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Graceful labeling of posets

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Abstract

The concept of graph labeling was introduced in mid-1960 by Rosa. In this paper, we introduce a notion of graceful labeling of a finite poset. We obtain graceful labeling of some postes such as a chain, a fence, and a crown. In 2002 Thakare, Pawar, and Waphare introduced the 'adjunct' operation of two lattices with respect to an adjunct pair of elements. We obtain the graceful labeling of an adjunct sum of two chains with respect to an adjunct pair (0, 1).

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1 Introduction

A graph labeling assigns integers to the vertices or edges (or both), subject to certain conditions. Interest in graph labeling began in mid- 1960 with the conjecture by Kotzing - Ringel [10] and a paper by Rosa [1]. There are different types of graph labeling such as prime labeling, magic labeling, antimagic labeling, graceful labeling [3], etc. Labeled graphs have wide applications in different fields such as circuit design, traffic control systems, communication network addressing, Automated Teller Machine (ATM) controlling devices, Local Area Network (LAN) network, radio astronomy, and Multiprotocol Label Switching (MPLS) protocols see [6, 8, 9, 12]. In this paper, we define graceful labeling of finite posets. We obtain in particular graceful labeling of some posets like a chain, a fence, and a crown. Thakare, Pawar and Waphare [4] introduced the 'adjunct' operation of two lattices with respect to a pair of elements. In this connection, We obtain the graceful labeling of an adjunct sum of two chains with respect to an adjunct pair (0, 1).

A non empty set P, together with a binary relation $\leq$ which is reflexive, antisymmetric and transitive is called a *partially ordered set or a Poset*. A Hasse diagram is a type of mathematical diagram used to represent a finite partially ordered set. Specifically, for a poset (P, $\leq$) each element of P represents a vertex in the plane, and whenever y covers x, it indicates that $x \leq y$ and there is no z such that $x < z < y$, which is represented by $x \prec y$. These curves (or lines) may cross each other but must not touch any vertex other than endpoints; we call such curves (or lines) as edges. Two elements $a, b \in P$ are said to be *comparable* if either $a \leq b$ or $b \leq a$; otherwise they are said to be *incomparable*. A poset in which every pair of elements is comparable is called a *chain*. A chain on n elements is denoted by C_n. In particular, see Figure I for C_3.

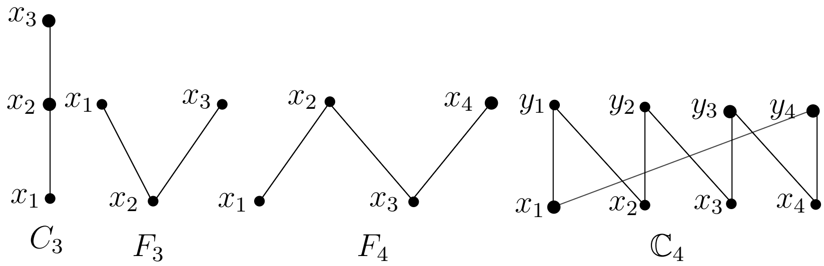

Figure I

Definition 1. [11] A partially ordered set $F_n = \{x_1, x_2, \dots, x_n\}$ is called a *fence* (of order $n \geq 3$), if either $x_1 < x_2$, $x_2 > x_3$, $\dots$, $x_{2m-1} < x_{2m}$, $x_{2m} > x_{2m+1}$, $\dots$, $x_{n-1} < x_n$, if n is even ($x_{n-1} > x_n$ if n is odd) or $x_1 > x_2$, $x_2 < x_3$, $\dots$, $x_{2m-1} > x_{2m}$, $x_{2m} < x_{2m+1}$, $\dots$, $x_{n-1} > x_n$, if n is even ($x_{n-1} < x_n$ if n is odd) are the only comparability relations. A fence F_n is called a *lower fence* if $x_1 < x_2$, and *upper fence* if $x_1 > x_2$. In particular, see Figure I for F_3 and F_4.

Definition 2. [11] *A* crown is a poset $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$ of order $n \geq 2$, whose elements satisfy precisely the comparabilities $x_1 < y_1$, $y_1 > x_2$, $x_2 < y_2$, $y_2 > x_3$, $x_3 < y_3$, $y_3 > x_4$, $\dots$, $x_{n-1} < y_{n-1}$, $y_{n-1} > x_n$, $x_n < y_n$, $y_n > x_1$. The crown of order n is denoted by $\mathbb{C}_n$. In particular, see Figure I for $\mathbb{C}_4$.

For other definitions, notation, and terminology, see [2, 5, 7]. In the following section, we introduce the notion of graceful labeling of a poset.

2 Graceful labeling of posets

On the line of graceful labeling of graphs, we define graceful labeling of a finite poset as follows.

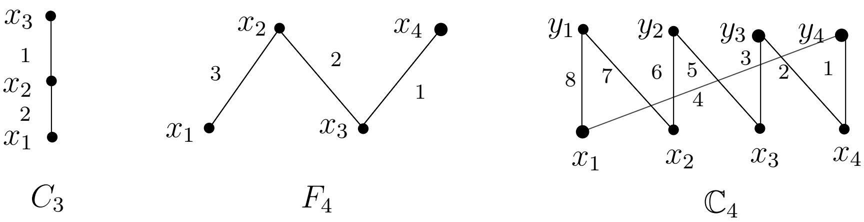

Figure II

Definition 3. Let P be a poset on n elements with m coverings, $x_1, x_2, \ldots, x_n$. Let V $= \{x_1, x_2, \ldots, x_n\}$ and E $= \{0, 1, 2, 3, \ldots, m\}$. If $\phi : V \to E$ is a one-to-one function, then when each covering, say $x_i \prec x_j$, is given the label $|\phi(x_i) - \phi(x_j)|$, the resulting cover labels are unique numbers from the set E. This is known as graceful labeling of P. A poset is called graceful if it has a graceful labeling. For example, C_3, F_4 and $\mathbb{C}_4$ are graceful (see Figure II).

Theorem 2.1. *A chain C_n is graceful for $n \geq 2$.*

Proof. Let $C_n : x_1 \prec x_2 \prec \cdots \prec x_n$ be a chain. Note that C_n contains n - 1 edges. Let V $= \{x_1, x_2, \ldots, x_n\}$ be the set of elements of C_n and E $= \{0, 1, 2, \ldots, n-1\}$. Define a map $\phi : V \to E$ as follows.

$$\phi(x_i) = \begin{cases} \frac{i-1}{2}, & \text{if i is odd} \\ n - \frac{i}{2}, & \text{if i is even.} \end{cases}$$

We claim that the map ϕ is the required graceful labeling of C_n. Firstly we prove that ϕ is one - one. One of the following four cases occurs.

1. $\phi(x_i) = \phi(x_j)$ and both i and j are odd. But then $\frac{i-1}{2} = \frac{j-1}{2}$ which implies that i = j and hence $x_i = x_j$.

2. $\phi(x_i) = \phi(x_j)$ and both i and j are even. But then $n - \frac{i}{2} = n - \frac{j}{2}$ implies that i = j and hence $x_i = x_j$.

3. i is odd and j is even. Then i $\neq$ j and $x_i \neq x_j \Rightarrow \phi(x_i) \neq \phi(x_j)$. For if, suppose $\phi(x_i) = \phi(x_j) \Rightarrow \frac{i-1}{2} = n - \frac{j}{2} \Rightarrow i - 1 = 2n - j \Rightarrow i + j = 2n + 1$. This is not possible, since $1 \leq i \leq n$ and $1 \leq j \leq n$.

4. i is even and j is odd. In this case we get the proof on the similar lines of Case (3). Thus, ϕ is one - one.

Secondly, we prove that the edge labels of C_n are all distinct. Now the edge label between the elements x_i and x_{i+1} is given by $|\phi(x_{i+1}) - \phi(x_i)|$.
Suppose for $i \neq j$, $|\phi(x_{i+1}) - \phi(x_i)| = |\phi(x_{j+1}) - \phi(x_j)|$. One of the following three cases occurs.

1. Both i and j are odd. Then we have $|n - (\frac{i+1}{2}) - (\frac{i-1}{2})| = |n - (\frac{j+1}{2}) - (\frac{j-1}{2})|$. This implies that $|n - i| = |n - j|$ and hence $i = j$, which is a contradiction.

2. Both i and j are even. Then we have $|\frac{(i+1)-1}{2} - (n - \frac{i}{2})| = |\frac{(j+1)-1}{2} - (n - \frac{j}{2})|$. This implies that $|n - i| = |n - j|$ and hence $i = j$ which is a contradiction.

3. Without loss of generality, if i is even and j is odd, then we have $|\frac{(i+1)-1}{2} - (n - \frac{i}{2})| = |n - \frac{(j+1)}{2} - (\frac{j-1}{2})|$. This implies that $|i-n| = |n-j|$ and hence $i = j$, which is a contradiction. Hence the edge labels of C_n are distinct.
   Therefore ϕ is a required graceful labeling of C_n.

$\square$

Remark 1. Let $C_n : x_0 \prec x_1 \prec \cdots \prec x_{n-1}$ be a chain where $n \geq 2$. Define a function $\psi : V(C_n) \to \{0, 1, 2, \ldots, n-1\}$ as follows.

1. If n is odd
$$\psi(x_i) = \begin{cases} (n-1) - \lfloor \frac{n-i}{2} \rfloor, & \text{if i is even} \\ \frac{n-i}{2} - 1, & \text{if i is odd.} \end{cases}$$
2. If n is even
$$\psi(x_i) = \begin{cases} (n-1) - \lfloor \frac{n-i}{2} \rfloor, & \text{if i is odd} \\ \frac{n-i}{2} - 1, & \text{if i is even.} \end{cases}$$

Then ψ is also a graceful labeling of C_n.

By the arguments similar to one as given in the proof of Theorem 2.1, we obtain the proof of the following result, since for F_n the edge labels are same as that of the chain C_n.

Corollary 2.2. *A fence F_n is graceful for $n \geq 3$.*

Note that, graceful labeling of a chain on n elements and a fence on n elements are the same. In fact we have the following.

Theorem 2.3. *A crown $\mathbb{C}_n$ is graceful if n is even.*

Proof. Suppose the set of elements of crown $\mathbb{C}_n$ is V $= \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ with 2n coverings $x_1 \prec y_1, x_2 \prec y_1, x_2 \prec y_2, x_3 \prec y_2, \cdots, x_{n-1} \prec y_{n-1}, x_n \prec y_{n-1}, x_n \prec y_n, x_1 \prec y_n$. Let E $= \{0, 1, 2, \ldots, 2n\}$. Define a map $\phi : V \to$ E as follows.
$\phi(x_i) = i - 1$, if $1 \leq i \leq n$, and $\phi(y_i) = \begin{cases} 2n - (i-1), & \text{if } 1 \leq \text{i} \leq \frac{n}{2}. \\ 2n - i, & \text{if } \frac{n}{2}+1 \leq \text{i} \leq \text{n}. \end{cases}$

We claim that the map ϕ is a graceful labeling of $\mathbb{C}_n$. Firstly we prove that ϕ is one-one. One of the following five cases occurs.

1. $\phi(x_i) = \phi(x_j)$ and $1 \leq i, j \leq n$. Then $i - 1 = j - 1$ implies that $i = j$ and hence $x_i = x_j$.
2. Suppose that $\phi(y_i) = \phi(y_j)$ and $1 \leq i, j \leq \frac{n}{2}$. Then $2n - (i-1) = 2n - (j-1)$ implies that $i = j$ and hence $y_i = y_j$.

3. that $\phi(y_i) = \phi(y_j)$ and $\frac{n}{2} + 1 \leq i, j \leq n$. Then $2n - i = 2n - j$ implies that $i = j$ and hence $y_i = y_j$.

4. $1 \leq i \leq \frac{n}{2}$ and $x_i \neq y_i$. Then $\phi(x_i) \neq \phi(y_i)$. For if, suppose $\phi(x_i) = \phi(y_i)$ implies that $i - 1 = 2n - (i - 1) \Rightarrow 2i = 2n + 2 \Rightarrow i = n + 1$ which is not possible.

5. $\frac{n}{2} + 1 \leq i \leq n$ and $x_i \neq y_i$. Then $\phi(x_i) \neq \phi(y_i)$. Since if $\phi(x_i) = \phi(y_i)$ then $i - 1 = 2n - i \Rightarrow 2i = 2n + 1$, which is not possible.
   Thus ϕ is one - one.

Secondly, we prove the edge labels of $\mathbb{C}_n$ are all distinct. Consider the edge labels of $\mathbb{C}_n$ for $1 \leq i \leq n$ as $|\phi(x_i) - \phi(y_i)|$, for $2 \leq i \leq n$ as $|\phi(x_i) - \phi(y_{i-1})|$, and $|\phi(y_n) - \phi(x_1)|$. One of the following five cases occurs.

1. $1 \leq i, k \leq \frac{n}{2}$ and $i \neq k$.
   Suppose $|\phi(x_i) - \phi(y_i)| = |\phi(x_k) - \phi(y_k)| \Rightarrow |i - 1 - (2n - (i - 1))| = |k - 1 - (2n - (k - 1))| \Rightarrow | - 2n + 2i - 2| = | - 2n + 2k - 2| \Rightarrow i = k$ which is a contradiction. Now let $\frac{n}{2} + 1 \leq i, k \leq n$ and $i \neq k$. Suppose $|\phi(x_i) - \phi(y_i)| = |\phi(x_k) - \phi(y_k)| \Rightarrow |i - 1 - (2n - i)| = |k - 1 - (2n - k)| \Rightarrow | - 2n + 2i - 1| = | - 2n + 2k - 1| \Rightarrow i = k$ which is a contradiction.

2. $2 \leq i, k \leq \frac{n}{2}$ and $i \neq k$.
   Suppose $|\phi(x_i) - \phi(y_{i-1})| = |\phi(x_k) - \phi(y_{k-1})| \Rightarrow |i - 1 - (2n - ((i - 1) - 1))| = |k - 1 - (2n - (k - 1))| \Rightarrow | - 2n + 2i - 3| = | - 2n + 2k - 3| \Rightarrow i = k$ which is a contradiction. Now let $\frac{n}{2} + 1 \leq i, k \leq n$ and $i \neq k$. Suppose $|\phi(x_i) - \phi(y_{i-1})| = |\phi(x_k) - \phi(y_{k-1})| \Rightarrow |i - 1 - (2n - (i - 1))| = |k - 1 - (2n - (k - 1)| \Rightarrow | - 2n + 2i - 2| = | - 2n + 2k - 2| \Rightarrow i = k$ which is a contradiction.

3. $1 \leq i \leq \frac{n}{2}$ and suppose $|\phi(x_i) - \phi(y_i)| = |\phi(x_1) - \phi(y_n)| \Rightarrow |i - 1 - (2n - (i - 1))| = |1 - 1 - (2n - (n - 1))| \Rightarrow |2i - 2n - 2| = | - n + 1| \Rightarrow 2i = n + 3$ which is not possible. Let $\frac{n}{2} + 1 \leq i \leq n$ and suppose $|\phi(x_i) - \phi(y_i)| = |\phi(x_1) - \phi(y_n)| \Rightarrow |i - 1 - (2n - (i))| = |1 - 1 - (2n - (n))| \Rightarrow |2i - 2n - 1| = | - n| \Rightarrow 2i - 1 = 3n$ which is not possible.

4. $2 \leq i \leq \frac{n}{2}$ and suppose $|\phi(x_i) - \phi(y_{i-1})| = |\phi(x_1) - \phi(y_n)| \Rightarrow |i - 1 - (2n - ((i - 1) - 1))| = |1 - 1 - (2n - (n))| \Rightarrow |i - 1 - 2n + i - 2| = | - n| \Rightarrow 2i - 2n - 3 = n \Rightarrow 2i - 3 = 3n$ which is not possible. Let $\frac{n}{2} + 1 \leq i \leq n$ and suppose $|\phi(x_i) - \phi(y_{i-1})| = |\phi(x_1) - \phi(y_n)| \Rightarrow |i - 1 - (2n - (i - 1))| = |1 - 1 - (2n - n)| \Rightarrow |2i - 2n - 2| = | - n| \Rightarrow 2i - 2 = 3n$ which is not possible.

5. $1 \leq i \leq \frac{n}{2}$ and $2 \leq k \leq \frac{n}{2}$, suppose $|\phi(x_i) - \phi(y_i)| = |\phi(x_k) - \phi(y_{k-1})| \Rightarrow |i - 1 - (2n - (i - 1))| = |k - 1 - (2n - ((k - 1) - 1)| \Rightarrow |2i - 2n - 2| = |2k - 2n - 3| \Rightarrow$

$2i-2=2k-3 \Rightarrow 2i-2k=-1 \Rightarrow i-k=\frac{-1}{2}$ which is not possible.

It is clear that the map ϕ gives the required graceful labeling of the $\mathbb{C}_n$.

□

Theorem 2.4. *A crown $\mathbb{C}_n$ has no graceful labeling if n is odd.*

Proof. Let $\mathbb{C}_n$ be crown with the set of elements $V=\{x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n\}$ with 2n Coverings $x_1 \prec y_1, x_2 \prec y_1, x_2 \prec y_2, x_3 \prec y_2, \cdots, x_{n-1} \prec y_{n-1}, x_n \prec y_{n-1}, x_n \prec y_n, x_1 \prec y_n$. Let $E=\{0,1,2,\ldots,2n\}$. Suppose $\mathbb{C}_n$ has p - labeling as $\phi: V \to E$. Taking sum of edge labels of the crown $\mathbb{C}_n$. We have $0 \le (|\phi(x_1)-\phi(y_1)|+|\phi(y_1)-\phi(x_2)|+|\phi(x_2)-\phi(y_2)|+\cdots+|\phi(x_{n-1})-\phi(y_{n-1})|+|\phi(x_n)-\phi(y_{n-1})|+|\phi(x_n)-\phi(y_n)|+|\phi(x_1)-\phi(y_n)|) \le (|\phi(x_1)|+|\phi(y_1)|+|\phi(y_1)|+|\phi(x_2)|+|\phi(x_2)|+|\phi(y_2)|+\cdots+|\phi(x_{n-1})|+|\phi(y_{n-1})|+|\phi(x_n)|+|\phi(y_{n-1})|+|\phi(x_n)|+|\phi(y_n)|+|\phi(x_1)|+|\phi(y_n)|) = 2(|\phi(x_1)|+|\phi(x_2)|+\cdots+|\phi(x_n)|+|\phi(y_1)|+|\phi(y_2)|+\cdots+|\phi(y_n)|)$.

Therefore, $(|\phi(x_1)-\phi(y_1)|+|\phi(y_1)-\phi(x_2)|+|\phi(x_2)-\phi(y_2)|+\cdots+|\phi(x_{n-1})-\phi(y_{n-1})|+|\phi(x_n)-\phi(y_{n-1})|+|\phi(x_n)-\phi(y_n)|+|\phi(x_1)-\phi(y_n)|) \equiv 0 (\mathrm{mod} 2)$

In crown $\mathbb{C}_n$, edge labels are from 1 to 2n. So, the sum of edge labels of $\mathbb{C}_n$ is $\sum_{k=1}^{2k} k = \frac{2n(2n+1)}{2} = n(2n+1)$ which is odd. As n is odd, $2n+1$ is odd, and therefore $n(2n+1)$ must be odd. Therefore we get a contradiction. Hence we conclude that $\mathbb{C}_n$ has no graceful labeling, if n odd. □

3 Adjunct sum of lattices

In 2002, Thakare, Pawar, and Waphare [4] introduced the concept an adjunct sum of lattices.

Definition 4. [4] Suppose L_1 and L_2 are two disjoint lattices and (a,b) is a pair of elements in L_1 such that $a<b$ and $a \not\prec b$. Define the partial order $\le$ on $L=L_1 \cup L_2$ with respect to the pair (a,b) as follows: $x \le y$ in L if $x,y \in L_1$ and $x \le y$ in L_1, or $x,y \in L_2$ and $x \le y$ in L_2, or $x \in L_1$, $y \in L_2$ and $x \le a$ in L_1, or $x \in L_2$, $y \in L_1$ and $b \le y$ in L_1.

It is easy to see that L is a lattice containing L_1 and L_2 as sublattices. The procedure for obtaining L in this way is called *adjunct operation (or adjunct sum)* of L_1 with L_2. We call the pair (a,b) as an *adjunct pair* and L as an *adjunct* of L_1 with L_2 with respect to the adjunct pair (a,b) and write $L=L_1]_a^b L_2$. A diagram of L is obtained by placing a diagram of L_1 and a diagram of L_2 side by side in such a way that the largest element 1 of L_2 is at lower position than b and the least element 0 of L_2 is at the higher position than a and then by adding the coverings $<1,b>$ and $<a,0>$, as shown in Figure III. This clearly gives $|E(L)|=|E(L_1)|+|E(L_2)|+2$.

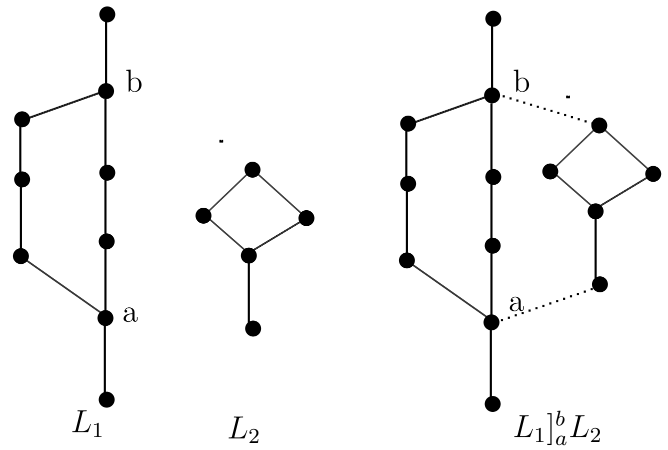

Figure III

The adjunct sum is often utilized to construct and analyze complex lattices from simpler, well-defined components while retaining the essential properties of a lattices. To obtain graceful labeling for lattices formed by the adjunct sum of two chains, we construct following sets. Let A $= \{1, 3, 5, \ldots, m-1\}, m \geq 3$, $B' = \{2, 4, 6, \ldots, \frac{m+n+1}{2}\}, B'' = \{\frac{m+n+1}{2}+2, \frac{m+n+1}{2}+4, \ldots, m\}$ and $B = B' \cup B''$. Also let $D = \{1, 3, 5, \ldots, n\}, n \geq 1, F' = \{2, 4, 6, \ldots, \frac{m+n+1}{2}-2\}, F'' = \{\frac{m+n+1}{2}, \frac{m+n+1}{2}+2, \ldots, n-1\}$, and $F = F' \cup F''$.

Theorem 3.1. *Let C and C' be the chains with $|C| = m \geq 3$, $|C'| = n \geq 1$ and $L = C]_0^1 C'$. Then L has graceful labeling if $m \equiv 2(mod\ 4)$ and $n \equiv 1(mod\ 4)$.*

Proof. Let C and C' be the chains with m and n elements, respectively. Suppose $m \equiv 2(mod\ 4)$ and $n \equiv 1(mod\ 4)$. Suppose $C = a_1 \prec a_2 \prec a_3 \cdots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \cdots \prec b_n$ and $L = C]_0^1 C'$. Clearly L has $m+n$ elements and $m+n$ coverings (edges). Suppose V $= \{a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3, \ldots, b_n\}$ and E $= \{0, 1, 2, \ldots, m+n\}$. Consider a map $\phi : V \to E$ defined as follows :

$$\phi(a_i) = \begin{cases} \frac{i-1}{2} & \text{if} \quad i \in A \\ m+n-\left\lfloor \frac{i-1}{2} \right\rfloor & \text{if} \quad i \in B' \\ m+n-\frac{i}{2} & \text{if} \quad i \in B'' \end{cases}$$

$$\phi(b_j) = \begin{cases} \frac{m+n-j}{2} & \text{if} \quad j \in D \\ \frac{m+n+j-1}{2} & \text{if} \quad j \in F' \\ \frac{m+n+j+1}{2} & \text{if} \quad j \in F'' \end{cases}$$

We claim that the map ϕ is the required graceful labeling for lattice L. Firstly we prove that ϕ is one - one. For this purpose we consider the following sets. Let $S_1 = \{a_i | i \in A\}$, $S_2' = \{a_i | i \in B'\}$, $S_2'' = \{a_i | i \in B''\}$ and $S_2 = S_2' \cup S_2''$. Also, let $T_1 = \{b_j | j \in D\}, T_2' = \{b_j | j \in F\}$, $T_2'' = \{b_j | j \in F''\}$ and $T_2 = T_1' \cup T_2''$. Let $a, b \in V$. Now to show ϕ is one-one. For this proof, one of the following five cases occurs depending on $a, b \in S_k$ (k = 1, 2) and $a, b \in T_k$ (k =1, 2):

1. Suppose $a, b \in S_k$ (k = 1, 2).

   (a) Suppose $a, b \in S_1$. Therefore $a = a_i$ and $b = a_j$ for $i, j \in A$. Consider $\phi(a) = \phi(b)$ i.e. $\phi(a_i) = \phi(a_j)$. $\Rightarrow \frac{(i-1)}{2} = \frac{(j-1)}{2}$ for $i, j \in A$. $\Rightarrow i = j$ $\Rightarrow a_i = a_j$ i.e $a = b$.

   (b) Let $a, b \in S_2$, here we have three parts.

      i. Suppose $a, b \in S_2'$. Therefore $a = a_i$ and $b = a_j$ for $i, j \in B'$. Consider $\phi(a) = \phi(b)$ i.e. $\phi(a_i) = \phi(a_j)$. $\Rightarrow m + n - \lfloor \frac{(i-1)}{2} \rfloor = m + n - \lfloor \frac{(j-1)}{2} \rfloor$ for $i, j \in B'$. $\Rightarrow \lfloor \frac{(i-1)}{2} \rfloor = \lfloor \frac{(j-1)}{2} \rfloor \Rightarrow i = j \Rightarrow a_i = a_j$ i.e. $a = b$.

      ii. Suppose $a, b \in S_2''$. Therefore $a = a_i$ and $b = a_j$ for $i, j \in B''$. Consider $\phi(a) = \phi(b)$ i.e. $\phi(a_i) = \phi(a_j)$. $m + n - \frac{i}{2} = m + n - \frac{j}{2} \Rightarrow i = j \Rightarrow a_i = a_j$ i.e. $a = b$.

      iii. Without loss of generality suppose that $a \in S_2'$ and $b \in S_2''$. Therefore $a = a_i$ and $b = a_j$ for $i \in B'$ and $j \in B''$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$. For if suppose $\phi(a) = \phi(b)$ i.e.$\phi(a_i) = \phi(a_j)$. Therefore $m + n - \lfloor \frac{(i-1)}{2} \rfloor = m + n - \frac{j}{2}$. $\Rightarrow \lfloor \frac{(i-1)}{2} \rfloor = \frac{j}{2}$. Which is not possible, since $\lfloor \frac{(i-1)}{2} \rfloor < \frac{j}{2}$ for $i \in B'$ and $j \in B''$. Thus, $\phi(a_i) \neq \phi(a_j)$ i.e. $\phi(a) \neq \phi(b)$.

   (c) Without loss of generality suppose that $a \in S_1$ and $b \in S_2$, then we have following two cases.

      i. Let $a \in S_1$ and $b \in S_2'$. Therefore $a = a_i$ and $b = a_j$ for $i \in A$ and $j \in B'$. Claim: $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$. For if suppose $\phi(a) = \phi(b)$ i.e. $\phi(a_i) = \phi(a_j)$. Therefore $\frac{i-1}{2} = m + n - \lfloor \frac{j-1}{2} \rfloor$. i.e. $m + n = \frac{i-1}{2} + \lfloor \frac{j-1}{2} \rfloor$. Which is not possible, since $m + n > \frac{i-1}{2} + \lfloor \frac{j-1}{2} \rfloor$, as $i \leq m - 1$ and $j \leq \frac{m+n+1}{2}$. Thus, $\phi(a_i) \neq \phi(a_j)$ i.e. $\phi(a) \neq \phi(b)$.

      ii. Let $a \in S_1$ and $b \in S_2''$. Therefore $a = a_i$ and $b = a_j$ for $i \in A$ and $j \in B''$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$ For, if suppose $\phi(a) = \phi(b)$ i.e. $\phi(a_i) = \phi(a_j)$. Therefore $\frac{i-1}{2} = m+n-\frac{j}{2}$. i.e. $i+j-1 = 2(m+n)$. Which is not possible since, $2(m + n) > i + j - 1$ as $i \leq m - 1, j \leq n$. Thus, $\phi(a_i) \neq \phi(a_j)$ i.e. $\phi(a) \neq \phi(b)$.

2. Suppose $a, b \in T_k$ (k = 1, 2.)

   (a) Suppose $a, b \in T_1$. Therefore $a = b_i$ and $b = b_j$ for $i, j \in D$. Suppose $\phi(a) = \phi(b)$ i.e. $\phi(b_i) = \phi(b_j)$. Therefore $\frac{m+n-i}{2} = \frac{m+n-j}{2}$. $\Rightarrow i = j \Rightarrow b_i = b_j$ i.e. $a = b$.

   (b) Suppose $a, b \in T_2$, here we have three parts.

      i. Suppose $a, b \in T_2'$. Therefore $a = b_i$ and $b = b_j$ for $i, j \in F'$. Consider $\phi(a) = \phi(b)$ i.e. $\phi(b_i) = \phi(b_j)$. $\Rightarrow \frac{m+n+i-1}{2} = \frac{m+n+j-1}{2}$. $\Rightarrow i = j \Rightarrow b_i = b_j$ i.e. $a = b$.

ii. Suppose $a, b \in T_2''$. Therefore $a = b_i$ and $b = b_j$ for $i, j \in F''$. Suppose $\phi(a) = \phi(b)$ i.e. $\phi(b_i) = \phi(b_j)$. $\Rightarrow \frac{m+n+i+1}{2} = \frac{m+n+j+1}{2} \Rightarrow i = j \Rightarrow b_i = b_j$ i.e. $a = b$.

iii. Without loss of generality $a \in T_2'$ and $b \in T_2''$. Therefore $a = b_i$ and $b = b_j$ for some $i \in F'$ and $j \in F''$. Claim: $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$ For, if suppose $\phi(a) = \phi(b)$ i.e. $\phi(b_i) = \phi(b_j)$. Therefore $\frac{m+n+i-1}{2} = \frac{m+n+j+1}{2}$. i.e. $i - 1 = j + 1 \Rightarrow i = j + 2$. Which is not possible since $j > i$ as $i \leq \frac{m+n+1}{2} - 2$, $j \leq n - 1$. $\Rightarrow \phi(b_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$.

(c) Without loss of generality, let $a \in T_1$ and $b \in T_2$, then we have following two parts.

i. Suppose $a \in T_1$ and $b \in T_2'$. Therefore $a = b_i$ and $b = b_j$ for some $i \in D$ and $j \in F'$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$ For, if suppose $\phi(a) = \phi(b)$ i.e. $\phi(b_i) = \phi(b_j)$. $\Rightarrow \frac{m+n-i}{2} = \frac{m+n+j-1}{2} \Rightarrow -i = j - 1$ which is not possible since $i \in D$ and $j \in F'$. $\Rightarrow \phi(b_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$.

ii. Suppose $a \in T_1$ and $b \in T_2''$. Therefore $a = b_i$ and $b = b_j$ for $i \in D$ and $j \in F''$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$. For, if suppose $\phi(a) = \phi(b)$ i.e. $\phi(b_i) = \phi(b_j)$. Therefore $\frac{m+n-i}{2} = \frac{m+n+j+1}{2}$ i.e. $-i = j + 1$, which is not possible since $i \in D$ and $j \in F''$. $\Rightarrow \phi(b_i) \neq \phi(b_j)$ i.e $\phi(a) \neq \phi(b)$.

3. Let $a \in S_k$ (k = 1,2) and $b \in T_k$ (k = 1, 2).

(a) Suppose $a \in S_1$ and $b \in T_1$. Therefore $a = a_i$ and $b = b_j$ for some $i \in A$ and $j \in D$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$ For, if suppose $\phi(a) = \phi(b)$ i.e. $\phi(a_i) = \phi(b_j)$. Therefore $\frac{i-1}{2} = \frac{m+n-j}{2}$. $\Rightarrow m + n = i + j - 1$ which is not possible since $m + n > i + j - 1$, since $i \leq m - 1$ and $j \leq n$. Thus, $\phi(a_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$.

(b) Suppose $a \in S_1$ and $b \in T_2$, then we have following two parts.

i. Suppose $a \in S_1$ and $b \in T_2'$. Therefore $a = a_i$ and $b = b_j$ for $i \in A$ and $j \in F'$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$. For, if suppose $\phi(a) = \phi(b)$ i.e. $\phi(a_i) = \phi(b_j)$. $\Rightarrow \frac{i-1}{2} = \frac{m+n+j-1}{2}$. $\Rightarrow m + n = i - j$ which is not possible since $m + n > i - j$, since $i \leq m - 1$ and $j \leq \frac{m+n1}{2} - 2$. Thus, $\phi(a_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$.

ii. Suppose $a \in S_1$ and $b \in T_2''$. Therefore $a = a_i$ and $b = b_j$ for $i \in A$ and $j \in F''$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$ For, if suppose $\phi(a) = \phi(b)$ i.e. $\phi(a_i) = \phi(b_j)$. $\Rightarrow \frac{i-1}{2} = \frac{m+n+j+1}{2} \Rightarrow m + n = i - j$ which is not possible since $m + n > i - j - 2$, as $i \leq m - 1$ and $j \leq n - 1$. $\Rightarrow \phi(a_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$.

(c) Suppose $a \in S_2$ and $b \in T_2$ then we have following parts.

i. Suppose $a \in S_2'$ and $b \in T_2'$. Therefore $a = a_i$ and $b = b_j$ for some $i \in B'$ and $j \in F'$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$. For, if suppose $\phi(a) = \phi(b)$ i.e. $\phi(a_i) = \phi(b_j)$. Therefore $m + n - \lfloor\frac{(i-1)}{2}\rfloor = \frac{m+n+j-1}{2}$. $\Rightarrow 2(m + n) - 2(\lfloor\frac{(i-1)}{2}\rfloor) = m + n + j - 1$. $\Rightarrow m + n = j + 2(\lfloor\frac{(i-1)}{2}\rfloor) - 1$ which is not possible since, $m + n > j - (\lfloor\frac{(i-1)}{2}\rfloor) - 1$ since, $i \leq \frac{m+n+1}{2}$ and $j \leq \frac{m+n+1}{2}$. Thus $\phi(a_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$.

ii. Suppose $a \in S_2'$ and $b \in T_2''$. Therefore $a = a_i$ and $b = b_j$ for $i \in B'$ and $j \in F'$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$. For, if suppose $\phi(a) = \phi(b)$ i.e.$\phi(a_i) = \phi(b_j)$. $\Rightarrow m + n - \lfloor\frac{(i-1)}{2}\rfloor = \frac{m+n+j+1}{2}$. $\Rightarrow 2(m+n) - 2(\lfloor\frac{(i-1)}{2}\rfloor) = m+n+j-1$. $m+n = j+2(\lfloor\frac{(i-1)}{2}\rfloor)+1$ which is not possible Since, $i \in B'$ and $j \in F'$. $\Rightarrow \phi(a_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$.

iii. Suppose $a \in S_2''$ and $b \in T_2'$. Therefore $a = a_i$ and $b = b_j$ for $i \in B''$ and $j \in F'$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$. For, if suppose $\phi(a) = \phi(b)$ i.e. $\phi(a_i) = \phi(b_j)$. Therefore $m + n - \frac{i}{2} = \frac{m+n+j-1}{2}$. $\Rightarrow (m + n) = j + i - 1$ which is not possible since, $i \in B''$ and $j \in F'$. Thus, $\phi(a_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$.

iv. Suppose $a \in S_2''$ and $b \in T_2''$. Therefore $a = a_i$ and $b = b_j$ for $i \in B''$ and $j \in F''$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$. For, if suppose $\phi(a) = \phi(b)$ i.e. $\phi(a_i) = \phi(b_j.)$ Therefore $m + n - \frac{i}{2} = \frac{m+n+j+1}{2}$. $\Rightarrow (m + n) = j + i + 1$ which is not possible since $i \in B''$ and $j \in F''$. $\Rightarrow \phi(a_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$. Hence ϕ is one - one function.

Secondly to show edge labels of L are distinct. We have edge labels of L are
$|\phi(a_i) - \phi(a_{i+1})| = m + n - (i - 1)$ for $1 \leq i \leq m - 2$.
$|\phi(a_i) - \phi(a_{i+1})| = m + n - (i)$ for $i = m - 1$.
$|\phi(b_j) - \phi(b_{j+1})| = j$ for $1 \leq j \leq n - 1$.
$|\phi(a_i) - \phi(b_j)| = \frac{(m+n-1)}{2}$ for $i = j = 1$.
$|\phi(a_m) - \phi(b_j)| = n$ for i = m and j = n. From the above labeling pattern, it is observed that the edge labels of L are distinct. Thus, lattice L has graceful labeling. □

Using proof of theorem 4.1, one can obtain the proof of the following theorems.

Theorem 3.2. *Let C and C' are chains with $|C| = m \geq 3$ and $|C'| = n \geq 1$ and $L = C]_0^1C'$. Then L has graceful labeling if $m \equiv 3(mod\ 4)$ and $n \equiv 1(mod\ 4)$.*

Proof. Let C and C' be the chains with m and n elements, respectively. Suppose $m \equiv 3(mod\ 4)$ and $n \equiv 1(mod\ 4)$. Suppose $C = a_1 \prec a_2 \prec a_3 \cdots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \cdots \prec b_n$ and $L = C]_0^1C'$. Clearly L has $m + n$ elements and $m + n$ coverings (edges). Suppose V = $\{a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3, \ldots, b_n\}$ and E = $\{0, 1, 2, \ldots, m+n\}$. Consider a map $\phi : V \to E$ defined as follows :

$$\phi(a_i) = \begin{cases} \frac{i-1}{2} & \text{if} \quad i = 1, 3, 5, \dots, m \\ m + n - \left\lfloor \frac{i-1}{2} \right\rfloor & \text{if} \quad i = 2, 4, 6, \dots, \frac{m+n}{2} \\ m + n - \frac{i}{2} & \text{if} \quad i = \frac{m+n}{2} + 2, \frac{m+n}{2} + 4 \dots, m - 1 \end{cases}$$

$$\phi(b_j) = \begin{cases} \frac{m+n+j-1}{2} & \text{if} \quad j = 1, 3, 5, \dots, \frac{m+n}{2} - 1 \\ \frac{m+n+j+1}{2} & \text{if} \quad j = \frac{m+n}{2} + 1, \frac{m+n}{2} + 3, \dots, n \\ \frac{m+n-j}{2} & \text{if} \quad j = 2, 4, 6, \dots, n - 1 \end{cases}$$

Clearly ϕ gives the required graceful labeling for L. □

Theorem 3.3. *Let C and C' are chains with $|C| = m \geq 3$ and $|C'| = n \geq 1$ and $L = C]_0^1 C'$. Then L has graceful labeling if $m \equiv 1 (mod\ 4)$ and $n \equiv 2 (mod\ 4)$.*

Proof. Let C and C' be the chains with m and n elements, respectively. Suppose $m \equiv 1 (mod\ 4)$ and $n \equiv 3 (mod\ 4)$. Suppose $C = a_1 \prec a_2 \prec a_3 \dots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \dots \prec b_n$ and $L = C]_0^1 C'$. Clearly L has $m + n$ elements and $m + n$ coverings (edges). Suppose V $= \{a_1, a_2, a_3, \dots, a_m, b_1, b_2, b_3, \dots, b_n\}$ and E $= \{0, 1, 2, \dots, m + n\}$. Consider a map $\phi : V \to E$ defined as follows :

$$\phi(a_i) = \begin{cases} \frac{i-1}{2} & \text{if} \quad i = 1, 3, 5, \dots, m \\ m + n - \left\lfloor \frac{i-1}{2} \right\rfloor & \text{if} \quad i = 2, 4, 6, \dots, \frac{m+n+1}{2} \\ m + n - \frac{i}{2} & \text{if} \quad i = \frac{m+n+1}{2} + 2, \frac{m+n+1}{2} + 4, \dots, m - 1 \end{cases}$$

$$\phi(b_j) = \begin{cases} \frac{m+n-j}{2} & \text{if} \quad j = 1, 3, 5, \dots, n - 1 \\ \frac{m+n+j-1}{2} & \text{if} \quad j = 2, 4, 6, \dots, \frac{m+n+1}{2} - 2 \\ \frac{m+n+j+1}{2} & \text{if} \quad j = \frac{m+n+1}{2}, \frac{m+n+1}{2} + 2, \dots, n \end{cases}$$

Clearly ϕ gives the required graceful labeling for L. □

Theorem 3.4. *Let C and C' are chains with $|C| = m \geq 3$ and $|C'| = n \geq 1$ and $L = C]_0^1 C'$. Then L has graceful labeling if $m \equiv 2 (mod\ 4)$ and $n \equiv 2 (mod\ 4)$.*

Proof. Let C and C' be the chains with m and n elements, respectively. Suppose $m \equiv 2 (mod\ 4)$ and $n \equiv 2 (mod\ 4)$. Suppose $C = a_1 \prec a_2 \prec a_3 \dots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \dots \prec b_n$ and $L = C]_0^1 C'$. Clearly L has $m + n$ elements and $m + n$ coverings (edges). Suppose V $= \{a_1, a_2, a_3, \dots, a_m, b_1, b_2, b_3, \dots, b_n\}$ and E $= \{0, 1, 2, \dots, m + n\}$. Consider a map $\phi : V \to E$ defined as follows :

$$\phi(a_i) = \begin{cases} \frac{i-1}{2} & \text{if} \quad i = 1, 3, 5, \dots, m - 1 \\ m + n - \left\lfloor \frac{i-1}{2} \right\rfloor & \text{if} \quad i = 2, 4, 6, \dots, \frac{m+n}{2} \\ m + n - \frac{i}{2} & \text{if} \quad i = \frac{m+n}{2} + 2, \frac{m+n}{2} + 4, \dots, m \end{cases}$$

$$\phi(b_j) = \begin{cases} \frac{m+n+j-1}{2} & \text{if} \quad j = 1, 3, 5, \dots, m - 1 \\ \frac{m+n+j+1}{2} & \text{if} \quad j = \frac{m+n}{2} + 1, \frac{m+n}{2} + 3, \dots, n - 1 \\ \frac{m+n-j}{2} & \text{if} \quad j = 2, 4, 6, \dots, n \end{cases}$$

Clearly ϕ gives the required graceful labeling for L. $\square$

Theorem 3.5. *Let C and C' are chains with $|C| = m \geq 3$ and $|C'| = n \geq 1$ and $L = C]_0^1C'$. Then L has graceful labeling if $m \equiv 0(mod\ 4)$ and $n \equiv 3(mod\ 4)$.*

Proof. Let C and C' be the chains with m and n elements, respectively. Suppose $m \equiv 0(mod\ 4)$ and $n \equiv 4(mod\ 4)$. Suppose $C = a_1 \prec a_2 \prec a_3 \cdots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \cdots \prec b_n$ and $L = C]_0^1C'$. Clearly L has $m+n$ elements and $m+n$ coverings (edges). Suppose V $= \{a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3, \ldots, b_n\}$ and E $= \{0, 1, 2, \ldots, m+n\}$. Consider a map $\phi : V \to E$ defined as follows :

$$\phi(a_i) = \begin{cases} \frac{i-1}{2} & \text{if} \quad i = 1, 3, 5, \ldots, m-1 \\ m+n-(\frac{i}{2}-1) & \text{if} \quad i = 2, 4, 6, \ldots, \frac{m+n+1}{2} \\ m+n-\frac{i}{2} & \text{if} \quad i = \frac{m+n+1}{2}+2, \frac{m+n+1}{2}+4, \ldots, m \end{cases}$$

$$\phi(b_j) = \begin{cases} \frac{m+n-j}{2} & \text{if} \quad j = 1, 3, 5, \ldots, n \\ \frac{m+n+j-1}{2} & \text{if} \quad j = 2, 4, 6, \ldots, \frac{m+n+1}{2}-2 \\ \frac{m+n+j+1}{2} & \text{if} \quad j = \frac{m+n+1}{2}, \frac{m+n+1}{2}+2, \ldots, n-1 \end{cases}$$

Clearly ϕ gives the required graceful labeling for L. $\square$

Theorem 3.6. *Let C and C' are chains with $|C| = m \geq 3$ and $|C'| = n \geq 1$ and $L = C]_0^1C'$. Then L has graceful labeling if $m \equiv 1(mod\ 4)$ and $n \equiv 3(mod\ 4)$.*

Proof. Let C and C' be the chains with m and n elements, respectively. Suppose $m \equiv 1(mod\ 4)$ and $n \equiv 3(mod\ 4)$. Suppose $C = a_1 \prec a_2 \prec a_3 \cdots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \cdots \prec b_n$ and $L = C]_0^1C'$. Clearly L has $m+n$ elements and $m+n$ coverings (edges). Suppose V $= \{a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3, \ldots, b_n\}$ and E $= \{0, 1, 2, \ldots, m+n\}$. Consider a map $\phi : V \to E$ defined as follows :

$$\phi(a_i) = \begin{cases} \frac{i-1}{2} & \text{if} \quad i = 1, 3, 5, \ldots, m \\ m+n-(\frac{i}{2}-1) & \text{if} \quad i = 2, 4, 6, \ldots, \frac{m+n}{2} \\ m+n-\frac{i}{2} & \text{if} \quad i = \frac{m+n}{2}+2, \frac{m+n}{2}+4, \ldots, m-1 \end{cases}$$

$$\phi(b_j) = \begin{cases} \frac{m+n+j-1}{2} & \text{if} \quad j = 1, 3, 5, \ldots, \frac{m+n}{2}-1 \\ \frac{m+n+j+1}{2} & \text{if} \quad j = \frac{m+n}{2}+1, \frac{m+n}{2}+3, \ldots, n \\ \frac{m+n-j}{2} & \text{if} \quad j = 2, 4, 6, \ldots, n-1 \end{cases}$$

Clearly ϕ gives the required graceful labeling for L. $\square$

Theorem 3.7. *Let C and C' are chains with $|C| = m \geq 3$ and $|C'| = n \geq 1$ and $L = C]_0^1C'$. Then L has graceful labeling if $m \equiv 0(mod\ 4)$ and $n \equiv 0(mod\ 4)$.*

Proof. Let C and C' be the chains with m and n elements, respectively. Suppose $m \equiv 0(mod\ 4)$ and $n \equiv 0(mod\ 4)$. Suppose $C = a_1 \prec a_2 \prec a_3 \cdots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \cdots \prec b_n$ and $L = C]_0^1C'$. Clearly L has $m+n$ elements and $m+n$ coverings (edges).

Suppose V $= \{a_1, a_2, a_3, \dots, a_m, b_1, b_2, b_3, \dots, b_n\}$ and E $= \{0, 1, 2, \dots, m+n\}$. Consider a map $\phi : V \to E$ defined as follows :

$$\phi(a_i) = \begin{cases} \frac{i-1}{2} & \text{if} \quad i = 1, 3, 5, \dots, m-1 \\ m+n-(\frac{i}{2}-1) & \text{if} \quad i = 2, 4, 6, \dots, \frac{m+n}{2} \\ m+n-\frac{i}{2} & \text{if} \quad i = \frac{m+n}{2}+2, \frac{m+n}{2}+4, \dots, m \end{cases}$$

$$\phi(b_j) = \begin{cases} \frac{m+n+j-1}{2} & \text{if} \quad j = 1, 3, \dots 5, \frac{m+n}{2}-1 \\ \frac{m+n+j+1}{2} & \text{if} \quad j = \frac{m+n}{2}+1, \frac{m+n}{2}+3, \dots, n-1 \\ \frac{m+n+j+1}{2} & \text{if} \quad j = 2, 4, 6, \dots, n \end{cases}$$

Clearly ϕ gives the required graceful labeling for L. □

Theorem 3.8. *Let C and C' are chains with $|C| = m \geq 3$ and $|C'| = n \geq 1$ and $L = C]_0^1 C'$. Then L has graceful labeling if $m \equiv 3(mod\ 4)$ and $n \equiv 0(mod\ 4)$.*

Proof. Let C and C' be the chains with m and n elements, respectively. Suppose $m \equiv 3(mod\ 4)$ and $0 \equiv 1(mod\ 4)$. Suppose $C = a_1 \prec a_2 \prec a_3 \cdots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \cdots \prec b_n$ and $L = C]_0^1 C'$. Clearly L has $m+n$ elements and $m+n$ coverings (edges). Suppose V $= \{a_1, a_2, a_3, \dots, a_m, b_1, b_2, b_3, \dots, b_n\}$ and E $= \{0, 1, 2, \dots, m+n\}$. Consider a map $\phi : V \to E$ defined as follows :

$$\phi(a_i) = \begin{cases} \frac{i-1}{2} & \text{if} \quad i = 1, 3, 5, \dots, m \\ m+n-(\frac{i}{2}-1) & \text{if} \quad i = 2, 4, 6, \dots, \frac{m+n+1}{2} \\ m+n-\frac{i}{2} & \text{if} \quad i = \frac{m+n+1}{2}+2, \frac{m+n+1}{2}+4, \dots, m-1 \end{cases}$$

$$\phi(b_j) = \begin{cases} \frac{m+n-j}{2} & \text{if} \quad j = 1, 3, \dots 5, m \\ \frac{m+n+j-1}{2} & \text{if} \quad j = 2, 4, 6, \dots, \frac{m+n+1}{2}-2 \\ \frac{m+n+j+1}{2} & \text{if} \quad j = \frac{m+n+1}{2}, \frac{m+n+1}{2}+2, \dots, n \end{cases}$$

Clearly ϕ gives the required graceful labeling for L. □

Conclusion

In this paper, we introduced graceful lableling for finite posets. We obtained graceful labeling of some finite posets such as a chain, a fence, and a crown. Also, we obtained graceful labeling of an adjunct sum of two chains with respect to an adjunct pair $(0, 1)$. We raise the problem of finding graceful labeling of an adjunct sum of two chains with respect to an adjunct pair (a, b) in general. Further, the problem may be extended to the class of finite dismantlable lattices/posets also.

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