\alpha \beta d(z) = 0 \tag{2}$$

for all $x, y \in U, z \in N, \alpha, \beta \in \Gamma$. Replace z by $z\gamma d(x)$ in (2) and using (2) we get

$$y\beta z\gamma\left[d(x), d^2(x)\right]_\alpha + y\beta[d(x), z]_\alpha \gamma d^2(x) + [d(x), y]_\alpha \beta z\gamma d^2(x) = 0 \tag{3}$$

Replace y by $t\delta y$ in (3) and using (3) we get

$$[d(x), t]_\alpha \delta y\beta z\gamma d^2(x) = 0 \text{ for all } x, y \in U, z, t \in N, \alpha, \beta, \gamma, \delta \in \Gamma.$$

This implies that

$$[d(x), t]_\alpha \delta U\gamma d^2(x) = 0$$

Thus either $[d(x), t]_\alpha = 0$ or $d^2(x) = 0$
By using the same technique as in the above theorem, we get the required result. ■

Theorem 3.4 *Let N be a 2-torsion free 3-prime distributive $\Gamma-$ near ring satisfying (*), U a semigroup ideal such that $U+U \subset U$ and f a generalized derivation associated with $d \neq 0$. If $(d(x) \circ f(y))_\alpha = (x \circ y)_\alpha$ for all $x, y \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z$.*

Proof. Given that

$$(d(x) \circ f(y))_\alpha = (x \circ y)_\alpha \quad \text{for all} \quad x, y \in U \quad \text{and} \quad \alpha \in \Gamma. \tag{4}$$

If $f = 0$, then $(x \circ y)_\alpha = 0$ for all $x, y \in U, \alpha \in \Gamma$. Replace y by $y\beta z$, we get $y\beta[x, z]_\alpha = 0$ for all $x, y \in U, z \in N, \alpha, \beta \in \Gamma$.In particular $[x, z]_\alpha \gamma y\beta[x, z]_\alpha = 0 \Rightarrow [x, z]_\alpha \gamma U\beta[x, z]_\alpha = 0$. This implies that $[x, z]_\alpha = 0$ for all $x \in U, z \in N, \alpha \in \Gamma$. Thus $U \subseteq Z$.
Now we assume that $f \neq 0$. Replace y by $y\beta z$ in (4) and using (4), we get

$$(d(x) \circ y)_\alpha \beta d(z) - f(y)\beta[d(x), z]_\alpha - y\beta[d(x), d(z)]_\alpha + y\beta[x, z]_\alpha = 0 \tag{5}$$

for all $x, y \in U, z \in N, \alpha, \beta \in \Gamma$. Replace z by $d(x)$ in (5),we get

$$(d(x) \circ y)_\alpha \beta d^2(x) - y\beta\left[d(x), d^2(x)\right]_\alpha + y\beta[x, d(x)]_\alpha = 0 \tag{6}$$

Replace y by $z\gamma y$ for all $y, z \in U$ in (6) and using (6) we get

$$[d(x), z]_\alpha \gamma y\beta d^2(x) = 0 \Rightarrow [d(x), z]_\alpha \gamma U\beta d^2(x) = 0$$

This implies that either $[d(x), z]_\alpha = 0$ or $d^2(x) = 0$. By using similar argument as in the Theorem 3.2, $U \subseteq Z$. ■

By using similar techniques, we also prove the following theorem.

Theorem 3.5 *Let N be a 2-torsion free 3-prime distributive $\Gamma-$ near ring satisfying (*), U a semigroup ideal such that $U+U \subset U$ and f a generalized derivation associated with $d \neq 0$. If $(d(x) \circ f(y))_\alpha + (x \circ y)_\alpha = 0$ for all $x, y \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z$.*

Theorem 3.6 *Let N be a 2-torsion free 3-prime distributive $\Gamma-$ near ring satisfying (*), U a semigroup ideal such that $U+U \subset U$ and f a non zero generalized derivation associated with $d \neq 0$. If $[d(x), f(y)]_\alpha = [x, y]_\alpha$ for all $x, y \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z$.*

Proof. Given that

$$[d(x), f(y)]_\alpha = [x, y]_\alpha \quad \text{for all } \ x, y \in U \ \text{ and } \ \alpha \in \Gamma. \tag{7}$$

Replace y by $y\beta z$ in (7) and using (7), we get

$$f(y)\beta[d(x), z]_\alpha + y\beta[d(x), d(z)]_\alpha + [d(x), y]_\alpha \beta d(z) - y\beta[x, z]_\alpha = 0 \tag{8}$$

for all $x, y, z \in U, \alpha, \beta \in \Gamma$.
Replace z by $z\gamma d(x)$ in (8) and using (8),we get

$$y\beta z\gamma\left[d(x), d^2(x)\right]_\alpha + y\beta[d(x), z]_\alpha \gamma d^2(x) + [d(x), y]_\alpha \beta z\gamma d^2(x) - y\beta z\gamma[x, d(x)]_\alpha = 0 \tag{9}$$

for all $x, y, z \in U, \alpha, \beta, \gamma \in \Gamma$
Replace y by $t\delta y$ for all $y, t \in U, \delta \in \Gamma$ in (9) and using (9) we get

$$[d(x), t]_\alpha \delta y\beta z\gamma d^2(x) = 0 \Rightarrow [d(x), t]_\alpha \delta U\gamma d^2(x) = 0$$

for all $x, y, z, t \in U, \alpha, \beta, \gamma, \delta \in \Gamma$ This implies that either $[d(x), z]_\alpha = 0$ or $d^2(x) = 0$. By using similar argument as in the proof of the above Theorems , $U \subseteq Z$. ■

We also prove the following theorem as in Theorem 3.6 with necessary variations.

Theorem 3.7 *Let N be a 2-torsion free 3-prime distributive $\Gamma-$ near ring satisfying (*), U a semigroup ideal such that $U+U \subset U$ and f a non zero generalized derivation associated with $d \neq 0$. If $[d(x), f(y)]_\alpha = -[x, y]_\alpha$ for all $x, y \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z$.*

Theorem 3.8 *Let N be a 2-torsion free 3-prime distributive $\Gamma-$ near ring satisfying (*), U a semigroup ideal such that $U+U \subset U$ and f a non zero generalized derivation associated with $d \neq 0$. If $f([u, v]_\alpha) = (u \circ v)_\alpha$ for all $u, v \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z$.*

Proof. For all $u, v \in U, \alpha \in \Gamma$,we have

$$f([u, v]_\alpha) = [x, y]_\alpha \Rightarrow f(u)\alpha v + u\alpha d(v) - f(v)\alpha u - v\alpha\alpha d(u) = u\alpha v + v\alpha u \tag{10}$$

Replace v by $v\beta u$ in (10) and using (10), we get

$$[u, v]_\alpha \beta d(u) = 0 \tag{11}$$

for all $u, v \in U, \alpha, \beta \in \Gamma$.
Replace v by $w\gamma v$ in (11) and using (11),we get

$$[u, w]_\alpha \gamma v\beta d(u) = 0 \Rightarrow [u, w]_\alpha \gamma U\beta d(u) = 0$$

for all $u, v \in U, w \in N, \alpha, \beta, \gamma \in \Gamma$
By [8,Lemma 3.3],$[u, w]_\alpha = 0$ or $d(u) = 0$ for all $u \in U, w \in N, \alpha \in \Gamma$
If $[u, w]_\alpha = 0$ then $U \subseteq Z$. On the other hand, if $d(u) = 0$ for all $u \in U$, by Lemma 3.1, $U \subseteq Z$. ■

Using the same technique with necessary variation we get the following

Theorem 3.9 *Let N be a 2-torsion free 3-prime distributive $\Gamma-$ near ring satisfying (*), U a semigroup ideal such that $U + U \subset U$ and f a non zero generalized derivation associated with $d \neq 0$. If $f([u, v]_\alpha) + (u \circ v)_\alpha = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z$.*

Theorem 3.10 *Let N be a 2-torsion free 3-prime distributive $\Gamma-$ near ring satisfying (*), U a semigroup ideal such that $U + U \subset U$ and f a non zero generalized derivation associated with $d \neq 0$. If $f((u \circ v)_\alpha) = [u, v]_\alpha$ for all $u, v \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z$.*

Proof. For all $u, v \in U, \alpha \in \Gamma$,we have

$$f((u \circ v)_\alpha) = [u, v]_\alpha \tag{12}$$

Replace v by $v\beta u$ in (12) and using (12), we get

$$f((u \circ v\beta u)_\alpha) = [u, v\beta u]_\alpha \Rightarrow (u \circ v)_\alpha \beta d(u) = 0 \tag{13}$$

for all $u, v \in U, \alpha, \beta \in \Gamma$.
Replace v by $w\gamma v$ in (13) and using (13),we get

$$[u, w]_\alpha \gamma v\beta d(u) = 0 \Rightarrow [u, w]_\alpha \gamma U\beta d(u) = 0$$

for all $u, v \in U, w \in N, \alpha, \beta, \gamma \in \Gamma$.
By using the same argument as in Theorem 3.8, we get the required result. ■

We also prove the following theorem by using the same technique with necessary variations.

Theorem 3.11 *Let N be a 2-torsion free 3-prime distributive $\Gamma-$ near ring satisfying (*), U a semigroup ideal such that $U + U \subset U$ and f a non zero generalized derivation associated with $d \neq 0$. If $f((u \circ v)_\alpha) + [u, v]_\alpha = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z$.*

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Department of Mathematics, Sri Sarada College for Women,Salem-636016,Tamilnadu,India.

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