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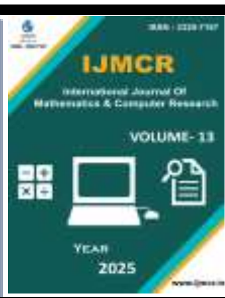

On Contra hc-Continuous Functions

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ABSTRACT

In this paper, by means of hc − open sets, we introduce and investigate certain ramifications of contra − continuous and allied functions, namely, contra hc − continuous , perfectly contra hc − $f$unctions, contra hc − open functions and contra hc − closed functions . along with their several properties, characterizations and mutual relationships. Further, we introduce new type of graphs, called contra hc − closed graphs via hc − open sets. The relationship between these graphs and contra hc − continuous $f$unctions are studied .

KEYWORDS: hc − open(closed) ; contra hc − continuous functions ; pre − hc − open(closed) functions ; contra hc − open(closed) functions ; perfectly hc − continuous function

1. INTRODUCTION

In 2021[1], Fadhil H. A. obtained a new class of sets in a topological space, known as h − open sets and in 2022 Elsaid L. , Amjad A. Al-Rehili [2] introduced and studied the notion of hc − open . In the year 1996 , Dontchev J. [3] introduced a concept known as contra − continuous . In 1999 [4], Jafari S. , Noiri T. introduced new generalization of contra continuity called contra super continuity. In 2009 [5], Jawad J. K. , Mustafa , H. J. introduced and studied Certain Types of contra − continuous $f$uction. Also in 2023 [6], Fadhil H. A introduced new type of continuity called contra h − continuous function .

The aim of this research is to study the properties of one types of contra − continuous functions , namely, contra hc − continuous, this type introduced by uses concept of hc − open sets . We define this class of functions by the requirement that the inverse image of each open set in the codomain is hc − closed set in the domain.

2. PRELIMINARIES

In this work X,Y,Z means topological space, If (X,τ) is a topological space and ω is subset of X then $\imath nt(\omega)$, $Cl(\omega)$ means the interior of ω, closure of ω respectively . A space X is said to be extremelly disconnected if the closure of any open set is open [3] . A subset ω of X is called regular open if $\omega = \imath nt\, Cl(\omega)$, respectively regular Closed if $\omega = Cl\,(\imath nt(\omega))$[7] . ω is called δ − open if $\omega = \cup_{i\in I}\mathcal{M}_i$ where $\mathcal{M}_i$ is regular open for each $i \in I$ [8] . A subset ω_0 of the topological space X is called h − open set if for each non-empty set ω_1 in X , $\omega_1 \neq X$ and $\omega_1 \in O(X)$, where , $\omega_0 \subseteq \imath nt(\omega_0 \cup \omega_1)$. The complement of the h − open set is said to be h − closed , every open [closed] in a topological space X is h − open[h − closed] [1] . For a topological space X and $\omega \subseteq X$, Intersection of all open sets of X containing ω is called kernel of ω and is denoted by $\kappa er(\omega)$ [3]. By a topological function $f : X \to Y$, it is meant a function from a topological spaceX to another topological spaceY and denoted by $(TO\,.\,f)$ O(X) [C(X)] is the set of all open [closed] sets in X , $O_h(X)$ $[C_h(X)]$ is the set of all h − open [h − closed] sets in X . We will use the symbol ∎ to indicate end of the proof.

Definition 2. 1[2] Let ω be a subset and h − open of a space X , ω is said to be hc − open if for every $x \in \omega$, there exists a closed set $\mathcal{M}$ and $x \in \mathcal{M} \subseteq \omega$. The set of all hc − open in a topological space X is defend by notation $O_{hc}(X)$, and $X - \omega$ in a topological space X is called hc − closed .

The denote $C_{hc}(X)$ means of the set of all hc − closed sets in a topological spaces X , ω is called hc − clopen if $\omega \in O_{hc}(X)$ and $\omega \in C_{hc}(X)$

Remark 2. 2, Let X be a topological space , clearly

1) If $\omega \in O(X)$, then $\omega \in O_h(X)$ [1]
2) If $\omega \in O_{hc}(X)$, then $\omega \in O_h(X)$ [2] .
3) If ω is clopen set in X , then $\omega \in O_{hc}(X)$ [2] .
4) If X is finite space and $\omega \in O_{hc}(X)$, then $\omega \in C(X)$[2].

But The converse of this statements , need not be true in general as explain in the following example.

Let $X = \{\ell_1, \ell_2, \ell_3\}$ with topology T = $\{X, \emptyset, \{\ell_1\}, \{\ell_2\}, \{\ell_1, \ell_2\}\}$, then

1) $\{\ell_3\} \in O_h(X)$ but $\{\ell_3\} \notin O(X)$
2) $\{\ell_2\} \in O_h(X)$ but $\{\ell_2\} \notin O_{hc}(X)$

3) $\{\ell_2,\ell_3\} \in O_{hc}(X)$ but not clopen .
4) If $T=\{X,\emptyset,\{\ell_1\}\}$, $\{\ell_2,\ell_3\} \in C(X)$, but $\{\ell_2,\ell_3\} \notin O_{hc}(X)$.

Remark 2.3 [2] If X is a topological space , then
1) If X is a regular space and ω is open , then $\omega \in O_{hc}(X)$
2) If X is an extremelly disconnected and ω is $\delta-$open , then $\omega \in O_{hc}(X)$

Remark 2.4[2] Let X be a topological space and $\omega_1, \omega_2 \in O_{hc}(X)$ then
1) $\omega_1 \cap \omega_2 \in O_{hc}(X)$.
2) $\omega_1 \cup \omega_2 \in O_{hc}(X)$.

Definition 2.5 Let X be a topological spacelet $\omega \subseteq X$, $p \in X$ we say that p is:
1) hc − adherent point of ω, if every $G \in O_{hc}(X)$ and containing p meets ω .The set of all hc − adherent point of ω is called the hc − closure of ω and denoted by $hc-Cl(\omega)$[2]
2) hc − interior point of ω, if there exists an $G \in O_{hc}(X)$ and containing p contained in ω. The set of all hc − interior points of ω is called hc − interior of ω , and denoted by $hc-\iota nt(\omega)$. [2]

Remark 2.6 [2]
i) $\omega \in C_{hc}(X)$ if and only if $\omega = hc-Cl(\omega)$.
ii) $\omega \in O_{hc}(X)$if and only if $\omega = hc-\iota nt(A)$.

Definition 2.7 A subset ω of a topological space X is said to be hc − dense in X if $hc-Cl(\omega) = X$. [2]

For example in remark (2.2) , $\omega = \{\ell_1,\ell_3\}$ is hc − dense , since $hc-Cl(\{\ell_1,\ell_3\}) = X$.

Definition 2.8 A topological space X is said to be
1) Urysohn space, if for each pair of distinct points x_1 & x_2 in X, there exists ω_1 and $\omega_2 \in O(X)$ such that $x_1 \in \omega_1$, $x_2 \in \omega_2$ & $Cl(\omega_1) \cap Cl(\omega_2) = \emptyset$. [7]
2) ultra − normal, if each pair of non–empty disjoint closed sets can be separated by disjoint closed sets.[9]
3) hc − Hausdorff space, if for each two points $x_1 \neq x_2$ in X ,there exists two ω_1 & $\omega_2 \in O_{hc}(X)$,such that $x_1 \in \omega_1$, $x_2 \in \omega_2$ & $\omega_1 \cap \omega_2 = \emptyset$.
4) hc − normal space, if for each disjoint H_1 , $H_2 \omega \in C(X)$,there exists two ω_1 & $\omega_2 \in O_{hc}(X)$ such that $H_1 \subseteq \omega_2$, $H_2 \subseteq \omega_2$ & $\omega_1 \cap \omega_2 = \emptyset$.

Definition 2.9 Let X be a topological space, X is said to be hc − disconnected if it is the union of two nonempty hc − open disjoint subsets , otherwise X called is hc − connected For example : let $X = \{\ell_1, \ell_2, \ell_3\}$ with topology $T = \{X, \emptyset, \{\ell_1\}\}$ is hc − connected .

Definition 2.10. A TO. f $f: X \to Y$ is called:
1) perfectly continuous, if every $\omega \in O(Y)$, then $f^{-1}(\omega)$is clopen subset of X . [5]
2) h − continuous , if $\omega \in O(Y)$, then $f^{-1}(\omega) \in O_h(X)$.**[1]**
3) hc − continuous , if $\omega \in O(Y)$, then $f^{-1}(\omega) \in O_{hc}(X)$.**[2]**
4) contra − continuous , if every $\omega \in O(Y)$, then $f^{-1}(\omega) \in C(X)$. [3]
5) contra h − continuous , if every $\omega \in O(Y)$, then $f^{-1}(\omega) \in C_h(X)$. **[6]**
6) pre − hc − open, if $\omega \in O_{hc}(X)$,then $f(\omega) \in O_{hc}(Y)$.
7) pre − hc − closed, if $\omega \in C_{hc}(X)$,then $f(\omega) \in C_{hc}(Y)$.
8) contra hc − open,if $\omega \in O_{hc}(X)$,then $f(\omega) \in C_{hc}(Y)$
9) contra hc − closed , if $\omega \in C_{hc}(X)$,then , $f(\omega) \in O_{hc}(Y)$.

3. CONTRA hc − CONTINUOUS FUNCTION

In this section, we introduce contra hc − continuous $f$unctions and perfectly hc − continuous $f$unctions , We studied their properties and the relationship between them.

Definition 3.1 Let $f: X \to Y$ be a TO. f f is said to be contra hc − continuous if for each $\omega \in O(Y)$, $f^{-1}(\omega) \in C_{hc}(X)$.

For example:

Let $f: (\Re, \tau_d) \to (\Re, \tau_d)$ be a $(\tau o . f)$, where τ_d is discrete topology on $\Re$ define, f by $f(x) = x$, then f is contra hc − continuous

Theorem 3.2 For a TO. f $f: X \to Y$, the following statements are equivalent:
i) $f$is contra hc − continuous .
ii) For every $H \in C(Y)$, $f^{-1}(H) \in O_{hc}(X)$.
iii) For each $x \in X$ and each $H \in C(Y)$ with $f(x) \in H$, there exists $\omega^* \in O_{hc}(X)$ such that $x \in \omega^* \subseteq \omega$, $f(\omega) \subseteq H$

Proof: i)→ ii) Obvious by definition (3.1).

ii) → iii) Let $H \in C(Y)$ and let $f(x) \in H$ where $x \in X$. Then by (ii), $f^{-1}(H) \in O_{hc}(X)$, Also $x \in f^{-1}(H)$ Take $\omega = f^{-1}(H)$. Then $\omega^* \in O_{hc}(X)$ and containing x , $\omega^* \subseteq \omega$ & $f(\omega) \subseteq H$.

iii) → i) Let $x \in X$ & $H \in O(Y)$,, $f(x) \in H$ & $(Y-H) \in C(Y)$, then $f^{-1}(Y-H) = X - f^{-1}(H)$, & $f(x) \in H$. Hence by iii) , there exists $\omega^* \in O_{hc}(X)$ of X with $x \in \omega^*$ such that $f(\omega^*) \subseteq H$. Then $f^{-1}(H) \in O_{hc}(X)$, therefore, $f$is contra hc − continuous . ■

Lemma 3.3 [**4**] The following properties hold for subsets ω_1 , ω_2 of a space X:
i) $x \in \kappa er(\omega_1)$ if and only if $\omega_1 \cap H \neq \emptyset$ for any $H \in C(X)$
ii) $\omega_1 \subseteq \kappa er(\omega_1)$ and $\omega_1 = \kappa er(\omega_1))$ if $\omega_1 \in O(X)$
iii) If $\omega_1 \subseteq \omega_2$, then $\kappa er(\omega_1) \subseteq \kappa er(\omega_2)$.

Theorem 3.4 Let $f: X \to Y$ be a bijective TO. f. Then the following statements are equivalent:
i) f is contra hc − continuous
ii) $f(hc-Cl(\omega_1)) \subseteq \kappa er(f(\omega_1))$ for every subset $\omega_1$of X

iii) hc $-\mathrm{Cl}(f^{-1}(\omega_2)) \subseteq f^{-1}(\kappa er(\omega_2))$ for every subset ω_2 of Y

Proof: (i) $\rightarrow$ (ii) Let ω_1 be any subset of X. Suppose that $y \in \kappa er(f(\omega_1))$. By Lemma (3.3 (i)), there exists $\mathcal{F} \in C(Y)$ & containing f(x) such that $(\omega_1) \cap \mathcal{F} = \emptyset$. Then $\omega_1 \cap f^{-1}(\mathcal{F}) = \emptyset$. Since f^{-1} (H) $\in O_{hc}(X)$ by (i), $hc-\mathrm{Cl}(\omega_1) \cap f^{-1}(\mathcal{F}) = \emptyset$. That implies $f(hc-\mathrm{Cl}(\omega_1) \cap \mathcal{F} = \emptyset$ and so $y \notin f(hc-\mathrm{Cl}(\omega_1)$. This shows that $f(hc-\mathrm{Cl}(\omega_1)) \subseteq \kappa er(f(\omega_1))$

(ii) $\rightarrow$ (iii) Let ω_2 be any subset of Y. Then by (ii), f $(hc-\mathrm{Cl}(f^{-1}(\omega_2)) \subseteq \kappa er\, f$ & $(f^{-1}(\omega_2)) = \kappa er(\omega_2)$. Therefore, $hc-\mathrm{Cl}(f^{-1}(\omega_2)) \subseteq f^{-1}(\kappa er(\omega_2))$.

(iii) $\rightarrow$ (i) Let $\omega_2 \in O(Y)$, then $hc-\mathrm{Cl}(f^{-1}(\omega_2)) \subseteq f^{-1}(\kappa er(\omega_2)) = f^{-1}(\omega_2)$ by (iii) and Lemma (3.3(ii)). But $f^{-1}(\omega_2) \subseteq hc-\mathrm{Cl}(f^{-1}(\omega_2))$. So $f^{-1}(\omega_2) = hc-\mathrm{Cl}((\omega_2))$. This means that $f^{-1}(\omega_2) \in C_{hc}(X)$ set in X so that f is cnotra $hc-$continuous. ■

Remark 3.5 :

1) If $f: X \rightarrow Y$ is a TO.f and cnotra $hc-$continuous function , then f cnotra $h-$continuous, since every $hc-$closed set is $h-$closed . But the converse need not be true. While the converses is not true in general as the following example:

Let $X = Y = \{\ell_1, \ell_2, \ell_3\}$, $\tau_1 = \{\emptyset, X, \{\ell_1\}, \{\ell_1, \ell_2\}\}$, $\tau_2 = \{\emptyset, Y, \{\ell_1\}, \{\ell_2, \ell_3\}\}$. Clearly, the identity function $f:(X, \tau_1) \rightarrow (Y, \tau_2)$ is a cnotra $h-$continuous , but is not cnotra $hc-$continuous, since $f^{-1}(\{\ell_2, \ell_3\}) = \{\ell_2, \ell_3\} \in C_h(X)$ but $\{\ell_2, \ell_3\} \notin C_{hc}(X)$.

2) There is no relation between the cnotra $-$continuous function and cnotra $hc-$continuous. Consider the following examples : Let $X = Y = \{\ell_1, \ell_2, \ell_3\}$, $\tau_1 = \{\emptyset, X, \{\ell_1\}, \{\ell_2\}, \{\ell_1, \ell_2\}\}$, $\tau_2 = \{\emptyset, Y, \{\ell_3\}\}$. Clearly, the identity TO.f $f:(X, \tau_1) \rightarrow (Y, \tau_2)$ is a cnotra $-$continuous , but is not cnotra $hc-$continuous , and a TO.f $f:(X, \tau_1) \rightarrow (Y, \tau_2)$ such that $f(\ell_1) = f(\ell_2) = \ell_3$, $f(\ell_3) = \ell_2$ is cnotra $hc-$continuous , but is not cnotra $-$continuous .

Theorem 3.6 Let $f: X \rightarrow Y$ be a TO.f ,

1) if X is T_1-space and f contra $h-$continuous ,then f is contra $hc-$continuous ,
2) if X is finite space and f contra $hc-$continuous , then f is continuous.
3) if X is regular space and f contra $-$continuous , then f is contra $hc-$continuous ,
4) if X is extremally disconnected and f contra $\delta-$continuous , then f is contra $hc-$continuous

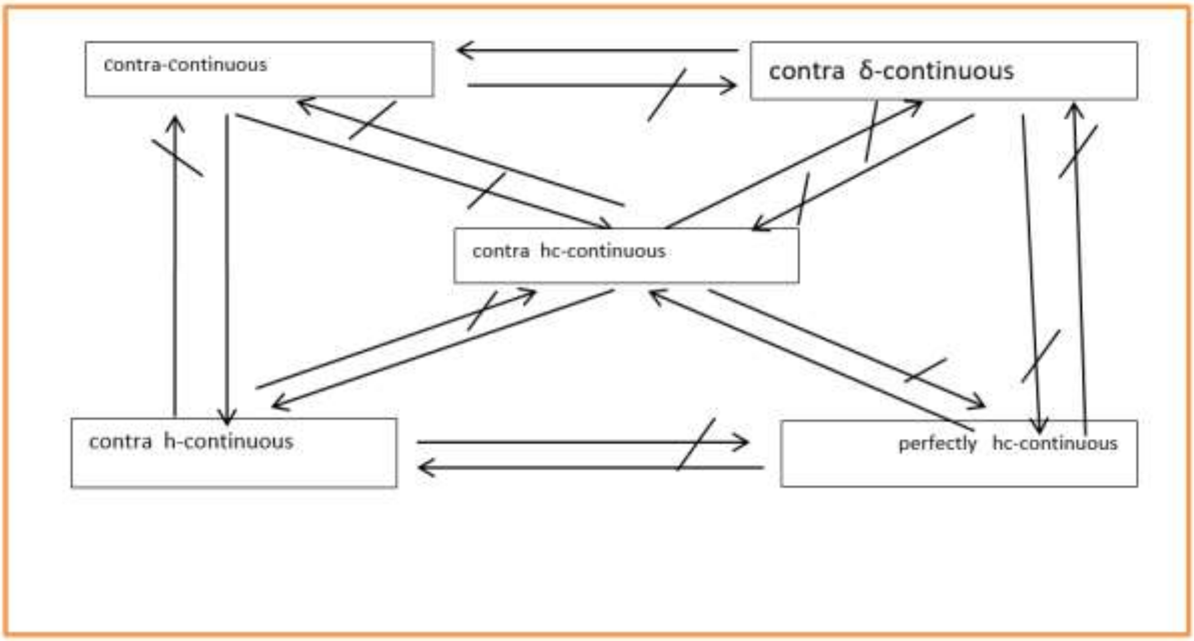
Proof: It directly follows from the definitions (3.1), (2.10) and remark (2.2), (2,3) . ■

Definition 3.10 A TO.f $f: X \rightarrow Y$ is called perfectly $hc-$continuous , if $f^{-1}(\omega)$ is $hc-$clopen in X for each open set ω in Y.

Theorem 3.11 Every perfectly $hc-$continuous function is $hc-$continuous and contra $hc-$continuous.

Proof: It directly follows from the definitions (3.1), (3.10), (2.10). ■

The next diagram explains the relations of these types of contra $-$ continuous functions



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Theorem 3.12 If $f_1 : X \to Y$ is contra hc − continuous TO.f, $f_2 : X \to Y$ is contra hc − continuous TO.f and Y is Urysohn, then $\pi = \{x \in X : f_1(x) = f_2(x)\}$ is hc − closed in X.

Proof: Let $x \in X \setminus \pi$. Then $f_1(x) \neq f_2(x)$. Since Y is Urysohn, there exists ω_1 & $\omega_2 \in C(Y)$ such that $f_1(x) \in \omega_1, f_2(x) \in \omega_2$ & $Cl(\omega_1) \cap Cl(\omega_2) = \emptyset$. Since f is contra hc − continuous, $f_1^{-1}(Cl(V)) \in O_{hc}(X)$. Since f_2 is contra hc − continuous, $f_2^{-1}(Cl(\omega)) \in O_{hc}(X)$. Let $O_1 = f_1^{-1}(Cl(\omega_1))$ & $O_2 = f_2^{-1}(Cl(\omega_2))$ & set $\rho = O_1 \cap O_2$. Then ρ is a hc − open set containing x in X.
Now
,$f_1(\rho) \cap f_2(\rho) \subseteq f_1(O_1) \cap f_2(O_2) \subseteq Cl(\omega_1) \cap Cl(\omega_2) = \emptyset$. This implies that $\rho \cap \pi = \emptyset$ where ρ is hc − open. So x is not a hc − cluster point of ρ. Hence $x \notin hc - Cl(\pi)$ & this completes the proof. ■

Theorem 3.13 Let $f_1 : X \to Y$ be a contra hc − continuous TO.f and $f_2 : X \to Y$ be a contra hc − continuous TO.f. If Y is Urysohn and $f_1 = f_2$ on hc − dense set $\omega \subseteq X$, then $f_1 = f_2$ on X.

Proof: Let $\pi = \{x \in X : f_1(x) = f_2(x)\}$. Since f_1 is contra hc − continuous, f_2 is contra hc − continuous & Y is Urysohn, by Theorem (3.12), π is hc − closed in X. By assumption, we have $f_1 = f_2$ on ω where ω is hc − dense in X. Since $\omega \subseteq \pi$, ω is hc − dense & π hc − closed, we have $X = hc - Cl(\omega) \subseteq hc - Cl(\pi) = \pi$. Hence $f_1 = f_2$ on X. ■

Theorem 3.14 If $f : X \to Y$ is closed TO.f injective and contra hc − continuous and Y is ultra− normal, then X is hc − normal.

Proof: Let $H_1, H_2 \in C(X)$ and disjoint. Since f is closed and injective, $f(H_1), f(H_1) \in C(Y)$and are disjoint . Since Y is ultra normal, there exists two clopen sets ω_1 & ω_1 in Y, such that $f(H_1) \subseteq \omega_1$, $f(H_2) \subseteq \omega_2$ & $\omega_1 \cap \omega_1 = \emptyset$. Since f is contra hc − continuous, $f^{-1}(\omega_1)$ & $f^{-1}(\omega_2) \in O_{hc}(X)$. Also $H_1 \subseteq f^{-1}(\omega_1)$, $H_2 \subseteq f^{-1}(\omega_2)$ & $f^{-1}(\omega_1) \cap f^{-1}(\omega_2) = \emptyset$. This shows that X is hc − normal. ■

Theorem 3.15 If a TO.f $f : X \to Y$ is injective, contra hc − continuous and Y is a Urysohn space, then X is hc − Hausdorff.

Proof: Let $x, y \in X$ with $x \neq y$. Since f is injective, $f(x) \neq f(y)$. Since Y is a Urysohn space, there exists two open sets O_1 & O_2 in Y such that $f(x) \in O_1, f(y) \in O_2$ & $Cl(O_1) \cap Cl(O_2) = \emptyset$. Since f is contra hc − continuous, by theorem (3.2) there exists ω_1 & $\omega_2 \in O_{hc}(X)$, such that $x \in \omega_1$, $y \in \omega_2$ & $f(\omega_1) \subseteq Cl(O_1)$, $f(\omega_2) \subseteq Cl(O_2)$. Then $f(\omega_1) \cap f(\omega_2) = \emptyset$ & so $f(\omega_1 \cap \omega_2) = \emptyset$. This implies that $\omega_1 \cap \omega_2 = \emptyset$ & hence X is hc − Hausdorff. ■

Lemma 3.16 For a topological space X, X is hc − connected if and only if the subsets of X which are both hc − open and hc − closed are the sets X and $\emptyset$.

Proof: ⇒ Let $\omega \in O_{hc}(X)$ & $\in C_{hc}(X)$. Then $X \setminus \omega \in O_{hc}(X)$ & $\in C_{hc}(X)$. Since X is hc − connected & X is the disjoint union of hc − open sets ω and $X \setminus \omega$, one of these must be empty. Hence either $\omega = \emptyset$ or $\omega = X$.
⇐ Suppose that X is not hc − connected . Then $X = \omega_1 \cup \omega_2$ where ω_1, ω_2 are nonempty & $\omega_1, \omega_2 \in O_{hc}(X)$, such that $\omega_1 \cap \omega_2 = \emptyset$. Since $\omega_2 = X \setminus \omega_1 \in O_{hc}(X)$, $\omega_1 \in O_{hc}(X)$ & $\in C_{hc}(X)$. By assumption , $\omega_1 = \emptyset$ or X. That is, either $\omega_1 = \emptyset$ or $\omega_2 = \emptyset$, which is a contradiction. Therefore X is hc − connected. ■

Theorem 3.17 For a topological space X, The only subsets of X which are both hc − open and hc − closed are the sets X and $\emptyset$ If and only if each contra hc − continuous TO.f of X into a discrete space Y with at least two points is a constant function .

Proof: ⇒ Let $f : X \to Y$ be a contra hc − continuous function from a topological space X into a discrete topological space Y. Then for each $y \in Y$, $\{y\}$is both open & closed in Y. Since f is Contra hc − continuous, $f^{-1}(y) \in O_{hc}(X)$ & $\in C_{hc}(X)$. Hence X is covered by hc − open and hc − closed covering $\{f^{-1}(y) : y \in Y\}$. $f^{-1}(y) = \emptyset$ or X for each $y \in Y$. If $f^{-1}(y) = \emptyset$ for each $y \in Y$, then f is fiasco function. Hence there exists only one point $y \in Y$ such that $f^{-1}(y) = X$, which shows that f is a constant function.
⇐ Let $\omega_0 \in O_{hc}(X)$ & $\in C_{hc}(X)$. Suppose $\omega_0 \neq \emptyset$. Let $f : X \to Y$ be a contra hc − continuous function from a topological space X into a discrete topological space Y defined by $f(\omega_0) = \{\gamma\}$ and $f(X \setminus \omega_0) = \{\delta\}$, where $\gamma, \delta \in Y$ and $\gamma \neq \delta$. Since f is constant so that $\omega_0 = X$. ■

Theorem 3.18 Let X be a hc − connected space and Y be any topological space . If $f : X \to Y$ is surjectiv and contra hc − continuous TO.f, then Y is not a discrete space.

Proof: Let Y be a discrete space & ω be any proper non-empty subset of Y. Then ω is both open & closed in Y. Since f is contra hc − continuous, $f^{-1}(A) \in O_{hc}(X)$ & $\in C_{hc}(X)$. Since X is hc − connected, by lemma (3.16), the only subsets of X which are both hc − open & hc − closed are the sets X and $\emptyset$. Hence $f^{-1}(\omega)$ is either X or $\emptyset$. If $f^{-1}(\omega) = \emptyset$, then it contradicts to the fact that $\omega \neq \emptyset$ and f is surjective. If $f^{-1}(\omega) = X$, then f is fails function. Hence Y is not a discrete space. ■

Theorem 3.19 If TO.f $f: X \to Y$ is surjective, contra hc − continuous and X is hc − connected, then Y is connected.

Proof: Assume that Y is not connected. Then $Y = \omega_1 \cup \omega_2$ where ω_1 & ω_2 are nonempty open subsets in Y such that $\omega_1 \cap \omega_2 = \emptyset$. Set $H_1 = Y \setminus \omega_1$ & $H_2 = Y \setminus \omega_1$. Then H_1 and H_2 are nonempty closed subsets in Y. Since f is surjective & contra hc − continuous, then $f^{-1}(H_1)$ & $f^{-1}(H_2) \in O_{hc}(X)$.
Now, $f^{-1}(H_1) \cap f^{-1}(H_2) = \emptyset$ & $f^{-1}(H_1) \cup f^{-1}(H_2) = X$. This contradicts to the $f$act that X is hc − connected & so Y is connected. ■

Theorem 3.20 If X is hc −connected space and $f: X \to Y$ is contra hc − continuous TO.f, Y is T_1 − space, then f is constant function.

Proof: Let X be hc − connected. Now, since Y is a T_1 − space, $\rho = \{f^{-1}(y) : y \in Y\}$ is disjoint hc − open partition of X. If $|\rho| \geq 2$ (where $|\rho|$ denotes the cardinality ρ), then X is the union of two nonempty disjoint hc − open sets. Since X is hc − connected, we get $|\rho| = 1$. Hence, f is constant. ■

Theorem 3.21 A space X is hc − connected if every contra hc − continuous TO.f from a space X into any T_0 − spaceY is constant.

Proof: suppose that X isn't hc − connected, Let $Y = \{\alpha^*, \alpha^{**}\}$ & $\sigma = \{Y, \emptyset, \{\alpha^*\}, \{\alpha^{**}\}\}$ be a topology $f$or Y. Let $f: X \to Y$ be a function such that $f(\omega) = \{\alpha^*\}$ & $f(X \setminus \omega) = \{\alpha^{**}\}$. Then f is non constant & contra hc − continuous such that Y is T_0, this is a contradiction , also implies that a space X have to hc −connected. ■

Theorem 3.22 Let $f_1 : X \to Y$ is and $f_2 : X \to Z$ are a TO.f, if

i) f_1 is contra hc − continuous and f_2 is continuous then $f_2 \circ f_1$ is contra hc − continuous

ii) f_1 is contra hc − continuous and f_2 is contra − continuous then $f_2 \circ f_1$ is hc − continuous function.

Proof: Clearly, It conduct derive from the definitions. Type equation here.

Let $f_1 : X \to Y$ & $f_2 : Y \to Z$ be two TO.f. The case when the composition $f_2 o f_1$ is contra hc − continuous has been studied in the following theorem :

Theorem 3.23 Let X , Y and Z be three topological spaces , $f_1 : X \to Y$ and $f_1 : Y \to Z$ be two TO.f if ,

i) $f_2 \circ f_1$ is contra hc − continuous and f_2 is open injection , then f_1 is contra n-continuous.

ii) $f_2 o f_1$ is contra hc − continuous and f_2 is closed injection , then f_1 is contra hc − continuous.

iii) $f_2 o f_1$ is contra hc − continuous and f_1 is pre − hc − closed surjection, then f_2 is contra hc − continuous.

iv) $f_2 o f_1$ is contra hc − continuous and f_1 is pre − hc − open surjection , then f_2 is contra hc − continuous.

proof :

i) Let f_2 be open & injection, let $\omega \in O(Y)$, then $f_2(\omega) \in O(Z)$. Since $f_2 o f_1$ is contra hc − continuous, then $(f_2 o f_1)^{-1}(f_2(\omega)) = f_1^{-1}(f_2^{-1}(f_2(\omega))) = f_1^{-1}(\omega) \in C_{hc}(X)$, therefore f is contra hc − continuous.

ii) Let $f_2 o f_1$ be contra hc − continuous & f_2 be closed and injection , let $\mathcal{M} \in C(Y)$, then $f_2(\mathcal{M}) \in C(Z)$. Since $f_2 o f_1$ is contra hc − continuous, then $(f_2 o f_1)^{-1}\left(f_2(\mathcal{M})\right) = f_1^{-1}(f_2^{-1}(f_2(\mathcal{M}))) = f_1^{-1}(\mathcal{M}) \in O_{hc}(X)$, therefore f is contra hc − continuous.

iii) Let $f_2 o f_1$ be contra hc − continuous & f_1 be pre − hc − closed surjection, let $\omega \in O(Z)$, then $(f_2 o f_1)^{-1}(\omega) \in C_{hc}(X)$. Since f_1 is pre − hc − closed & surjection which implies $\left(f_1(f_2 o f_1)^{-1}(\omega)\right) = f_1\left(f_1^{-1}(f_2^{-1}(\omega))\right) = f_2^{-1}(\omega) \in C_{hc}(Y)$, therefore f_2 is contra hc − continuous.

iv) Let $f_2 o f_1$ be contra hc − continuous & f_1 be pre − hc − open surjection, let $\mathcal{M} \in C(Z)$, then $\left(f_2 o f_1\right)^{-1}(\mathcal{M}) \in O_{hc}(X)$. Since f_1 is pre − hc − open & surjection which implies $f_1(\left(f_2 o f_1\right)^{-1}(\mathcal{M})) = f_1(f_1^{-1}(f_2^{-1}(\mathcal{M}))) = f_2^{-1}(\mathcal{M})) \in O_{hc}(Y)$, therefore f_2 is contra hc − continuous.■

4. CONTRA hc − CLOSEd GRAPH

In this section, have presented hc − closed and contra hc − closed graph , and we studied relationship with contra hc − continuous functions .

Definition 4.1 The graph $G(f)$ of a TO.f $f: X \to Y$ is called:

i) hc − closed in X × Y, if and only if for each $(x, y) \in \{(X \times Y) \setminus G(f)\}$, there exist $\omega_1 \in O_{hc}(X)$, ω_1 containing the element x and $\omega_2 \in O(Y)$, ω_2 containing the element y such that $f(\omega_1) \cap \omega_2 = \emptyset$.

ii) contra hc − closed in X × Y if and only if for each $(x, y) \in \{(X \times Y) \setminus G(f)\}$ there exist $\omega_1 \in O_{hc}(X)$, ω_1 containing the element x and a $\omega_2 \in C(Y)$, ω_2 containing the element y such that $f(\omega_1) \cap \omega_2 = \emptyset$.

For example : Let $X = \{\ell_1, \ell_2\}$, $\tau_1 = \{\emptyset, X, \{\ell_1\}, \{\ell_2\}\}$,$Y = \{\alpha_1, \alpha_2, \alpha_3\}$ $\tau_2 = \{\emptyset, Y, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_2, \alpha_3\}\}$, such that $f(\ell_1) = f(\ell_2) = \alpha_1$, then $G(f) = \{(\ell_1, \alpha_1), (\ell_2, \alpha_1)\}$ is hc − closed in X × Y
But not contra hc − closed in X × Y.

If $(\ell_1) = f(\ell_2) = \alpha_3$, then $G(f) = \{(\ell_1, \alpha_3), (\ell_2, \alpha_3)\}$ is contra $hc -$ closed in $X \times Y$,while not $hc -$ closed in $X \times Y$.

Theorem 4.2 If $f: X \to Y$ is $hc -$ continuous $TO. f$ and Y is T_1 , then $G(f)$ is contra $hc -$ closed in $X \times Y$.

Proof: Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ & since $Y$is T_1 there exist open set O^* of Y, such that $f(x) \in O^*$, $y \notin O^*$. Since f is $hc -$ continuous, there exist $hc -$ open set O^{**} of X containing x such that $f(O^{**}) \subseteq O^*$. Therefore $f(O^{**}) \cap (Y \setminus O^*) = \emptyset$ & $\{Y \setminus O^*\}$ is a closed set in Y containing y. Hence by above definition, $G(f)$is contra $hc -$ closed in $X \times Y$. ■

Theorem 4.3 If Y is a Urysohn and $f: X \to Y$ is contra $hc -$ continuous $TO. f$, then $G(f)$is contra $hc -$ closed in $X \times Y$.

Proof: assume that $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ & since Y is Urysohn, there exist open sets ω_1 , ω_1 in Y such that $f(x) \in \omega_1$, $y \in \omega_2$ & $Cl(\omega_1) \cap Cl(\omega_2) = \phi$. Now, since f is contra $hc -$ continuous, there exist O^* $hc -$ open set such that $f(O^*) \subseteq Cl(\omega_1)$ which implies that $f(O^*) \cap Cl(\omega_2) = \emptyset$. Hence by definition (4.1) , $G(f)$ is contra $hc -$ closed in $X \times Y$. ■

Theorem 4.4 Let $f_1 : X \to Y$ be a $TO. f$ and $f_2 : X \to X \times Y$ be a graph function of f_1, defined by $f_2(x) = (x, f_1(x))$ for every $x \in X$. If f_2 is contra $hc -$ continuous, then f_1 is contra $hc -$ continuous.

Proof: Let ω be an open set in Y. Then $X \times \omega$ is open in $X \times Y$. Since f_2 is contra $hc -$ continuous, $f_2^{-1}(X \times \omega) = f_1^{-1}(\omega)$is $hc -$ closed in X. This shows that f is contra $hc -$ continuous.

5. CONCLUSIONS

In this work , we have presented the idea of $hc -$ open and $hc -$ closed sets and learned about its master properties. Then, we have used this idea to show a kind of contra $-$ continuity . We discussed the master properties of this continuity and we have revealed the relationship between this type of continuity and other types. In addition, we have introduced graph functions using $hc -$ open and $hc -$ closed sets and checked their main properties. Our next works will concentrate on studying further topological concepts associated with the contra $hc -$ continuous .

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