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ON COMBINATORIAL IDENTITIES OF LUCAS AND FIBONACCI NUMBERS CONNECTED TO TCHEBYSHEV POLYNOMIALS

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Abstract. The present paper is on derivation of many combinatorial identities of $x_n = T_n(9)$, $y_n = U_{n-1}(9)$, $L_{6n+k} = L_k\ x_n + 20\ F_k\ y_n$ and $F_{6n+k} = F_k\ x_n + 4\ L_k\ y_n$, $k = 0, 1, 2, 3, 4, 5$ *and* $n = 0, 1, 2, \ldots$, where $T_n(x)$ and $U_{n-1}(x)$ are the well known Tchebyshev polynomials of first and second kind and L_n and F_n are the well known Lucas and Fibonacci numbers.

Keywords: Lucas and Fibonacci numbers, Tchebyshev polynomials of first and second kind, Combinatorial identities.

1. Introduction

In the recent literature, many combinatorial studies on Combinatorial entities such as Catalan numbers, Lucas numbers, Fibonacci numbers, Tchebyshev polynomials, Pell and Pell-Lucas polynomials, Brahmagupta polynomials and so on are available in abundance [2, 6, 11, 12, 13, 14, 15]. Combinatorial study combined with number theory will be an ideal platform for the derivation of many combinatorial identities such as recurrence relations, generating functions, matrix power identities, cassini determinant identities, different types of summation identities and convolution identities [1, 2, 3, 4, 5, 7, 8, 9, 10, 15].

The main motivation for the present paper is the beautiful connection between (L_{6n}, F_{6n}) and $(T_n(x), U_{n-1}(x))$ given below.

$$\begin{aligned} L_{6n} &= \left(\frac{1+\sqrt{5}}{2}\right)^6 + \left(\frac{1-\sqrt{5}}{2}\right)^6 \\ &= \left(9+\sqrt{9^2-1}\right)^n + \left(9-\sqrt{9^2-1}\right)^n \\ &= 2\ T_n(9) \end{aligned}$$

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and

$$\begin{aligned} F_{6n} &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^6 + \left(\frac{1-\sqrt{5}}{2}\right)^6\right] \\ &= \frac{8}{2\sqrt{9^2-1}}\left[\left(9+\sqrt{9^2-1}\right)^n + \left(9-\sqrt{9^2-1}\right)^n\right] \\ &= 8\,U_{n-1}(9) \end{aligned}$$

For the sake of simplicity, one may put $x_n = T_n(9)$ and $y_n = U_{n-1}(9)$. Then they fit so well that one may derive $L_{6n+k} = L_k\ x_n + 20\ F_k\ y_n$ and $F_{6n+k} = F_k\ x_n + 4\ L_k\ y_n,\ k = 0,1,2,3,4,5\ and\ n = 0,1,2,\ldots.$ The number theory of L_n and F_n enables one to derive many Combinatorial identities mentioned in the end of the above paragraph.

2. Recurrence Relations

For the sake of smooth computation of $\{(L_{6n+k}, F_{6n+k}) : k = 0,1,2,3,4,5\ and\ n = 0,1,2,3,.....\}$, a pair of sequences, namely (x_n, y_n) described by Tchebyshev polynomials is extensively applied [4, 5, 6].

Definition:

$$x_n = T_n(9),\ y_n = U_{n-1}(9) \qquad n = 0,1,2\cdots.$$

The pair (x_n, y_n) has the following binet form :

$$\begin{aligned} x_n &= \frac{1}{2}\left[\alpha^n + \beta^n\right] \\ y_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \end{aligned}$$

where $\alpha = 9 + \sqrt{80}$, $\beta = 9 - \sqrt{80}$ and $n = 0,1,2\cdots$.
α and β satisfy the following relations : $\alpha+\beta = 18$, $\alpha-\beta = 8\sqrt{5}$, $\alpha\,\beta = 1$. By applying above binet form, one can show that (x_n, y_n) satisfy the following relations which are directed by Tchebyshev polynomials [4, 5, 6].

Identities 2.1

$$\begin{aligned}(1)\ x_{n+1} &= 18\ x_n - x_{n-1}\\ &= 9\ y_{n+1} - y_n\\ &= 9\ x_n + 80\ y_n\\ (2)\ y_{n+1} &= 18\ y_n - y_{n-1}\\ &= \frac{9\ x_{n+1} - x_n}{80}\\ &= x_n + 9\ y_n\end{aligned}$$

$n = 1, 2, 3 \dots$

Identities 2.2

$$\begin{aligned}(1)\ L_{6n+k} &= \frac{F_k\ x_{n+1} + (-1)^k F_{6-k}\ x_n}{4}\\ &= L_k\ y_{n+1} + (-1)^{k-1}\ L_{6-k}\ y_n\\ &= L_k\ x_n + 20\ F_k\ y_n\\ (2)\ F_{6n+k} &= F_k\ y_{n+1} + (-1)^k F_{6-k}\ y_n\\ &= \frac{L_k\ x_{n+1} + (-1)^{k-1} L_{6-k}\ x_n}{20}\\ &= F_k\ x_n + 4\ L_k\ y_n\end{aligned}$$

$k = 0, 1, 2, 3, 4, 5\ and\ n = 0, 1, 2, \dots$

Proof

For $k = 0$, the six identities of result (2.2), follows directly by the binet forms of x_n, y_n, x_{n+1} and y_{n+1}.

For $k = 1$, one may apply following identities [10, 13, 15].

$$\begin{aligned}(i)&\quad 8\ L_{6n+1} = L_{6n+6} - 5\ L_{6n}\\ (ii)&\quad 8\ L_{6n+1} = F_{6n+6} + 11\ F_{6n}\\ (iii)&\quad 2\ L_{6n+1} = L_{6n} + 5\ F_{6n}\\ (iv)&\quad 8\ F_{6n+1} = F_{6n+6} - 5\ F_{6n}\\ (v)&\quad 40\ F_{6n+1} = L_{6n+6} + 11\ L_{6n}\\ (vi)&\quad 2\ F_{6n+1} = L_{6n} + F_{6n}\\ &\quad n = 0, 1, 2, \dots\end{aligned}$$

and express interms of x_n and y_n. The identities for $k = 2, 3, 4, 5$ simply follows by the well known relations

$$L_{6n+(k-1)} + L_{6n+k} = L_{6n+(k+1)} \;;\; F_{6n+(k-1)} + F_{6n+k} = F_{6n+(k+1)},$$
$$k = 1, 2, 3, 4 \; and \; n = 0, 1, 2, \ldots$$

3. Matrix power identities, Cassini determinant identities and Generating functions

In the literature, a matrix power identity called Brahmagupta power identity ([9], [12]) is described

$$\begin{bmatrix} \xi_n & \eta_n \\ t\,\eta_n & \xi_n \end{bmatrix} = \begin{bmatrix} \xi & \eta \\ t\,\eta & \xi \end{bmatrix}^n$$

for

$$\xi_n = \frac{1}{2}\left[\left(\xi + \eta\sqrt{t}\right)^n + \left(\xi - \eta\sqrt{t}\right)^n\right]$$
$$\eta_n = \frac{1}{2\sqrt{t}}\left[\left(\xi + \eta\sqrt{t}\right)^n - \left(\xi - \eta\sqrt{t}\right)^n\right]$$

When $\xi = 9$, $\eta = 1$ and $t = 80$ we obtain x_n and y_n. As a consequence we have the following Matrix power identities and Cassini determinant identities described ih the following results

Identities 3.1

(1) $$\begin{bmatrix} x_n & y_n \\ 80\,y_n & x_n \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 80 & 9 \end{bmatrix}^n$$

(2) $$\begin{bmatrix} y_{n-1} & y_n \\ y_n & y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 18 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 18 \end{bmatrix}^{n-1}$$

(3) $$\begin{bmatrix} x_{n-1} & x_n \\ x_n & x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 9 & 161 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 18 \end{bmatrix}^{n-1}$$

$n = 1, 2, 3, \ldots$

Identities 3.2

(1) $x_n^2 - 80\,y_n^2 = 1$

(2) $y_{n-1}\,y_{n+1} - y_n^2 = -1$

(3) $x_{n-1}\,x_{n+1} - x_n^2 = 80$

$n = 1, 2, 3, \ldots$

Identities 3.3

$$\begin{bmatrix} 4\ L_{6n+k} & F_{6n+k} \\ 20\ F_{6n+k} & L_{6n+k} \end{bmatrix} = \begin{bmatrix} 4\ L_k & F_k \\ 20\ F_k & L_k \end{bmatrix} \begin{bmatrix} 9 & 1 \\ 80 & 9 \end{bmatrix}^n$$
$$k = 0, 1, 2, 3, 4, 5 \ and \ n = 0, 1, 2, 3, \ldots$$

Next we state the generating function of x_n, y_n, L_{6n+k} and F_{6n+k}, $k = 0, 1, 2, 3, 4, 5$ with the help of the identities given by (2.1) and (2.2).

Identities 3.4

$$(1) \qquad \sum_{n=0}^{\infty} x_n\ t^n = \frac{1-9t}{1-18t-t^2}$$

$$(2) \qquad \sum_{n=0}^{\infty} y_n\ t^n = \frac{t}{1-18t-t^2}$$

$$(3) \qquad \sum_{n=0}^{\infty} L_{6n+k}\ t^n = \frac{L_k + (L_{6+k} - 18\ L_k)\ t}{1-18t-t^2}$$

$$(4) \qquad \sum_{n=0}^{\infty} F_{6n+k}\ t^n = \frac{F_k + (F_{6+k} - 18\ F_k)\ t}{1-18t-t^2}$$
$$k = 0, 1, 2, 3, 4, 5.$$

4. Summation identities

By suitable rearranging recurrence relations one like below the summation identities can be derived non trivially.

$$x_k \quad = \quad 18\ x_{k-1} - x_{k-2},\ \ k = 2, 3, ..., n$$

Adding all we get

$$16\ (x_0 + x_1 + ... + x_n) = (x_{n+1} - x_n) + 8$$

Following the same ideas, one can derive the following summation identities :

Identities 4.1

$$(1) \qquad 16\ \sum_{k=0}^{n} x_k = x_{n+1} - x_n + 8$$

$$(2) \qquad 16\ \sum_{k=0}^{n} y_k = y_{n+1} - y_n - 1$$
$$n = 0, 1, 2, 3, \ldots$$

Next by employing

$$y_{2k+2} + 19\ y_{2k} \quad = \quad 18\ (y_{2k} + y_{2k+1}),\ \ k = 0, 1, 2, ..., n$$

We obtain

$$\sum_{k=0}^{n} y_{2k} = \frac{1}{320}\left[y_{2n+2} - y_{2n} - 18\right],\ \ n = 0, 1, 2, 3, \ldots$$

Similarly one can also derive an identity for $\sum_{k=0}^{n} x_{2k}$. Together the result is stated below.

Identities 4.2

$$\begin{aligned} &(1) \qquad 320 \sum_{k=0}^{n} x_{2k} = x_{2n+2} - x_{2n} + 160 \\ &(2) \qquad 320 \sum_{k=0}^{n} y_{2k} = y_{2n+2} - y_{2n} - 18 \\ &\qquad\qquad n = 0, 1, 2, 3, \ldots \end{aligned}$$

Since $\sum_{k=0}^{n}(x_{2k}+x_{2k+1}) = \sum_{k=0}^{2n+1} x_k$, the summation identities for $\sum_{k=0}^{n} x_{2k+1}$ and $\sum_{k=0}^{n} y_{2k+1}$ can be derived directly. So we just leave it for the interested reader to figure it out. More intersting ones are square sums of x_k's and y_k's. Using $2L_{12k} = L_{6k}^2 + 5F_{6k}^2$ and $x_k^2 - 80\ y_k^2 = 1$, one can derive $x_{2k} = 2\ x_k^2 - 1 = 160\ y_k^2 + 1$. As a result one may directly derive the following two square sum identities :

Identities 4.3

$$\begin{aligned} &(1) \qquad 640 \sum_{k=0}^{n} x_k^2 = (x_{2n+2} - x_{2n}) + 160\ (2n+3) \\ &(2) \qquad 640 \sum_{k=0}^{n} y_k^2 = \frac{1}{80}[(x_{2n+2} - x_{2n}) - 160\ (2n+1)] \\ &\qquad\qquad n = 0, 1, 2, 3, \ldots \end{aligned}$$

By making use of $L_{6n+k}^2 = L_k^2\ x_n^2 + 400\ F_k^2\ y_n^2 + 20\ F_{2k}\ y_{2k}$ and $F_{6n+k}^2 = F_k^2\ x_n^2 + 16\ L_k^2\ y_n^2 + 4\ F_{2k}\ y_{2k} : k = 0, 1, 2, 3, 4, 5$. One can directly derive the following square sum identities :

Identities 4.4

(1) $$\sum_{r=0}^{n} L_{6r+k}^2 = \frac{L_{2k}}{320}(x_{2n+2} - x_{2n}) + \frac{F_{2k}}{16}(y_{2n+2} - y_{2n}) + \frac{L_k^2}{4}(2n+3) - 100\ F_k^2(2n+1) - \frac{9}{8}F_{2k}$$

(2) $$\sum_{r=0}^{n} F_{6r+k}^2 = \frac{L_{2k}}{1600}(x_{2n+2} - x_{2n}) + \frac{F_{2k}}{80}(y_{2n+2} - y_{2n}) + \frac{F_k^2}{4}(2n+3) - 4L_k^2(2n+1) - \frac{9}{40}F_{2k}$$

$k = 0, 1, 2, 3, 4, 5$ *and* $n = 0, 1, 2, 3, \ldots$

5. Convolution identities

The binet forms (2.2) will guide the following convolution identities :

Identities 5.1

(1) $$\sum_{k=0}^{n} x_k\ x_{n-k} = \frac{1}{2}\left[(n+1)x_n + y_{n+1}\right]$$

(2) $$\sum_{k=0}^{n} y_k\ y_{n-k} = \frac{1}{160}\left[(n+1)x_n - y_{n+1}\right]$$

(3) $$\sum_{k=0}^{n} x_k\ y_{n-k} = \sum_{k=0}^{n} y_k\ x_{n-k} = \frac{n+1}{2}\ y_n$$

$n = 0, 1, 2, \ldots$

Identities 5.2

(1) $$\sum_{r=0}^{n} L_{6r+k}\ L_{6(n-r)+k} = L_{2k}(n+1)x_n + (-1)^k\ 2\ y_{n+1} + 20(n+1)F_{2k}y_n$$

(2) $$\sum_{r=0}^{n} F_{6r+k}\ F_{6(n-r)+k} = \frac{L_{2k}}{5}(n+1)\ x_n + (-1)^{k+1}\frac{2}{5}\ y_{n+1} + 4\ (n+1)\ F_{2k}\ y_n$$

(3) $$\sum_{r=0}^{n} L_{6r+k}\ F_{6(n-r)+k} = F_{2k}(n+1)x_n + 4L_{2k}(n+1)y_n$$

$k = 0, 1, 2, 3, 4, 5$ *and* $n = 0, 1, 2, 3, \ldots$

The identities 5.1 and 5.2 can be derived directly using the results of section 2. For more details about computation and application of Convolution identities one may refer to [2].

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