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Optimal Control of Linear Delay Systems with Delays in State and Control

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ABSTRACT

This paper extends the work of [6] to establish sufficient conditions for the existence and uniqueness of optimal control of linear delay systems with distributed delays in state and control. It is shown that if the system is relatively controllable, the optimal control is unique and bang bang and is of the form

$$u^* = sgn\, c^T \int_{-h}^{0} X(t_1, t-s)dH(t-s,s)$$

KEYWORDS: Optimal control, Distributed delays, Null controllability.

1. INTRODUCTION

The study of optimal control by the control theorist is fast becoming fundamental as it presents the best amongst alternatives. This has led to a thorough and careful presentations of the current status of control theory. The objective here is to present an organized treatment of the optimal control of linear systems with distributed delays in the state and control. There are many definitions of optimal control derivable from controllability which strongly depends on the class of systems we are dealing with. However, it should be stressed that, whatever method is used the result is achieved in minimum time and with minimum energy. In the same view, the problem reaching the origin in time t, corresponds to null controllability of the given system. E.N. Chukwu [3] settled the time optimal control problem of linear Neutral Functional systems without delay in the control given by

$$\frac{d}{dt}D(t, x_t) = L(t, x_t) + Bu(t)$$

where the control set is in a unit cube in the m-dimensional Euclidean space and the target is a continuous function in an n-dimensional Euclidean space also.

In this work, necessary and sufficient conditions for the existence and uniqueness of optimal control is given. However, in some cases stability of the systems under study have been established as in [2] and [7].

In a related work, Onwuatn [6] studied the system;

$$\dot{x}(t) = Ax(t) + \sum_{j=0}^{p} B_j x(t-j) + \sum_{j=0}^{p} D_j u(t-j)$$

and resolved the problem of optimal control of discrete systems in which he showed that, if a system is relatively controllable, then it is sufficient for it to be optimally controllable.

Eke and Nse also diagnosed the optimal control of Neutral systems with a non-linear base given by

$$\frac{d}{dt}(D(t, x_t) = A(x,t) + B(t)u(t)$$

They employ the method of the maximum principle of Pontryagin to be able to obtain the term of the optimal control. They were able to show that, if the optimal control exists, then it is unique and bang - bang. Klamka investigated the system,

$$\dot{x}(t) = A(t)x(t) + \int_{-h}^{0} [dsH(t,s)u(t+s)]$$

and gave conditions for the relative controllability of the system. In this work, we considered that the delays are distributed in both the state and the control variables. We followed Onwuatu to attempt the relative controllable using square integrable controls

2. NOTATIONS AND PRELIMINARIES

In this study, we consider the linear delay system given by

$$\dot{x}(t) = L(t, x_t) + \int_{-h}^{0} [dsH(t,s)u(t+s) \qquad (1.1)$$

$$x(t) = E^n, \qquad u(t) = E^m,$$

where

$$L(t, x_t) = \sum_{k=0}^{\infty} A_k x(t - W_k) + \int_{-h}^{0} A(t,\theta)x(t,\theta) \qquad (1.2)$$

satisfied almost everywhere on $[t_0, t_1]$. Let n, m be positive integers, $E = (-\infty, +\infty)$be the real line. $E^n$is the n-dimensional Euclidean space with the Euclidean norm denoted by $\|\cdot\|$; j is any interval in E. The usual Lebesque space of square integrable (equivalent class of) functions from $J \to E^n$ be denoted by $L_2(J, E^n) \cdot L_1([t_0, t_1], E^n)$ denotes the space of integrable functions from $L_2(t_0, t_1)$ to $E^n \cdot N_{n,m}$ will be used for the collection of all $n \times m$matrices with a suitable norm. let $h > 0$be given. For a function $x: [t_0 - h, t] \to E^n$ and $t \in [t_0, t_1]$, we use the symbol x_t to denote the function on $[-h, 0]$ defined by $x_t(s) = x(t + s)$ for $s \in [-h, 0]$. The symbol $C = C([-h, 0], E^n)$ denotes the space of continuous functions mapping the interval $[-h, 0]$, $h > 0$ into E^n. Similarly, for functions, $U: [t_0 - h, t_1] \to E^n$, $t \in [t_0, t_1]$, We use u_t to denote the function on [-h,0] defined by $u(s) = u(t + s)$ for $s \in [-h, 0]$. $x(t) \in C$; $u \in L_2([t_0, t_1], E^m)$; $L(t, \varphi)$ is continuous in t and linear in φ; $H(t, s)$ is an $n \times m$matrix valued function which is measurable in (t, s). We shall assume that $H(t, s)$ is of bounded variation in S on $[-h, 0]$ for each $t \in [t_0, t_1]$; $A(t) \in L_1([t_0, t_1], N_{n,m})$. Throughout the sequel, the control sets of interest are

$B = L_2([t_0, t_1], N_{n,m})$, $U \subseteqq L_2([t_0, t_1], E^m)$ a closed and bounded subset of B with zero in the interior relative to B. If X and Y are linear spaces and $T: X \to Y$is a mapping, we shall use the symbols D(T), R(T) and N(T) to denote the domain, range and null spaces of T respectively.

Definition 1.1: The complete stale of system (1.1) at time t is given by

$$z(t) = \{x(t), x_t, u_t\} \qquad (1.3)$$

Definition 1.2: System (1.1) is relatively controllable on $[t_0, t_1]$ if for every initial complete state z(t_0) and every $x_1 \in E^n$, there exists a control $U \in B$ such that the corresponding trajectory of system (1.1) satisfies $x(t_1) = x_1$ whenever $x(t_0) = x_0$

Definition 1.3: System (1.1) is said to be relatively null controllable if in definition (1.2), the response $x(t)$ of the system satisfies $x(t_1) = 0$

Definition 1.4: System (1.1) is said to be optimally controllable if in the set of admissible controls $U \in B$, there exists a $u^* \in U$ such that the trajectory of system (1.1) satisfies $x(t_1) = x_1$ in minimum time.

The solution of system (1.1) is of the form

$$x(t, t_0, \varphi, u) = X(t_0, t_1)\varphi(0) + \int_{t_0}^{t} X(t, \tau)[\int_{-h}^{0} dXH(t, s)u(\tau + s)]d\tau \qquad (1.4)$$

where

$$X(\cdot, s) + X(t + \theta, s); -h < \theta < 0 \qquad (1.5)$$

and $X(t, s)$ is the fundamental matrix solution of

$$\dot{x} = L(t, x_t) \qquad (1.6)$$

satisfying

$$\frac{dX(t,s)}{dt} = L(t, X_t(\cdot, s)) \qquad (1.7)$$

almost everywhere in (t, s) and

$$X(t, s) = \begin{cases} 0 & s - h \le t < s \\ I & t = s, \ I = identity \end{cases} \qquad (1.8)$$

We now define the $n \times m$ controllability matrix of system (1.1) by

$$W(t_0, t_1) = \int_{t_0}^{t_1} \left\{ [\int_{-h}^{0} X(t_1, \tau - s)d\overline{H}(\tau - s, s)][\int_{-h}^{0} X(t_1, \tau - s)d\overline{H}(\tau - s, s)]^T d\tau \qquad (1.9)$$

where

$$\overline{H}(t, s) = \begin{cases} H(t, s) & for \ t \le t_1, \ s \in E \\ 0 & for \ t > t_1, \ s \in E \end{cases} \qquad (1.10)$$

and T denotes the matrix transpose.

Definition 1.5: The Reachable set $R(t_1, t_0)$ of system (1.1) is the subset of E^n given by

$$R(t_1, t_0) = \left\{ \int_{t_0}^{t_1} [\int_{-h}^{0} X(t_1, \tau - s)d\overline{H}(\tau - s, s)]u(\tau)d\tau, u \in U \right\} \qquad (1.11)$$

Definition 1.6: System (1.1) is said to be proper in E^non $[t_0, t_1]$ if

$$C^T[\int_{-h}^{0} X(t_1, \tau - s)d\overline{H}(\tau - s, s)] = 0 \qquad (1.12)$$

at most everywhere, $t \in [t_0, t_1]$, $c \in E^n$ implies that $c = 0$.

Definition 1.7: The attainable set $A(t)$ of system (1.1) is given by $A(t) = \{x(t, u); u \in U\}$ and is the set of all possible solutions of the system. The optimal control problem seeks to determine an admissible control u^*such that the solution $x(t, \varphi, u^*)$ of a given system hits a target point in minimum time t^*. Here $u^*$is the optimal control and t^*, the optimal time. The optimal question can thus be answered: u^* is an optimal control if there exist $t^* = [t_1, t_0]$ for $t_1 \ge t_0$ and $u^* = \{min\, U : A(t, x) \cap G(t, u) \neq 0$ for some $t \ge t_1$. Let $z(t)$ be a continuous target of the general control system given by system (1.1), if there exists an admissible control $u \in U$and a time $t \ge 0$ for which $x(t_1, u) = z(t)$, then there exists an optimal control, that is, the solution hits the target in minimum time.

3. MAIN RESULTS

Here we state one proposition and one theorem for the relative controllability of system (1.1)

Proposition 2.1: The following statements are equivalent

(i) $W(t_0,t_1)$ is non-singular for each $t_1 > t_0$

(ii) System (1.1) is proper in E^n for each interval $[t_0,t_1]$

(iii) System (1.1) is relatively controllable on each interval $[t_0,t_1]$

Proof: $(i) \Rightarrow (ii)$

Let

$$W(t_0,t_1) = \int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)][\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)^T d\tau \quad (3.1)$$

Define the operator $K: L_2([t_0,t_1],E^n) \to E^n$.

$$K(u) = \int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)][\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]u(\tau)d\tau \quad (3.2)$$

K is a continuous linear operator from a Hilbert space to another. Thus $R(k) \subset E^*$is a linear subspace and its orthogonal complement satisfies the relation

$$(R(k))' = N(k^*) \quad (3.3)$$

where k^* is the adjoint of K. By the non – singularity of $W(t_0,t_1)$, the symmetric operator

$KK^T = W(t_0,t_1)$ is positive definite and hence

$$\{R(k)' = \{0\}\} \quad (3.4)$$

for any $c \in E^n$, $u \in L_2$; $<c,ku> = <k^*c,u>$

$$<c,ku> = <c\int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]u(\tau)d\tau> \quad (3.5)$$

$$= \int_{t_0}^{t_1} C^T \int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]u(\tau)d\tau \quad (3.6)$$

Thus, k is given by

$$C \to C^T \int_{t}^{t_1}[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s);\tau \in [t_0,t_1]$$

$N(k^*)$ is therefore the set of all such $c \in E^n$ such that

$$C^T\left[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)\right] = 0 \quad (3.7)$$

almost everywhere in $[t_0,t_1]$. since $\{N(k^*) = \{0\}\}$, all such c are equal to zero; that is $c = 0$. This establishes properness of system (1.1), *That is* $(ii) \to (iii)$

We now show that if system (1.1) is proper, then it is relatively controllable on each interval $[t_0,t_1]$ Let $c \in E^n$, if system (1.1) is proper, then

$$C^T\left[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)\right] = 0 \quad (3.8)$$

almost everywhere implies $c = 0$

Thus,

$$\int_{t_0}^{t_1} C^T[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]u(\tau)d\tau = 0$$

For $u \in L_2$, It follows that, the only vector orthogonal to the set

$$R(t_1,t_0) = \{\int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau - s)dH(\tau - s,s)]u(\tau)d\tau; u \in L_2\}$$

is the zero vector. Hence $R(t_1,t_0) = 0$, that is, $R(t_1,t_0) = E^n$. This means that the system is Euclidean controllable and hence relative controllable on $[t_1,t_0]$. That is $(iii) \Rightarrow (i)$.

We now show that, if the system is relatively controllable then the controllability grammian $W = W(t_0,t_1)$ is non – singular

Let us assume for a contradiction that W is singular. Then there exists an n – vector v such that $vWU^T = 0$. Then

$$\int_{t_0}^{t_1}\left\|v[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]\right\|^2 d\tau = 0 \quad (3.9)$$

This implies that

$$\left\|v[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]\right\|^2 d\tau = 0$$

almost everywhere. Hence

$$v[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)] = 0,$$

almost everywhere for $t \in [t_0,t_1]$. This contradicts the assumption of properness since $v \neq 0$. This completes the proof.

Theorem 3.1

System (1.1) is relatively controllable if and only if $0 \in intR(t_0,t_1)$ for each $t_1 > t_0$.

Proof:

$R(t_0,t_1)$ is a closed and convex subset of E^n. Therefore a point y_1 on the boundary of $R(t_0,t_1)$ implies that there is a support plane π of $R(t_0,t_1)$ through y_1. That is $C^T(y - y_1) \leq 0$ for each $y \in R(t_0,t_1)$ where $c \neq 0$ is an outward normal to π. If u, is the control corresponding to y_1, we have

$$C^T\int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau,s)d\overline{H}(\tau - s,s)]u(\tau)d\tau \leq C^T\int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]u(\tau)d\tau \quad (3.10)$$

For each $u \in U$. This last inequality holds if and only if

$$C^T\int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]u(\tau)d\tau \leq C^T\int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]u_1(\tau)d\tau$$

$$= \int_{t_0}^{t_1}\left|C^T\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]u(\tau)\right|d\tau \quad (3.11)$$

and

$$u(t) = sgn[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]u(\tau) \quad (3.12)$$

As $y_1$is on the boundary, since we always have $0 \in R(t_0,t_1)$, if zero were not in the interior of $R(t_0,t_1)$, then it is on the boundary. Hence from preceding argument, this implies that

$$0 = \int_{t_0}^{t_1}\left|C^T[\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s)]u(\tau)\right|d\tau \quad (3.13)$$

So that

$$C^T\int_{-h}^{0} X(t_1,\tau - s)d\overline{H}(\tau - s,s) = 0$$

at most everywhere on $t \in [t_0, t_1]$. This by definition of properness if systems implies that the system is not proper, since $c \neq 0$.

Hence, if $0 \in int\ \ R(t_0, t_1)$, then

$$C^T \int_{-h}^{0} X(t_1, \tau - s) d\overline{H}(\tau - s, s) = 0$$

almost everywhere on $t \in [t_0, t_1]$ would imply $c = 0$ proving properness of system (1.1). We conclude that the system is relatively controllable on each interval.

4. OPTIMALITY CONDITION OF THE SYSTEM

We now return to our original goal of hitting a continuously moving target $z(t)$ in minimum time. Consider the trajectory of the system (1.1) given by

$$x(t_1, t_0, \varphi, u) = X(t_1, t_0, \varphi(0)) + [\int_{-h}^{0} ds\overline{H}(\tau - s, s) u(\tau, s) d\tau$$

or equivalently

$$W(t) = z(t) - x(t) \qquad (4.1)$$

Then reaching $z(t)$ at time t corresponds to $z(t) - x(t) \cong W(t) \in R(t, 0)$

We now show that if $u^*$is the optimal control with time t^* the optimal time, then

$$z(t^*) = X(t^*, 0)\varphi(0) - \int_{-h}^{0} X(t^*, \tau - s)[\int_{-h}^{0} ds\overline{H}(t^*, \tau) u_0 ds \equiv u(t^\tau) t \partial R(t^*, 0) \qquad (4.2)$$

That is $u(t^*)$ is on the boundary of the constrained reachable set.

Theorem 4.1

Let $u^*(t)$ be the optimal control with t^* the minimum time, then $u(t^*) \in \partial R(t^*, 0)$ is on the boundary of $R(t^*, 0)$

Proof:

Assume $u^*$is used to hit $\omega(t)$ in time t^*, then

$$z(t^*) - X(t^*, 0)\varphi(0) - \int_{t_0}^{t_1} X(t^*, \tau - s) \left[\int_{-h}^{0} ds\overline{H}(t^*, \tau\right) u_0 d\tau \equiv u(t^*) \in R(t^*, 0) \qquad (4.3)$$

Assume $u(t^*)$ is not on the boundary then $u(t^*) \in int$ $R(t^*, 0)$, $t^* > 0$. Hence, there exists a ball $B(u(t^*), r)$ of radius r, about $u(t^*)$ such that $B(u(t^*), r) \in R(t^*, 0)$. Since $R(t^*, 0)$ is a continuous function of t, there exists a $\delta > 0$ such that $B(u(t^*), r) \in R(t^*, 0)$ for

$t^* - \delta \leq t \leq t^*$. Therefore $u(t^*) \in \partial R(t^*, 0)$ for $t - \delta \leq t$.

This contradicts the optimality of t^*, Hence

$$u(t^*) \in \partial R(t^*, 0)$$

Theorem 4.2

If u^*be an optimal control transferring system (1.1) from $x(0)$ to $Z(t^*)$ in minimum time, t^*, then there exists a non-zero function $C \in E^n$ such that

$$u^*(t) = sgn\{C^T X(t, \tau - s)\overline{H}\} \qquad (4.4)$$

Proof:

Define $y(t) - X(t, \tau - s)\overline{H}$

$$u(t^*) = z(t^*) - X(t^*, 0)\varphi(0) - \int_{t_0}^{t_1} X(t^*, \tau - s) \left[\int_{-h}^{0} ds\overline{H}(t^*, \tau\right) u_0 d\tau \qquad (4.5)$$

That is,

$$u(t^*) = \int_{t_0}^{t_1} X(t^*, \tau - s) H u^*(\tau) d\tau$$

$$H = \int_{t_0}^{t^*} X(t^*, \tau - s) H u^*(\tau) d\tau$$

From theorem (3.1), $u(t^*)$ is on the boundary $\partial R(t^*, 0)$ of constrained reachable set. The supporting hyper plane theorem (see Hermes and Lasalle[4]) then implies the existence of a non trivial hyperplane with outward normal c (say) supporting $\partial R(t^*, 0)$ at $u(t^*)$. In other words,

$C^T u(t^*) \geq c^T y$ for all $y \in R(t^*, 0)$

That is

$C^T \int_{t_0}^{t_1} X(t^*, \tau - s) H u^*(\tau) d\tau \geq C^T \int_{t_0}^{t^*} X(t^*, \tau - s) H u(\tau) d\tau$ for all $u \notin U$

Rearranging gives

$$C^T \int_{t_0}^{t^*} X(t^*, s) H[u^*(\tau) - u(\tau)] d\tau \geq 0$$

This can happen only if

$$u^* = sgn\{C^T X(t^*, \tau - s) H\} \qquad (4.6)$$

5. APPLICATION AND CONCLUSION

Consider the Simple Harmonic Oscillator given by

$$\ddot{x} + x = u(t); \ \ |u| \leq 1$$

The principal matrix solution of (4.1) above is

$$X(t) = \begin{pmatrix} cost & sint \\ -sint & cost \end{pmatrix}$$

From which we infer that

$$X^{-1}(t) = \begin{pmatrix} cost & -sint \\ sint & cost \end{pmatrix}$$

We can easily verify by the Kalman Rank condition that the system is controllable, that is rank $(B, AB) = n = 2$

Also, the eigenvalues are $\pm i$ indicating non – negative real parts. Hence, by Brunday [10], the solution is uniformly asymptotically stable. This solution goes to zero as $t \to \infty$. Since the system is controllable, there exists an optimal control $u^*(t)$ that drives the solution to the origin in finite time t. this optimal control is of the form of Hermes and Lasalle in [4]

That is $sgn(C^T Y(t))$, were

$$C^T Y(t) = (C_1 \quad C_2) \begin{pmatrix} cost & -sint \\ sint & cost \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= C_1 S\,int + C_2 Cost \equiv (C_1^2 + C_2^2)(Sin(t + \delta)); t \leq \delta \leq \pi$$

$$sgn(C^TY(t) = sgn[sin(t+\delta)] = \begin{cases} 1 \ \ if \ sin(t+\delta) > 0 \\ 0 \ \ if \ sin(t+\delta) = 0 \\ -1 \ \ if \ sin(t+\delta) < 0 \end{cases}$$

This illustrate that the Simple Harmonic Oscillator can optimally be controlled.

In conclusion, we have shown that a linear delay system with distributed delays in state and control can be relatively controlled if the system is proper and the controllability Grammian is non-singular. We also show that a necessary condition for existence of the optimal control is that it must be on the boundary of the reachable set. We proceeded by showing the form of the optimal control for the system in question. Finally, we join Chukwu in [3] and Onwuata in [6] to conclude affirmatively that if a system is relatively controllable, then optimal control is unique and Bang-Bang

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