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Common Fixed Point under New Contraction Framework in Digital Metric Spaces

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ARTICLE INFO	ABSTRACT
Published Online: **10 February 2025** **Corresponding Author:** **Jyoti Jhade**	This paper presents a common fixed-point theorem for a pair of weakly commutative mappings within the framework of a digital metric space, satisfying a specific contractive condition. The results obtained expand and generalize several well-established findings in the existing literature

KEYWORDS: Fixed point, Digital Metric Space, Digital Image, weakly commutative mapping.

2020 Mathematics Subject Classification: Primary: 54H25; Secondary: 54E50, 47H10.

1. INTRODUCTION

Let {un} is a sequence in complete digital metric space

In fixed point theory, various generalizations of metric spaces have been explored, one of which is the digital metric space (DMS) introduced by Ozgur Ege and Ismet Karaca [8]. This concept is closely tied to digital topology, which focuses on examining the topological and geometrical digital characteristics of images. In computer graphics and other technology-driven industries, images are treated as objects consisting of arranged points called pixels (in 2D) or voxels (in 3D). Digital topology studies these points and their adjacency relations within the image. Rosenfeld [10] was the pioneer in utilizing digital topology as a framework, investigating the properties of almost fixed points in digital images. Later, Boxer [6, 7] advanced this field by incorporating topological concepts into a digital context. Building on these developments, Ozgur Ege and Ismet Karaca [8] introduced the notion of a DMS in 2015. They demonstrated the "Banach Contraction Principle" and established various fixed-point results in this newly defined space. The study of common fixed points for various types of mappings has long been a captivating area in fixed-point theory. In 1976, Jungck [3] pioneered the introduction of commutative mappings in complete metric spaces and used their properties to derive some common fixed-point results. Over time, numerous researchers have generalized and extended these results for commutative mappings under various contractive conditions. Sessa [13] introduced the concept of weakly commutative mappings in 1982, which generalizes commutative mappings; every commutative mapping is weakly commutative, although the reverse is not necessarily true.

In recent years, weakly commutative and commutative mappings have been applied to digital metric spaces by Asha Rani et al. [1]. Additionally, researchers such as Sunjay Kumar et al. [2, 12], Sumitra Dalal [11], and Rashmi Rani [9] have presented significant findings regarding commutative and weakly commutative mappings in DMS. Inspired by these studies, this paper establishes a common fixed-point theorem for a pair of weakly commutative mappings satisfying a specific contractive condition in DMS. Our results not only broaden but also extend numerous existing findings in the literature. Prior to presenting our main result, the following definitions are provided to ensure clarity.

2. PRELIMINARIES

Definition 2.1. "Let $F \subseteq \mathbb{Z}^n$, n ∈ ℕ where $\mathbb{Z}^n$ is a lattice point set in the Euclidean n - dimensional space and (F, Υ) represent a digital image, with Υ -adjacency relation between the members of F and (F, Φ, Υ) represent a *DMS*, where (F, Φ) is a metric space."

Definition 2.2. [7] "Let l, n be two positive integers, where $1 \leq l \leq n$ and g, h are two distinct points,

$$g = (g_1, g_2, \ldots\ldots g_n),\ h = (h_1, h_2, \ldots\ldots h_n) \in \mathbb{Z}^n.$$

Then the points g and h are said to be Υ_1– adjacent if there are at most l indices i such that $|g_i - h_i| = 1$ and for all other indices j, $|g_j - h_j| \neq 1$, $g_j = h_j$."

Definition 2.3. [7] **"**Let $\kappa \in \mathbb{Z}^n$, then the set –

$$N_{\Upsilon}(\kappa) = \{ \sigma \mid \sigma \text{ is } \Upsilon - \text{adjacent to } \kappa\}$$

Represent the Υ – neighbourhood of κ for $n \in \{1, 2, 3\}$. Where $\Upsilon \in \{2, 4, 6, 8, 18, 26\}$."

Definition 2.4. [7] **"**Let $\delta, \sigma \in \mathbb{Z}$ where $\delta < \sigma$, then the digital interval is -

$$[\delta, \sigma]_{\alpha} = \{\alpha \in \mathbb{Z} \mid \delta \leq \alpha \leq \sigma \}."$$

Definition 2.5. [8] "The digital image $(F, \Upsilon) \subseteq \mathbb{Z}^n$ is called Υ –connected if and only if for every pair of different points $g, h \in F$, there is a set $\{g_0, g_1, \ldots\ldots g_s\}$ of points of digital image (F, Υ), such that $g = g_0$, $h = g_s$, and g_e and g_{e+1} are Υ -neighbours where $e = 0,1,2\ldots\ldots$ s-1."

Definition 2.6. [8] **"**Let K*:* $F \to$ K is a function and $(F, \Upsilon_0) \subset \mathbb{Z}^n{}_0$, $(K, \Upsilon_1) \subset \mathbb{Z}^n{}_1$ are two digital images. Then –

(i). K is (Υ_0, Υ_1) - continuous if there exists Υ_0 - connected subset σ of F, for every $K(\sigma)$, Υ_1- connected subset of K.

(ii). K is (Υ_0, Υ_1) - continuous if for every Υ_0 - adjacent point $\{\sigma_0, \sigma_1\}$ of F, either $K(\sigma_0) = K(\sigma_1)$ or $K(\sigma_0)$ and $K(\sigma_1)$ are Υ_1 - adjacent in K.

(iii). K is said to be (Υ_0, Υ_1) - isomorphism, if K is (Υ_0, Υ_1) - continuous bijective and K^{-1} is (Υ_0, Υ_1)- continuous, also it is denoted by $F \cong K_{(\Upsilon_0, \Upsilon_1)}$."

Definition 2.7. [8] **"**Let a $(2, \Upsilon)$ continuous function K*:* $[0, \sigma]_z \to F$ s.t. $K(0) = \alpha$ and $K(\sigma) = \beta$. Then in the digital image (F, Υ), it is called a digital Υ - path from α to β*."*

Definition 2.8. [10] **"**Let K*:* $(F, \Upsilon) \to (F, \Upsilon)$ be a (Υ, Υ) - continuous function on a digital image (F, Υ), then we said that the property of fixed point satisfied by the digital image (F, Υ) if for every (Υ, Υ) - continuous function K*:* $F \to F$ there exists $\alpha \in F$ such that $K(\alpha) = \alpha$*."*

Definition 2.9. [8] "Let $\{u_n\}$ is a sequence in digital metric space (F, Φ, Υ), then the sequence $\{u_n\}$ is called-

(i). Cauchy sequence if and only if there exists $\varrho \in$ N such that, $\Phi(u_n, u_m) < \epsilon, \forall\, n, m > \varrho$.

(ii). Converge to a limit point $\ell \in F$ if for every $\epsilon > 0$, there exists $\varrho \in N$ such that for all $n > \varrho$, $\Phi(u_n, \Upsilon) < \epsilon$."

Theorem 2.10. [8] *"A digital metric space* (F, Φ, Υ) *is complete."*

Definition 2.11. [8] **"**Let K*:* $(F, \Phi, \Upsilon) \to (F, \Phi, \Upsilon)$ be a self-map. Then K is called a digital contraction if, for all $u, \sigma \in F$ there exist $\tau \in [0, 1)$ such that,

$$\Phi(K(u), K(\sigma)) \leq \tau\, \Phi(u, \sigma)."$$

Proposition 2.12. [8] *"Every digital contraction map* K*:* $(F, \Phi, \Upsilon) \to (F, \Phi, \Upsilon)$ *is digitally* Υ *- continuous."*

Definition 2.13. [11] **"***Let* J, K*:* $F \to F$ are two self-mappings on (F, Φ, Υ). Then the point $\sigma \in F$ is said to be a coincidence point of J and K if $J(\sigma) = K(\sigma)$. Furthermore, if $J(\sigma) = K(\sigma) = \eta$ then η is said to be a point of coincidence for mappings J and K."

Definition 2.14. [1] **"**Let J, K*:* $(F, \Phi, \Upsilon) \to (F, \Phi, \Upsilon)$ are two mappings defined on the digital metric space (F, Φ, Υ). Then these mappings are called weakly commutative mappings if

$$\Phi(J(K(\sigma)), K(J(\sigma))) \leq \Phi(J(\sigma), K(\sigma)),\ \forall\, \sigma \in F."$$

Lemma 2.15. [1] "Let $\{u_n\}$ is a sequence in complete digital metric space (F, Φ, Υ), and if there exists $\rho \in (0, 1)$, such that $\Phi(u_{n+1}, u_n) \leq \Phi(u_n, u_{n-1})$ for all n then, sequence $\{u_n\}$ converges to a point in F."

3. MAIN RESULT

Theorem 3.1 Let (F, Φ, Υ) be a complete Digital Metric Space (DMS), where Υ is an adjacency, and Φ is the usual Euclidean metric on $\mathbb{Z}^n$. Let $J, K: F \to F$ be two self-mappings satisfying:

1. K is continuous.
2. The pair $\{J, K\}$ is weakly commutative.
3. For all $u, q \in F$, the following contraction condition holds:
   $$\Phi(Ju, Jq) + \Phi(Ku, Kq) \leq \xi \Phi(Ku, Kq),$$
   where $0 \leq \xi < 1$.

Then J and K have a unique common fixed point in F.

Proof: Let $u_0 \in F$ be an arbitrary point. Since $J(F) \subseteq K(F)$, there exists $u_1 \in F$ such that:

$$J(u_0) = K(u_1).$$

Construct the sequence $\{u_n\}$ in F such that:

$$K(u_n) = J(u_{n-1}), \quad \forall n \geq 1.$$

For $u_n, u_{n-1} \in F$, apply the given contraction condition:

$$\Phi(Ju_n, Ju_{n-1}) + \Phi(Ku_n, Ku_{n-1}) \leq \xi \Phi(Ku_n, Ku_{n-1}).$$

Denote $d_n = \Phi(Ku_n, Ku_{n-1})$. Then:

$$\Phi(Ju_n, Ju_{n-1}) + d_n \leq \xi d_n.$$

Rearrange to obtain:

$$\Phi(Ju_n, Ju_{n-1}) \leq (\xi - 1) d_n.$$

Since $0 \leq \xi < 1$, $(\xi - 1) < 0$, which implies:

$$\Phi(Ju_n, Ju_{n-1}) \leq 0.$$

Hence, $\Phi(Ju_n, Ju_{n-1}) = 0$, and $d_n = \Phi(Ku_n, Ku_{n-1}) \to 0$ as $n \to \infty$.

Since $d_n = \Phi(Ku_n, Ku_{n-1}) \to 0$, it follows that:

$$\Phi(Ku_n, Ku_{n-1}) \to 0 \quad \text{and} \quad \Phi(Ju_n, Ju_{n-1}) \to 0.$$

By the continuity of K, the sequence $\{K(u_n)\}$ converges to some point $\beta \in F$. From the construction, $J(u_n) \to \beta$ as well. To show uniqueness, Suppose there are two fixed points β_1 and β_2 such that:

$$J(\beta_1) = K(\beta_1) = \beta_1 \quad \text{and} \quad J(\beta_2) = K(\beta_2) = \beta_2.$$

Applying the contraction condition for β_1 and β_2:

$$\Phi(J\beta_1, J\beta_2) + \Phi(K\beta_1, K\beta_2) \leq \xi\Phi(K\beta_1, K\beta_2).$$

Since $K\beta_1 = \beta_1$ and $K\beta_2 = \beta_2$, this becomes:

$$\Phi(\beta_1, \beta_2) + \Phi(\beta_1, \beta_2) \leq \xi\Phi(\beta_1, \beta_2).$$

Simplify:

$$2\Phi(\beta_1, \beta_2) \leq \xi\Phi(\beta_1, \beta_2).$$

Rearranging gives:

$$(2 - \xi)\Phi(\beta_1, \beta_2) \leq 0.$$

Since $2 - \xi > 0$, it follows that $\Phi(\beta_1, \beta_2) = 0$, implying $\beta_1 = \beta_2$.

Using the contraction condition $\Phi(Ju, Jq) + \Phi(Ku, Kq) \leq \xi\Phi(Ku, Kq)$, we have proved that the mappings J and K have a unique common fixed point in F.

4. CONCLUSION

This paper focuses on introducing the concept of weakly commutative mappings in the context of DMS. By utilizing these mappings and their variations, a digital common fixed-point theorem is established. The results presented expand and generalize several existing findings in the literature. This result has applications in fixed-point theory, particularly in the compression of digital images, offering potential benefits in image processing and optimizing image storage.

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