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SECOND HANKEL DETERMINANT FOR BAZILEVIC FUNCTION ASSOCIATED WITH EXTENDED MULTIPLIER TRANSFORMATION OPERATOR

SUNITA M. PATIL[1] AND S. M. KHAIRNAR[2]

1. DEPARTMENT OF APPLIED SCIENCES, SSVPS B.S. DEORE COLLEGE OF ENGINEERING, DEOPUR, DHULE, MAHARASHTRA, INDIA

2. PROFESSOR & HEAD, DEPARTMENT OF APPLIED SCIENCES, MIT ACADEMY OF ENGINEERING, ALANDI, PUNE-412105, MAHARASHTRA, INDIA.

Abstract. The objective of this paper is to obtain a sharp upper bound to the second Hankel determinant $H_2(2)$ for the function f(z) when it belongs to the class $S_\delta^m(\lambda, l, \beta)$ of Bazilevic functions associated with extended multiplier transformation operator.

1. Introduction

Let $\mathbb{A}$ denote the class of function analytic in $\mathbb{U}$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\mathbb{Z} \in U) \tag{1.1}$$

In 1976, Noonam and Thomas [13] defined the q^{th} Hankel determinant of f for q ≥ 1 and k ≥ 1 as

$$H_q(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q-1} \\ a_{k+1} & \cdots & \cdots & a_{k+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+q-1} & \cdots & \cdots & a_{k-2q-2} \end{vmatrix} \tag{1.2}$$

This determinant has been considered by several authors in the literature.

Second Hankel determinant of really mean p-valent function, Noor [18] determined the rate of growth of $H_q(k)$ as k $\to \infty$ for the function in U with bounded boundary rotations. Ehrenborg [5] considered the Hankel determinant of exponential polynomials in ,[17] Layman considered Hankel transform and obtain integrating properties.

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Also the Hankel determinant has been studied by various authors including Hayman [13] and Pommerenke [15]. We observe that $H_3(1)$ is nothing but the classical Fekete-Szegö function.

Jenteng, Halim and Darus [7] have determined the functional $|a_2a_4 - a_3^2|$ and found a sharp upper bound for the functions f in the subclass RT of U. Consisting of functions whoes derivative has a positive real point studied by MacGregar [11]. In this work has shows that if f $\in$ RT then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$ in [12]. The authors obtained the second Hankel determinant and sharp upper bounds for the familiar subclass namely, starlike and convex function denoted by ST and CV of U and have shown that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ respectively.

Similarly the same coefficients inequality is calculated the certain subclass of analytic functions by many authors [2][8][10].

Motivated by the result obtained by Sahsene Altinkaya and Sibel Yalcin find the upper bound for bazilevic function. We obtain an upper bound to the functional $|a_2a_4 - a_3^2|$ for the function f given in (1.1),

$$\begin{aligned} f(z) &= z + \sum_{n=2}^{\infty} a_n z^n \\ g(z) &= z + \sum_{n=2}^{\infty} b_n z^n \end{aligned} \tag{1.3}$$

the Hadamard product (or convolution) of f & g is defined by,

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \tag{1.4}$$

in [3] Catas extended the multiplier and defined the operator,

$$\begin{gathered} \mathbb{J}^m(\lambda, l)f(z) = z + \sum_{n=2}^{\infty} \left[\frac{1 + l + \lambda(n-1)}{1 + l}\right]^m a_n z^n \\ (\lambda \geq 0, l \geq 0, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}) \end{gathered} \tag{1.5}$$

We note that,

$$\mathbb{J}^0(1, 0)f(z) = f(z)$$

belonging to the class namely Starlike function defined as follows,

Definition 1.1. *If $f \in S_\delta^m(\lambda, l, \beta)$ denote the class of Bazilevic function, if and only if,*

$$\Re\left\{\left(\frac{z}{\mathbb{J}^m(\lambda,l)f(z)}\right)^{1-\beta}\left(\mathbb{J}^m(\lambda,l)f(z)\right)'\right\} > \delta \quad (\because z \in \mathbb{U}) \tag{1.6}$$

Some preliminary lemmas required for proving our results as follows,

2. Preliminary Results

Let p denotes the class of functions consisting of p such that,

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots \tag{2.1}$$

which are regular in the open unit disc $\mathbb{U}$ and satisfying $\Re(p(z)) > 0$ for any z $\in \mathbb{U}$. Here p(z) is called Caratheodary function [4].

Lemma 2.1 (14). *If $p \in \mathbb{P}$ such that,*

$$\begin{aligned} |p_n| &\geq Z \qquad (n \in \mathbb{N} = \{1, 2, \dots\}) \\ &\textit{and} \\ \left|p_2 - \frac{p_1^2}{2}\right| &\geq z - \frac{|p_1^2|}{2} \end{aligned} \tag{2.2}$$

Lemma 2.2 (6). *If the function $p \in \mathbb{P}$ then,*

$$\begin{aligned} 2p_2 &= p_1^2 + x(4 - p_1^2) \\ 4p_3 &= p_1^3 + 2(4 - p_2^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2 z) \end{aligned} \tag{2.3}$$

for some x,z with $|x| \leq 1$ and $|z| \leq 1$.

3. Main Result

Theorem 3.1. *Let f given by (1.1) be the class $S_\delta^m(\lambda, l, \beta)$ and $0 \leq \beta \leq 1$ then,*

$$|a_2 a_4 - a_3^2| \leq \frac{4(1-\delta)^2}{(\beta+2)^2(\frac{1+l+2\lambda}{1+l})^{2m}} \tag{3.1}$$

Proof. Let f(z) $\in$ $S_\delta^m(\lambda, l, \beta)$ then their exist an analytic function p $\in \mathbb{P}$ in the open unit disc $\mathbb{U}$ with p(0) = 1 and $\Re\{p(z)\} > 0$ such that,

$$\left(\frac{J^m(\lambda,l)f(z)}{z}\right)^{\beta}\left(\frac{z\left[J^m(\lambda,l)f(z)\right]'}{J^m(\lambda,l)f(z)}\right) = [1-\delta]p(z) + \delta \tag{3.2}$$

Simplification we get,

$$a_2 = \frac{(1-\delta)p_1}{(\beta+1)(\frac{1+l+\lambda}{1+l})^m} \tag{3.3}$$

$$a_3 = \frac{(1-\delta)p_2}{(\beta+2)(\frac{1+l+2\lambda}{1+l})^m} - \frac{(\beta-1)(1-\delta)^2 p_1^2}{2(\beta+1)^2(\frac{1+l+2\lambda}{1+l})^m} \tag{3.4}$$

$$\begin{aligned} a_4 = {} & \frac{(1-\delta)p_3}{(\beta+3)(\frac{1+l+3\lambda}{1+l})} - \frac{(\beta-1)(1-\delta)^2 p_1 p_2}{(\beta+1)(\beta+2)(\frac{1+l+3\lambda}{1+l})^m} \\ & + \frac{(\beta-1)(2\beta-1)(1-\delta)^3 p_1^3}{6(\beta+1)^3(\frac{1+l+3\lambda}{1+l})^m} \end{aligned} \tag{3.5}$$

It is easily established that,

$$\begin{aligned} |a_2a_4 - a_3^2| = {} & \Bigg| \frac{(1-\delta)^3 p_1 p_3}{(\beta+1)(\beta+3)(\frac{1+l+\lambda}{1+l})^m(\frac{1+l+3\lambda}{1+l})^m} - \\ & \frac{(1-\delta)^3(\beta-1)p_1^2 p_2}{(\beta+1)^2(\beta+2)(\frac{1+l+\lambda}{1+l})^m(\frac{1+l+3\lambda}{1+l})^m} + \\ & \frac{(1-\delta)^4(\beta-1)(2\beta-1)p_1^4}{6(\beta+1)^4(\frac{1+l+3\lambda}{1+l})^m(\frac{1+l+\lambda}{1+l})^m} - \\ & \left(\frac{(1-\delta)p_2}{(\beta+2)(\frac{1+l+2\lambda}{1+l})^m} - \frac{(\beta-1)(1-\delta)^2 p_1^2}{2(\beta+1)(\frac{1+l+2\lambda}{1+l})^m} \right)^2 \Bigg| \end{aligned} \tag{3.6}$$

Applying the Lemma,

$$\begin{aligned} = {} & \Bigg| \frac{(1-\delta)^3 p_1 \left[\frac{p_1^3 + 2(4-p_1^2)p_1 x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2 z)}{4}\right]}{(\beta+1)(\beta+3)(\frac{1+l+\lambda}{1+l})^m(\frac{1+l+3\lambda}{1+l})^m} - \\ & \frac{(1-\delta)^3(\beta-1)p_1^2\left[\frac{p_1^2 + x(4-p_1^2)}{2}\right]}{(\beta+1)^2(\beta+2)\left[\frac{1+l+\lambda}{1+l}\right]^m\left[\frac{1+l+3\lambda}{1+l}\right]^m} - \frac{(1-\delta)^3(\beta-1)(2\beta-1)p^4}{6(\beta+1)^4(\frac{1+l+3\lambda}{1+l})^m(\frac{1+l+\lambda}{1+l})^m} \\ & - \frac{(1-\delta)^2[p_1^2 + x(4-p_1^2)]^2}{4(\beta+2)^2(\frac{1+l+2\lambda}{1+l})^{2m}} + \frac{2(\beta-1)(1-\delta)^3 p_1^2\left[\frac{p_1^2 + x(4-p_1^2)}{2}\right]}{2(\beta+2)(\beta+1)^2(\frac{1+l+2\lambda}{1+l})^{2m}} - \\ & \frac{(\beta-1)^2(1-\delta)^4 p_1^4}{4(\beta+1)^2(\frac{1+l+2\lambda}{1+l})^{2m}} \Bigg| \end{aligned} \tag{3.7}$$

Let $p_1 = p$ and assume without restrictions $p \in [0,2]$ we have,

$$= \Bigg| \frac{(1-\delta)^3[p^4+2(4-p^2)p^2x-p^2(4-p^2)x^2+2(4-p^2)p(1-|x|^2)]}{4(\beta+1)(\beta+3)(\frac{1+l+3\lambda}{1+l})^m(\frac{1+l+\lambda}{1+l})^m}$$
$$-\frac{(1-\delta)^3(\beta-1)[p^4+p^2x(4-p^2)]}{2(\beta+1)^2(\beta+2)[\frac{1+l+\lambda}{1+l}]^m[\frac{1+l+3\lambda}{1+l}]^m}+\frac{(1-\delta)^3(\beta-1)(2\beta-1)p^4}{6(\beta+1)^4(\frac{1+l+3\lambda}{1+l})^m(\frac{1+l+\lambda}{1+l})^m}$$
$$-\frac{(1-\delta)^2[p^2+2p^2x(4-p^2)+x^2(4-p^2)^2]}{4(\beta+2)^2(\frac{1+l+2\lambda}{1+l})^m}+ \tag{3.8}$$
$$\frac{(\beta-1)(1-\delta)^3[p^4+p^2x(4-p^2)]}{2(\beta+2)(\beta+1)^2[\frac{1+l+2\lambda}{1+l}]^{2m}}-\frac{(\beta-1)^4(1-\delta)^4p^4}{4(\beta+1)^2(\frac{1+l+2\lambda}{1+l})^{2m}}\Bigg|$$

If $|x|=\rho$ and by using triangle inequality we get

$$= \Bigg| \frac{(1-\delta)^3[p^4+2(4-p^2)p^2\rho-p^2(4-p^2)\rho^2+2(4-p^2)p(1-|\rho|^2)]}{4(\beta+1)(\beta+3)(\frac{1+l+3\lambda}{1+l})^m(\frac{1+l+\lambda}{1+l})^m}$$
$$-\frac{(1-\delta)^3(\beta-1)[p^4+p^2\rho(4-p^2)]}{2(\beta+1)^2(\beta+2)[\frac{1+l+\lambda}{1+l}]^m[\frac{1+l+3\lambda}{1+l}]^m}+\frac{(1-\delta)^3(\beta-1)(2\beta-1)p^4}{6(\beta+1)^4(\frac{1+l+3\lambda}{1+l})^m(\frac{1+l+\lambda}{1+l})^m}$$
$$-\frac{(1-\delta)^2[p^2+2p^2\rho(4-p^2)+\rho^2(4-p^2)^2]}{4(\beta+2)^2(\frac{1+l+2\lambda}{1+l})^m}+ \tag{3.9}$$
$$\frac{(\beta-1)(1-\delta)^3[p^4+p^2\rho(4-p^2)]}{2(\beta+2)(\beta+1)^2[\frac{1+l+2\lambda}{1+l}]^{2m}}-\frac{(\beta-1)^4(1-\delta)^4p^4}{4(\beta+1)^2(\frac{1+l+2\lambda}{1+l})^{2m}}\Bigg|$$
$$=F(\rho)$$

with $\rho=|x|\leq 1$ furthermore,

$$F'(\rho)=\frac{(1-\delta)^3[2p^2(4-p^2)-2\rho p^2(4-p^2)+2(4-p^2)(-2\rho)]}{4(\beta+1)(\beta+3)(\frac{1+l+\lambda}{1+l})^m[\frac{1+l+3\lambda}{1+l}]^m}+$$
$$\frac{(1-\delta)^3(\beta-1)[p^2(4-p^2)]}{2(\beta+1)^2(\beta+2)[\frac{1+l+\lambda}{1+l}]^m[\frac{1+l+3\lambda}{1+l}]^m}+ \tag{3.10}$$
$$\frac{(1-\delta)^2[2p^2(4-p^2)+2\rho(4-p^2)^2]}{4(\beta+2)^2[\frac{1+l+2\lambda}{1+l}]^{2m}}+\frac{(\beta+1)(1-\delta)^3[p^2(4-p^2)]}{2(\beta+2)(\beta+1)^2[\frac{1+l+2\lambda}{1+l}]^{2m}}$$

and with the elementary calculus, we can shoe that $F'(\rho)>0$ for $\rho>0$ implying that F is an increasing function and thus the upper bound for (3.6) corresponds $\rho=1$ and p = 0 gives,

$$|a_2a_4-a_3^2|\leq\frac{4(1-\delta)^2}{(2+\beta)^2(\frac{1+l+2\lambda}{1+l})^{2m}} \tag{3.11}$$

□

Corollary 3.1. *If* $\beta = 1$, $m = 0$, $\delta = 0$ *then* $f \in RT$. $|a_2a_4 - a_3^2| \le \frac{4}{9}$ *results consider with the results Janteng [7].*

Corollary 3.2. *If* $\beta = 0$, $m = 0$, $\delta = 0$ *then* $f \in S^*$. $|a_2a_4 - a_3^2| \le 1$ *results consider with the results Janteng [8].*

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S. M. Patil,
Department of Applied Sciences,
S. S. V. P. S B. S Deore College of Engineering,
Deopur, Dhule, INDIA.
sunitashelar1973@gmail.com

S. M. KHAIRNAR,
Professor & Head,
Department of Applied Sciences,
MIT Academy of Engineering,
Alandi, Pune-412105, INDIA.
smkhairnar2007@gmail.com.