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# The Maximum Reduced Sombor Index of Unicyclic Graphs in Terms of the Girth

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**ABSTRACT**

The Reduced sombor index is defined as $RSO(G) = \sum_{uv \in E(G)} \sqrt{(d_G(u)-1)^2 + (d_G(v)-1)^2}$. A unicyclic graph is a graph with exactly one cycle. The unicyclic graphs are well-studied for other topological indices. The Reduced sombor index is studied in this article and proved novel results on the bounds with the restrictions to the unicyclic graphs. This work concentrates on the extreme values of the Reduced sombor index. We propose a maximum value for the Reduced sombor index of unicyclic graph and the unicyclic graphs attaining the maximum Reduced Sombor index are achieved.

**KEYWORDS:** Reduced Sombor index; topological index; graph invariant; extremal problem; characterization.

## I. INTRODUCTION

The topological indices (also known as graph invariants) play a major role in the chemical graph theory because they are used to analyze the behaviour of the molecule structures and their inter-relationships. There are numerous topological indices available in the literature, a few of them are the Sombor index, Zagreb index, and so on. The topological indices were defined with minor and major modifications in the past and several classes of topological indices are available for the Sombor index and Zagreb index.

By a graph G, in this article, we mean an ordered pair ($V_G$ , $E_G$) and the members of the sets $V_G$ and $E_G$ respectively are the vertices and edges of the graph. The set of vertices that are adjacent to a vertex u in G is denoted as $N_G(u)$and called by "the open neighbourhood" of u in G. The term "closed neighbourhood" is $N_G[v] = N_G(v) \cup \{v\}$ and by a (v,w)-Path $vv_1v_2 \ldots w$ is a sequence of distinct members of the set $V_G$ and the vertices v,w are usually known as the origin and the terminus of the path P respectively. The concept of distance between any two vertices x,y ∈ VG is usually defined as the length of the smallest (x,y)-path that exists in G. If $D_G[v] = 1$, then v is a pendant vertex and it is adjacent to a unique vertex in G, say u which is called a support vertex. For more on graphs and related works, the reader is referred to [1,2].

Given a graph G, the Sombor (SO) index is defined (by Gutman [3]) as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{(d_G(u))^2 + (d_G(v))^2}.$$

The Sombor index, in recent years, received numerous attention from academics and researchers throughout the globe [4–6]. For some recent surveys in Sombor index, one can refer to the articles [7,8]. Chemical applications have been carried out in the articles [9–11].

For various results and versions of Sombor index, one can refer [12,13].

The reduced Sombor index is defined as

$$RSO(G) = \sum_{uv \in E(G)} \sqrt{(d_G(u)-1)^2 + (d_G(v)-1)^2}$$

The reduced Sombor index is a recently introduced term and some of the works can be found in [14,15]. The reduced Sombor index of unicyclic graphs is studied in this article and characterized the graphs with maximum Reduced Sombor index.

## II. RESULTS

The main results on the maximum Reduced Sombor index for the class of unicyclic graphs is studied in this section. If the order n of G is one, there is no edge in the graph, hence by a graph G, throughout this article, we mean a graph with minimum two vertices. Also, by a graph G, we mean only a connected graph, unless explicitly stated.

### *A. The Graphs With The Maximum Reduced Sombor Index*

In this section, the results on the maximum Reduced Sombor index of unicyclic graphs and a characterization of unicyclic graphs with the maximum values of RSO(G) is provided. First, without loss of generality, let the order n of G be minimum three. Because otherwise, the graph contains no cycle. For any integer n, we found the unicyclic graphs of order n with maximum RSO(G). This characterization provides a unique unicyclic graph for any positive integer n ≥ 3 with maximum RSO(G). The section starts with the following lemma:

**Lemma 1**: Let u,v be vertices of a unicyclic graph G = Cn. Let H be a graph constructed from G by deleting the vertex u and it's incident edges, and attaching it to v. Then, RSO(H) > RSO(G).

**Proof:** Consider the vertex u on the cycle with u′ being a pendant neighbour of u and v is a neighbour of u in the cycle of G. Let w be another neighbour on the cycle of G.

Now,

$$RSO(H) = RSO(G) - \sqrt{(d_G(u)-1)^2 + (d_G(u')-1)^2} - \sqrt{(d_G(u)-1)^2 + (d_G(u'')-1)^2} - \sqrt{(d_G(v)-1)^2 + (d_G(v')-1)^2} - \sqrt{(d_G(v)-1)^2 + (d_G(v'')-1)^2} + \sqrt{(d_H(v)-1)^2 + (d_H(v')-1)^2} + \sqrt{(d_H(v)-1)^2 + (d_H(v'')-1)^2} + \sqrt{(d_H(u)-1)^2 + (d_H(v)-1)^2}$$

$$= RSO(G) - \sqrt{(2-1)^2 + (2-1)^2} - \sqrt{(2-1)^2 + (2-1)^2} - \sqrt{(2-1)^2 + (2-1)^2} - \sqrt{(2-1)^2 + (2-1)^2} + \sqrt{(3-1)^2 + (2-1)^2} + \sqrt{(3-1)^2 + (2-1)^2} + \sqrt{(3-1)^2 + (2-1)^2}$$

$$= RSO(G) - 4\sqrt{2} + 2\sqrt{5} + \sqrt{2^2}$$

$$= RSO(G) + k, \text{say}$$

>RSO(G)  Since k > 0.  Hence, it is proved.

**Lemma 2:** Let $u_1, u_2, \ldots u_k, v$, where $k \le n-3$, be vertices of a unicyclic graph G = $C_n$. Let H be a graph constructed from G by deleting the vertices $u_1, u_2, \ldots u_k$ and it's incident edges, and attaching them to v. Then, RSO(H) > RSO(G).

**Proof:** The proof follows from the repeated application of Lemma-1.

**Lemma 3:** Let u be a vertex on the cylce of a unicyclic graph G with the property that it is adjacent to a pendant vertex u′ and v be a vertex on the cycle of G with $d_G(v) = 2$. Let $H_1$ be a graph obtained from G by deleting the vertex v on the cycle of G and attaching it to the pendant vertex u′ of G and making the neighbours v′, v″ of v to be adjacent in $H_1$. *Then*, $RSO(H_1) > RSO(G)$.

**Proof:** Consider,

$$RSO(H_1) = RSO(G) - \sqrt{(d_G(v)-1)^2 + (d_G(v')-1)^2} - \sqrt{(d_G(v)-1)^2 + (d_G(v'')-1)^2} - \sqrt{(d_G(u)-1)^2 + (d_G(u')-1)^2} + \sqrt{(d_H(u)-1)^2 + (d_H(u')-1)^2} + \sqrt{(d_H(v)-1)^2 + (d_H(v'')-1)^2} + \sqrt{(d_H(u')-1)^2 + (d_H(v')-1)^2}$$

$$= RSO(G) - \sqrt{(2-1)^2 + (2-1)^2} - \sqrt{(2-1)^2 + (2-1)^2} - \sqrt{(3-1)^2 + (1-1)^2} - \sqrt{(3-1)^2 + (2-1)^2} + \sqrt{(2-1)^2 + (2-1)^2} + \sqrt{(2-1)^2 + (1-1)^2}$$

$$= RSO(G) + \sqrt{5} + \sqrt{2} - 1$$

=RSO(G)+ k,  say

>RSO(G)   Since k > 0

Hence, it is proved.

Now, let us define a class of graphs $G = C_{k,n-k}$, as the unicyclic graph with $k$ vertices on the cycle and remaining $n-k$ vertices are adjacent to a vertex on the cycle of $G$. For the case $k = 3$, the graph $G = C_{3,n-3}$ is illustrated in the following Figure:

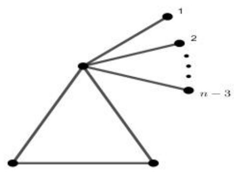


**Figure 1: Graph $G = C_{3,n-3}$**

For $G = C_{k,n-k}$, we have the following propositions as an observation:

**Proposition 4:** If $G = C_{k,n-k}$, *then* $RSO(G) \le 2\sqrt{(n-k+1)^2+1} + (n-k)\sqrt{(n-k+1)^2+2} + (k-2)\sqrt{2}$.

**Proposition 5:** If $G = C_{3,n-3}$, then $RSO(G) \le \sqrt{2} + 2\sqrt{(n-2)^2+1} + (n-3)\sqrt{(n-2)^2}$

The following theorem is a consequence of the lemmas we proved. For any positive integer *n*, this theorem provides a unique unicyclic graph (up to isomorphism) of order *n* with maximum *RSO*(*G*) and hence holds the validity of the existence of the result we provided for unicyclic graphs.

**Theorem 6:** Given a positive integer n, the graph G is a unicyclic graph of order n and $G = C_{k,n-k}$ then $RSO(G) \leq 2\sqrt{(n-k+1)^2+1} + (n-k)\sqrt{(n-k+1)^2} + (k-2)\sqrt{2}$.
with equality if and only if $G \in C_{3,n-3}$.

**Proof:** If $G = C_{3,n-3}$, then by Proposition-5, the result is true. Let us prove the converse of the theorem on induction on the number of vertices $n$ of $G$. First, if $n = 3$, then $G = C_3$ which is the only unicyclic graph on 3 vertices and $RSO(G) = B$ and hence it is true. If $n = 4$, then $g = 4$ or $g = 3$. If $g = 4$, then $RSO(G) = 4\sqrt{2}$ but for the case $g = 3$, $RSO(G) = \sqrt{2} + 2\sqrt{5} + 2$. Thus, the induction is true for base cases $n = 3,4$.

Now, let us assume that the induction is true for any unicyclic graph with its order less than $n$. Now, assume $G$ is a random unicyclic graph on $n$-vertices. If $g = n$, then by Lemma-1 there exists a graph $H$ such that $RSO(H) > RSO(G)$. By repeated application of Lemma-1 and Lemma-2, we can construct a graph $H_1$ such that $RSO(H1) > RSO(G)$. Moreover, the graph $H_1$ is of the form $C_{3,n-3}$. Thus, the result is true. Thus, if $g = n$, the result is proved.

Now, let us assume that $g \neq n$. Then, $G$ is of the form $C_{n,n-g}$. That is, there are $g$ vertices on the cycle of $G$ and the remaining $n-g$ vertices are adjacent to either vertices on the cycle or they can be connected by a path to the vertices on the cycle of $G$. Thus, there exists at least one vertex such that $d_G(v) \geq 3$. We have two cases now.

**Case-1:** There exists two vertices $u$ and $v$ such that $d_G(u), d_G(v) \geq 3$.

Suppose there are two vertices on the cycle of $G$, say $u, v \in V(G)$ such that $dG(u), dG(v) \geq 3$. Then by Lemma-3, we have a graph $H_1$ such that $RSO(H_1) > RSO(G)$ with the property that $H_1$ have exactly one vertex with degree 3. Now, by Lemma-1, we have a graph $H_2$ such that $RSO(H_2) > RSO(G)$ with the property that $H_2$ has only one vertex with degree 3. Then applying Lemma-1 and Lemma-2 we can construct a graph of the form
$C_{3,n-3}$. Thus, it is proved.

**Case-2:** There exists only one vertex $u$ on the cycle of $G$ with $d_G(u) \geq 3$.

Assume that there is only one vertex on the cycle of $G$ with $d_G(u) \geq 3$. Now, by Lemma-1, we have a graph $H_2$ such that $RSO(H_2) > RSO(G)$ with the property that $H_2$ has only one vertex with degree 3. Then applying Lemma-1 and Lemma-2 we can construct a graph of the form $C_{3,n-3}$. Thus, it is proved.

We have listed a few unicyclic graphs with orders $n = 4,5,6,7$ which have maximum reduced sombor index in the following Figure:

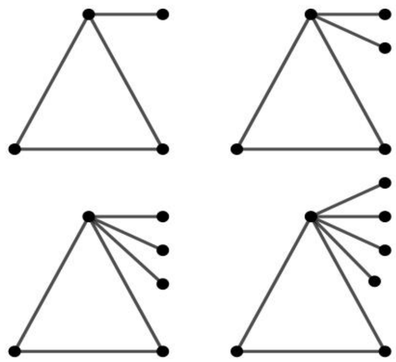

**Figure 2: Unicyclic graphs of orders $n = 4,5,6,7$**

## III. CONCLUSION

In this article, we found the maximum values of the Reduced Sombor index for the class of unicyclic graphs. The unicyclic graphs achieving the maximum values are characterized.

## REFERENCES


1. Haynes, T., Hedetniemi, S., Slater, P., Fundamentals of Domination in Graphs. Marcel Dekker, New York (1998).
2. Haynes T.W. Hedetniemi S. Slater P., Domination in Graphs: Advanced Topics, Marcel Dekker, 1998.
3. Gutman I., Geometric approach to degree-based topological indices: Sombor indices, MATCH Commun. Math. Comput. Chem. 86 (2021) 11–16.
4. Das K. C., Gutman I., On Sombor index of trees, Appl. Math. Comput. 412 (2022) #126575.
5. Gutman I., Kulli V. R., Redˇzepovi´c I., Sombor index of Kragujevac trees, Sci. Publ. Univ. Novi Pazar Ser. A 13 (2021) 61–70.
6. Liu H., Gutman I., You L., Huang Y., Sombor index: review of extremal results and bounds, J. Math. Chem. 60 (2022) 771–798.
7. Chen H., Li W., Wang J., Extremal values on the Sombor index of trees, MATCH Commun. Math. Comput. Chem. 87 (2022) 23–49.
8. Gutman I., Sombor index – one year later, Bull. Acad. Serb. Sci. Arts 153 (2020) 43–55.
9. Redˇzepovi´c I., Chemical applicability of Sombor indices, J. Serb. Chem. Soc. 86 (2021) 445–457.
10. Zhou T., Lin Z., Miao L., The Sombor index of trees and unicyclic graphs with given maximum degree, Discrete Math. Lett. 7 (2021) 24–29.
11. Deng H., Tang Z., Wu R., Molecular trees with extremal values of Sombor indices, Int. J. Quantum Chem. 121 (2021) #e26622.
12. Gutman I., Redzepovic I., Furtula B., On the product of Sombor and modified Sombor index, Open Journal of Applied Discrete Mathematics, 6(2) (2023) 1-6.

13. Yufei Huang, Hechao Liu, Bounds of modified Sombor index, spectral radius and energy, AIMS Mathematics, 6(10), (2021) 11263-11274.
14. Fangxia Wang, Baoyindureng Wu, The Proof of a Conjecture on the Reduced Sombor Index, MATCH Commun. Math. Comput. Chem. 88 (2022) 583-591.
15. Hechao Liu, Lihua You, Zikai Tang, Jia–Bao Liu, On the Reduced Sombor Index and Its Applications, MATCH Commun. Math. Comput. Chem. 86 (2021) 729-753.