International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 13 Issue 02 February 2025, Page no. – 4822-4826

Index Copernicus ICV: 57.55, Impact Factor: 8.615

DOI: 10.47191/ijmcr/v13i2.04

Properties of a Certain Subclass of Multivalent Function Defined by using Generalized Ruscheweyh Derivative

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ARTICLE INFO

Published Online:
06 February 2025

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ABSTRACT

In this research paper, we work on various analytic and geometric properties of a new subclass of analytic and multivalent function defined under the open unit disk by using generalized ruscheweyh derivative operator. These properties mainly include Radii of close – to – convexity, starlikeness and convexity, arithmetic mean property and convex set property for the analytic and multivalent function belonging to this new subclass.

KEYWORDS: Generalised Ruscheweyh Derivative, multivalent function, radii of close to convexity, starlikeness and convexity.

I. INTRODUCTION

Many researchers like I. Al Dawish et al.[1], E. Deniz and H. Orhan [4], A.R.S. Juma and S.R. Kulkarni [5], B.S. Keerthi et al.[7] S. Khosravianarab [8], S. Najafzadeh and S.R. Kulkarni [9], O. Ozkan and O. Altintas [10], J. Patel and P. Sahoo [11], R.K. Raina and H.M. Srivastava [12], P. Vijaywargiya [13], J. Yang and S.Li [14] and S. Yun et al.[15] discussed various properties of subclasses of univalent and multivalent function defined by using Generalised Ruscheweyh derivative operator. M.K. Aouf et al. [2], W.G. Atshan et al. [3] and H. Ohan [9] derived a new subclass of univalent and multivalent function by making use of Generalised Ruscheweyh Derivative. Properties like radii of close to convexity, starlikeness and convexity, arithmetic mean property, convex set property have been derived in research article.

Definition 1: A function $f(z)$ is said to be in a class $S(P,\gamma,\lambda,\delta)$ if it satisfy the following condition [6]

$$\operatorname{Re}\left\{\frac{z^2(J_P^{\delta,\mu}f(z))''+z(1-\gamma)(J_P^{\delta,\mu}f(z))'}{(1-\gamma)(J_P^{\delta,\mu}f(z))+\gamma z^2(J_P^{\delta,\mu}f(z))''}\right\}>\lambda \quad (1)$$

For

$$z\in U=\{z\in c:|z|<1\},\ 0\le\gamma<1,\ 0\le\lambda<p,\ \delta>-1$$

II. RADII OF CLOSE – TO – CONVEXITY, STARLIKENESS AND CONVEXITY

In this section, we have derive result related to radii of close-to-convexity, starlikeness and convexity for function $f(z)$ belonging to new subclass $S(p,\gamma,\lambda,\delta)$ of multivalent function.

Theorem 1: Let us consider $f(z)=z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k$ belonging to class $S(p,\gamma,\lambda,\delta)$ then the function $f(z)$ is p-valent close to convex of order m; $0\le m<p$ in $|z|<R$; where

$$R_1'=\inf_{k\ge n+p}\left[\left(\frac{p-m}{k}\right)\left(B_p^{\delta,\mu}(k)\right)\left(\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right)\right]^{\frac{1}{k-p}}$$

(2)

Proof: To show $f(z)$ is p-valent close-to-convex of order m; $0\le m<p$ in $|z|<R_1'$ it is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le p-m \qquad |z|<R'_1 \quad (3)$$

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|=\left|\frac{pz^{p-1}-\sum_{k=n+p}^{\infty}\alpha_k kz^{k-1}}{z^{p-1}}-p\right|$$

$$=\left|\frac{pz^{p-1}-\sum_{k=n+p}^{\infty}\alpha_k kz^{k-1}-pz^{p-1}}{z^{p-1}}\right|$$

$$=\left|\frac{\sum_{k=n+p}^{\infty}\alpha_k kz^{k-1}}{z^{p-1}}\right| \qquad \le \sum_{k=n+p}^{\infty}k\alpha_k\,|z|^{k-p}$$

The inequality (3) is less than p-m if

$$\sum_{k=n+p}^{\infty}\frac{k}{p-m}\alpha_k\,|z|^{k-p}<1$$

Since $f(z)\in S(p,\gamma,\lambda,\delta)$ if and only if

$$\sum_{k=n+p}^{\infty}\alpha_k B_p^{\delta,\mu}(k)\left\{\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right\}<1$$

The inequality (3) hold true if

$$\frac{k}{p-m}|z|^{k-p}<B_p^{\delta,\mu}(k)\left\{\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right\}$$

or

$$|z|^{k-p}<\left(\frac{p-m}{k}\right)\left(B_p^{\delta,\mu}(k)\right)\left(\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right)$$

Thus

$$|z|<R'_1=\inf_{k\ge n+p}\left[\left(\frac{p-m}{k}\right)\left(B_p^{\delta,\mu}(k)\right)\left(\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right)\right]^{\frac{1}{k-p}}$$

Hence the theorem is proved.

Theorem 2: Let us consider $f(z)=z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k$ belonging to the class $S(p,\gamma,\lambda,\delta)$ then the function $f(z)$ is p-valent starlike of order m, $0\le m<p$ in $|z|<R'_2$

$$R'_2=\inf_{k\ge n+p}\left[\left(\frac{p-m}{k-m}\right)\left(B_p^{\delta,\mu}(k)\right)\left(\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right)\right]^{\frac{1}{k-p}} \quad (4)$$

Proof: To show the function $f(z)$ is p-valent starlike function of order m; $0\le m<p$ in $|z|<R'_2$ it is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-p\right|\le p-m \qquad |z|<R'_2 \qquad (5)$$

$$\left|\frac{zf'(z)}{f(z)}-p\right|=\left|\frac{z\left[pz^{p-1}-\sum_{k=n+p}^{\infty}k\alpha_k z^{k-1}\right]}{z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k}-p\right|$$

$$=\left|\frac{pz^p-\sum_{k=n+p}^{\infty}k\alpha_k z^k-pz^p+p\sum_{k=n+p}^{\infty}\alpha_k z^k}{z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k}\right|$$

$$=\left|\frac{\sum_{k=n+p}^{\infty}(k-p)\alpha_k z^k}{z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k}\right|$$

$$\le\sum_{k=n+p}^{\infty}\frac{(k-p)\alpha_k\,|z|^{k-p}}{1-\sum_{k=n+p}^{\infty}\alpha_k\,|z|^{k-p}}$$

The above inequality (5) is less than p – m if

$$\frac{\sum_{k=n+p}^{\infty}(k-p)\alpha_k\,|z|^{k-p}}{1-\sum_{k=n+p}^{\infty}\alpha_k\,|z|^{k-p}}<p-m$$

or $$\sum_{k=n+p}^{\infty}\left(\frac{k-m}{p-m}\right)\alpha_k\,|z|^{k-p}<1$$

Since $f(z)\in S(p,\gamma,\lambda,\delta)$ if and only if

$$\sum_{k=n+p}^{\infty}\alpha_k B_p^{\delta,\mu}(k)\left\{\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right\}<1$$

The inequality (5) hold true if

$$\left(\frac{k-m}{p-m}\right)|z|^{k-p} < B_p^{\delta,\mu}(k)\left\{\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right\}$$

$$|z|<R'_2=\inf_{k\geq n+p}\left[\left(\frac{p-m}{k-m}\right)\left(B_p^{\delta,\mu}\right)\left(\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right)\right]^{\frac{1}{k-p}}$$

Hence the theorem is proved.

Theorem 3: Let us consider $f(z)=z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k$ belonging to the class $S(p,\gamma,\lambda,\delta)$ then the function $f(z)$ is p-valent convex function of order m; $0\leq m<p$ in $|z|<R_3^{'}$.

$$R_3^{'}=\inf_{k\geq n+p}\left[\left(\frac{p(p-m)}{k(k-m)}\right)B_p^{\delta,\mu}(k)\left(\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right)\right]^{\frac{1}{k-p}} \quad (6)$$

Proof: To show $f(z)$ is p-valent convex function of order m; $0\leq m<p$ in $|z|<R_3^{'}$, it is sufficient to show that

$$\left|\frac{z f''(z)}{f'(z)}+(1-p)\right|\leq p-m \qquad (7)$$

$$\left|\frac{z f''(z)}{f'(z)}+(1-p)\right|$$

$$=\left|\frac{z\left[p(p-1)z^{p-2}-\sum_{k=n+p}^{\infty}k(k-1)\alpha_k z^{k-2}\right]}{pz^{p-1}-\sum_{k=n+p}^{\infty}k\alpha_k z^{k-1}}+(1-p)\right|$$

$$=\left|\frac{p(p-1)z^{p-1}-\sum_{k=n+p}^{\infty}k(k-1)\alpha_k z^{k-1}+(1-p)pz^{p-1}-(1-p)\sum_{k=n+p}^{\infty}k\alpha_k z^{k-1}}{pz^{p-1}-\sum_{k=n+p}^{\infty}k\alpha_k z^{k-1}}\right|$$

$$=\left|\frac{-\sum_{k=n+p}^{\infty}[k^2-k+k-kp]\alpha_k z^{k-1}}{pz^{p-1}-\sum_{k=n+p}^{\infty}k\alpha_k z^{k-1}}\right|$$

$$\leq\sum_{k=n+p}^{\infty}\frac{k(k-p)\alpha_k|z|^{k-p}}{p-\sum_{k=n+p}^{\infty}k\alpha_k|z|^{k-p}}$$

The inequality (7) is less than or equal to p-m if

$$\sum_{k=n+p}^{\infty}\frac{k(k-p)\alpha_k|z|^{k-p}}{p-\sum_{k=n+p}^{\infty}k\alpha_k|z|^{k-p}}\leq p-m$$

or $\sum_{k=n+p}^{\infty}\frac{k(k-m)}{p(p-m)}\alpha_k|z|^{k-p}\leq 1$

Since $f(z)\in S(p,\gamma,\lambda,\delta)$ if and only if

$$\sum_{k=n+p}^{\infty}\alpha_k B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]<1$$

The inequality (7) hold true if

$$\frac{k(k-m)}{p(p-m)}|z|^{k-p}<B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]$$

Thus we get

$$|z|<R'_3=\inf_{k\geq n+p}\left[\left(\frac{p(p-m)}{k(k-m)}\right)\left(B_p^{\delta,\mu}(k)\right)\left(\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right)\right]^{\frac{1}{k-p}}$$

Hence the theorem is proved.

III. ARITHMETIC MEAN PROPERTY

Theorem 4: Let us consider two functions $f(z)$ and $g(z)$ such that $f(z),g(z)\in S(p,\gamma,\lambda,\delta)$ then the function $h(z)$ defined as $h(z)=\frac{1}{2}(f(z)+g(z))$ is also belonging to the class $S(p,\gamma,\lambda,\delta)$.

Proof: Let us consider $f(z),g(z)\in S(p,\gamma,\lambda,\delta)$ and defined as

$$f(z)=z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k$$

and $$g(z)=z^p-\sum_{k=n+p}^{\infty}\beta_k z^k$$

then we have

$$\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]\alpha_k<1$$

and

$$\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]\beta_k<1$$

To prove: $h(z)=\frac{1}{2}\left(f(z)+g(z)\right)$

$$h(z)=z^p-\sum_{k=n+p}^{\infty}\left(\frac{\alpha_k+\beta_k}{2}\right)z^k\in S(p,\gamma,\lambda,\delta)$$

For this we show that

$$\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]\frac{(\alpha_k+\beta_k)}{2}<1$$

$$\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]\frac{(\alpha_k+\beta_k)}{2}$$

$$=\frac{1}{2}\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]\alpha_k$$

$$+\frac{1}{2}\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]\beta_k$$

$$<\frac{1}{2}+\frac{1}{2}=1$$

Hence the theorem is proved.

IV. CONVEX SET PROPERTY

Theorem 5: Let us consider two functions $f(z)$ and $g(z)$ such that $f(z),g(z)\in S(p,\gamma,\lambda,\delta)$ then the function $h(z)$ defined as

$$h(z)=z^p-\sum_{k=n+p}^{\infty}\left(\varphi\alpha_k+(1-\varphi)\beta_k\right)z^k$$

also belonging to class $S(p,\gamma,\lambda,\delta)$

Proof: Let $f(z)=z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k$

and $g(z)=z^p-\sum_{k=n+p}^{\infty}\beta_k z^k$

belonging to class $S(p,\gamma,\lambda,\delta)$ so we have

$$\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]\alpha_k<1$$

and

$$\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]\beta_k<1$$

To prove: $h(z)=z^p-\sum_{k=n+p}^{\infty}\left(\varphi\alpha_k-(1-\varphi)\beta_k\right)z^k$

belonging to class $S(p,\gamma,\lambda,\delta)$ for this we show that

$$\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right](\varphi\alpha_k+(1-\varphi)\beta_k)<1$$

From

$$\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right](\varphi\alpha_k+(1-\varphi)\beta_k)$$

$$=\varphi\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]\alpha_k$$

$$+(1-\varphi)\sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k)\left[\frac{k[(k-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+1-\gamma]-\lambda(1-\gamma)}\right]\beta_k$$

$$<\varphi+(1-\varphi)=1$$

Hence the theorem is proved.

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