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An Investigation on Jacobian Integral of the Equations of Motion of the System in the Elliptic Orbit of the Centre of Mass

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ARTICLE INFO	ABSTRACT
Published Online: **03 January 2025** **Corresponding Author:** **Avinash Kumar Sharma**	In this article, we discuss the effect of earth's oblateness and magnetic force on the motion and stability of the system for elliptic orbit of the centre of mass. We have got a set of non-linear, non-homogenous and non-autonomous equations. Generally, a system moving in space has to face some perturbative forces. These forces compel the system to change its orbit from circular to elliptic orbit. On account of this we have studied in details the problem in elliptic orbit of the centre of mass. This is simply the generalization of the particular case of the Keplerian orbit i.e. circular orbit. So, we analysed the effect of the earth's oblateness and magnetic force on the existence and behavior of different equilibrium position of the system. Also, discuss the Jacobian Integral of the averaged equations of motion of the system, Two equilibrium solutions of the problem and after Hooke's modulus of elasticity.

KEYWORDS: Hooke's modulus of elasticity, Non-Linear and Non-Homogenous, Equilibrium position of the system, Jacobian integral.

1.1 INTRODUCTION

We have searched in this article that the effect of earth's oblateness and magnetic force on the motion and stability of the system for elliptic orbit of the centre of mass. This is simply the generalization of deals with the particular case of the Keplerian orbit i.e, circular orbit.

Generally, a system moving in space has to face some Perturbative forces. These forces compel the system to change its orbit from circular to elliptic one. On account of this we have studied in details the problem in elliptic orbit of the centre of mass.

We have considered the two-dimensional case. Periodic terms have been averaged with respect to the true anomaly of the path. We have found there are two equilibrium positions $(a_1, 0)$ and $(0, b_1)$. But the first equilibrium position $(a_1, 0)$ alone is stable. It has been shown that the presence of earth's oblateness and magnetic force does not affect the stability of the first equilibrium position. Only this equilibrium position gives the significant value of λ, the Hooke's modulus of elasticity. This equilibrium position is stable in the Liapunov's sense.

1.2) JACOBIAN INTEGRAL OF THE EQUATION OF MOTION OF THE SYSTEM IN THE ELLIPTIC ORBIT OF THE CENTRE OF MASS

we have obtained the system of equations can be written as

$$x'' - 2y' - 3\rho x - \frac{4\beta}{\rho}x = -\bar{\lambda}_\alpha\left[\rho^4 - \frac{\rho^3 I_0}{r}\right]x + \left(\frac{A}{\rho}cos i\right)$$

$$y'' + 2x' + \frac{\beta}{\rho}y = -\bar{\lambda}_\alpha\left[\rho^4 - \frac{\rho^3 I_0}{r}\right]y \qquad \ldots \qquad [1.2.1]$$

Where $\bar{\lambda}_\alpha = \lambda_\alpha \frac{\rho^3}{\mu}$ $\qquad \beta = 3k_2/\rho^2$

$$A = \left\{\frac{m_2}{m_1+m_2}\right\}\left(\frac{Q_1}{m_1} - \frac{Q_2}{m_2}\right)\left(\frac{\mu_E}{\sqrt{\mu p}}\right) \qquad r^2 = x^2 + y^2$$

In the case of elliptic orbit of the centre of mass. The dashes denote differentiation w.r. t. the true anomaly v of the orbit. The condition for constraint will be given by

$$x^2+y^2 \le \frac{I_0^2}{\rho^2} \qquad \ldots \qquad [1.2.2]$$

Now, we obtain the averaged values of the following terms

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\rho}dv = I$$

$$\frac{1}{2\pi}\int_0^{2\pi}\rho\, dv = \frac{1}{(1-e^2)^{1/2}}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\rho^2\, dv = \frac{1}{(1-e^2)^{3/2}}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\rho^3\, dv = \frac{(2+e^2)}{2(1-e^2)^{5/2}}$$

And

$$\frac{1}{2\pi}\int_0^{2\pi}\rho^4\, dv = \frac{(2+3e^2)}{2(1-e^2)^{7/2}} \qquad \ldots \qquad [1.2.3]$$

Putting these values in the system of equations (1.2.1), we have

$$x''-2y'-3x\left\{\frac{1}{(1-e^2)^{1/2}}\right\}-4\beta x = -\bar{\lambda}_\alpha\left[\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\}-\left(\frac{I_0}{2r}\right)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}\right]x + A\cos i$$

$$y''+2x'+\beta y = -\bar{\lambda}_\alpha\left[\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\}-\left(\frac{I_0}{2r}\right)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}\right]y \qquad \ldots \qquad [1.2.4]$$

The condition of the constraints [1.2.2] reduces to

$$x^2+y^2 \le I_0^2(1-e^2)^{3/2} \qquad \ldots \qquad [1.2.5]$$

Thus, the system of equation [1.2.4] will represent the motion of the system with the condition of constraints [1.2.5]

Now, we have discussed three types of motion

i) Free motion, when $\bar{\lambda}_\alpha = 0$

ii) Constrained motion, when $\bar{\lambda}_\alpha \ne 0$

iii) Evolutional motion (combination of free and constrained motion)

We have that even in the particular case of circular orbit of the centre of mass free motion is bound to be converted into a constrained motion with the lapse of time. The free motion is easily integrated in terms of simple functions. Hence, we assume that the system is describing constrained motion. Here, equality sign will hold in the condition [1.2.5]. therefore, the satellite m_1 will move on the sphere.

$$x^2+y^2 = I_0^2(1-e^2)^{3/2} \qquad \ldots \qquad [1.2.6]$$

We see that true anomaly is not present in the system of equations [1.2.4] explicitly. Therefore, there exists Jacobi’s integral for the system of equations [1.2.4].

We multiply two equations of [1.2.4] by $2x'$ and $2y'$ respectively and then adding and integrating, we have,

$$x'^2+y'^2-\left\{\frac{3x^2}{(1-e^2)^{1/2}}\right\}-4\beta x^2+\beta y^2+\bar{\lambda}_\alpha\left[\left\{\left(\frac{(2+3e^2)}{2(1-e^2)^{3/2}}\right)\right\}(x^2+y^2)\right]- \bar{\lambda}_\alpha I_0\left[\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}(x^2+y^2)^{\frac{1}{2}}\right]-2A\cos i = h_e \qquad \ldots \qquad [1.2.7]$$

Where h_e is the constant of integration. Equation [1.2.7] is the Jacobi’s Integral and h_e is called Jacobian constant.

For, obtaining the equations of the surface of zero velocity, we put

$x'^2+y'^2 = 0$ in [1.2.7]

∴ Surface of zero velocity is given by –

$$\left\{\frac{3x^2}{(1-e^2)^{1/2}}\right\}+4\beta x^2-\beta y^2-\bar{\lambda}_\alpha\left[\left\{\left(\frac{(2+3e^2)}{2(1-e^2)^{3/2}}\right)\right\}(x^2+y^2)\right]+ \bar{\lambda}_\alpha I_0\left[\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}(x^2+y^2)^{\frac{1}{2}}\right]+2A\cos i + h_e = 0 \qquad \ldots \qquad [1.2.8]$$

From this it follows that the satellite m_1 will be moving within the boundary of the curve of zero velocity represented by [1.2.8] for different values of the Jacobian constant h_e.

1.3) EQUILIBRIUM SOLUTION OF THE PROBLEM

We have deduced the system of equations [1.2.4] of motion of the system in rotating frame of reference for the case of elliptic orbit of the centre of mass. We have assumed that the system is moving under the effective constraint.

The general solutions of the system of equations [1.2.4] is beyond our reach and hence we shall obtain some particular solution called "Equilibrium Position" for the system. Let the co-ordinates of the equilibrium position be

$x = x_1 = constant,\qquad y = y_1 = constant$ … [1.3.1]

$\therefore x' = x'' = 0,\qquad y' = y'' = 0$ … [1.3.2]

Hence, at the equilibrium position new co-ordinates [1.3.1] and their derivatives [1.3.2] will satisfy [1.2.4]

∴ Equations [1.2.4] will take the from

$$-\left\{\frac{3x_1}{(1-e^2)^{1/2}}\right\} - 4\beta x_1 = -\bar{\lambda}_\alpha \left[\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\} - \frac{I_0}{2r_1}\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}\right] x_1 + Acosi$$

$$\beta y_1 = -\bar{\lambda}_\alpha \left[\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\} - \frac{I_0}{2r_1}\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}\right] y_1 \quad \ldots \quad [1.3.3]$$

Where, $r_1 = \sqrt{x^2 + y^2}$ … [1.3.4]

Now, we shall discuss two particular solutions to the system of equations [1.3.3]

i) The system may be wholly extended along x-axis. In this case, y = 0. Let the first equilibrium position be $(a_1, 0)$.

ii) The system may be wholly extended along y-axis. In this case, x =0. Let this equilibrium position be $(0, b_1)$.

Now, we shall calculate the values of constant a_1 and b_1 in the problem.

i) FIRST EQUILIBRIUM POSITION $(a_1, 0)$:

$$-\left\{\frac{3x_1}{(1-e^2)^{1/2}}\right\} - 4\beta x_1 = -\bar{\lambda}_\alpha \left[\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\} - \frac{I_0}{2x_1}\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}\right] x_1 + Acos\, i$$

$$or, -\left\{\frac{3x_1}{(1-e^2)^{1/2}}\right\} - 4\beta x_1 + \bar{\lambda}_\alpha \left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\}x_1 = \bar{\lambda}_\alpha I_0\left\{\frac{(2+e^2)}{2(1-e^2)^{5/2}}\right\} x_1 + Acos\, i$$

$$or, x_1\left[\left\{\frac{-3}{(1-e^2)^{1/2}}\right\} - 4\beta + \bar{\lambda}_\alpha\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\}\right] = \frac{\bar{\lambda}_\alpha I_0(2+e^2)+2Acos\, i\,(1-e^2)^{5/2}}{2(1-e^2)^{5/2}}$$

$$or, x_1\left[\frac{-6(1-e^2)^3 - 8\beta(1-e^2)^{7/2} + \bar{\lambda}_\alpha(2+3e^2)}{2(1-e^2)^{7/2}}\right] = \frac{\bar{\lambda}_\alpha I_0(2+e^2)+Acos\, i\; 2(1-e^2)^{5/2}}{2(1-e^2)^{5/2}}$$

$$or, x_1 = \frac{\bar{\lambda}_\alpha I_0(1-e^2)(2+e^2) + 2\,Acos\, i\,(1-e^2)^{7/2}}{\bar{\lambda}_\alpha(2+3e^2) - 2(1-e^2)^3\{3+4\beta(1-e^2)^{1/2}\}}$$

The first equilibrium position is

$$\left[\frac{\bar{\lambda}_\alpha I_0(1-e^2)(2+e^2)+2\,Acos\, i\,(1-e^2)^{7/2}}{\bar{\lambda}_\alpha(2+3e^2)-2(1-e^2)^3\{3+4\beta(1-e^2)^{1/2}\}}, 0\right] \quad \ldots \quad [1.3.5]$$

ii) SECOND EQUILIBRIUM POSITION $(0, b_1)$

We have, from the second equation of [1.2.4]

$$\beta = -\bar{\lambda}_\alpha\left[\left\{\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}}\right\} - \frac{I_0}{2y_1}\left\{\frac{(2+e^2)}{(1-e^2)^{\frac{5}{2}}}\right\}\right]$$

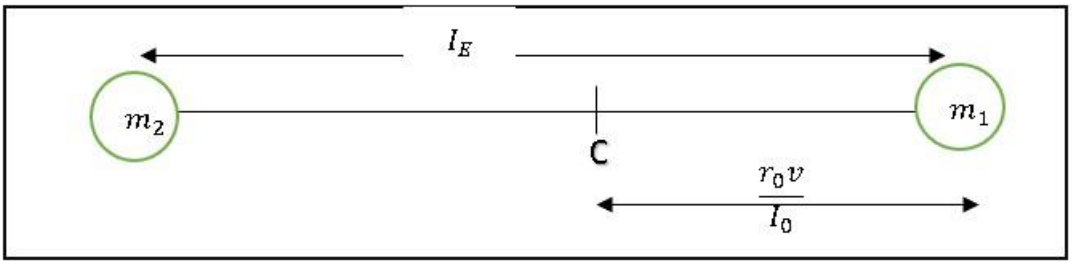
$$y_1 = \frac{\bar{\lambda}_\alpha I_0(1-e^2)(2+e^2)}{\bar{\lambda}_\alpha(2+3e^2)+2\beta\,(1-e^2)^{7/2}}$$

Second equilibrium position is

$$\left[0, \frac{\bar{\lambda}_\alpha I_0(1-e^2)(2+e^2)}{\bar{\lambda}_\alpha(2+3e^2)+2\beta\,(1-e^2)^{7/2}}\right] \quad \ldots \quad [1.3.6]$$

Thus, we have obtained the co-ordinates of the points of the two equilibrium positions of the system as given in [1.3.5] and [1.3.6].

1.4) THE VALUE OF THE MODULUS OF ELASTICITY λ:



Let us assume that the extended length of the cable connecting the two satellites is I_E at any equilibrium position and r_0 the length of the normalized extended cable between the centre of mass of the system and the satellite m_1. Hence, actual extended length of the cable between m_1 and the centre C of the mass will be $\frac{r_0\, v}{I_0}$, where $v = \frac{I_0 m_2}{M}$

Now, taking the moments of different masses about the satellite m_1 in amy equilibrium position,

$$m_2 I_E = (m_1 + m_2)\left(\frac{r_0 v}{I_0}\right)$$

Putting the value of v, we get

$$r_0 = I_E \quad \ldots \quad [1.4.1]$$

Therefore, we shall consider the two equilibrium positions of the system separately for obtaining the value of Hooke's modulus of elasticity λ.

i) First Equilibrium Position $(a_1, 0)$:

In this case,

$$r_0 = a_1 = \frac{\bar{\lambda}_\alpha I_0(1-e^2)(2+e^2)+2\,A\cos i\,(1-e^2)^{7/2}}{\bar{\lambda}_\alpha(2+3e^2)-2(1-e^2)^3\left\{3+4\beta(1-e^2)^{1/2}\right\}} \quad \ldots \quad [1.4.2]$$

Comparing [4.4.1] and [4.4.2] , we have

$$I_{E_1} = \frac{\bar{\lambda}_\alpha I_0(1-e^2)(2+e^2)+2\,A\cos i\,(1-e^2)^{7/2}}{\bar{\lambda}_\alpha(2+3e^2)-2(1-e^2)^3\left\{3+4\beta(1-e^2)^{1/2}\right\}} \quad \ldots \quad [1.4.3]$$

Where I_{E_1} is the stretched length of the cable in the first equilibrium position.

Simplifying the relation [1.4.3] and putting the value of $\bar{\lambda}_\alpha = \left(\frac{\rho^3\lambda}{\mu I_0}\right)\left\{\frac{(m_1+m_2)}{m_1.m_2}\right\}$, we get

$$\lambda = \frac{2\mu I_0.m_1.m_2}{\rho^3(m_1+m_2)}\;\frac{I_{E_1}(1-e^2)^3\left\{3+4\beta(1-e^2)^{1/2}\right\}+2\,A\cos i\,(1-e^2)^{7/2}}{2(I_{E_1}-I_0)+(3I_{E_1}-I_0)e^2+I_0\,e^4} = +ve \quad \ldots \quad [1.4.4]$$

The relation [1.4.4] gives a meaningful value of λ in this case.

ii) Second Equilibrium Position $(0, b_1)$

In this case,

$$r_0 = b_1 = \frac{I_0\bar{\lambda}_\alpha(1-e^2)(2+e^2)}{\bar{\lambda}_\alpha(2+3e^2)+2\beta(1-e^2)^{7/2}} \quad \ldots \quad [1.4.5]$$

$$I_{E_2} = r_0 = b_1 = \frac{I_0\bar{\lambda}_\alpha(1-e^2)(2+e^2)}{\bar{\lambda}_\alpha(2+3e^2)+2\beta(1-e^2)^{7/2}} \quad \ldots \quad [1.4.6]$$

Where I_{E_2} being the extended length of the cable in the second equilibrium position.

Again, simplifying the relation [1.4.6] and putting the value of $\bar{\lambda}_\alpha = \left(\frac{\rho^3 I}{\mu I_0}\right)\left\{\frac{(m_1+m_2)}{m_1.m_2}\right\}$,

We get

$$\lambda = -\left(\frac{2\mu\beta I_0 I_{E_2}}{\rho^3}\right)\left\{\frac{m_1.m_2}{(m_1+m_2)}\right\}\left\{\frac{(1-e^2)^{7/2}}{2\left(I_{E_2}-I_0\right)+3\left(I_{E_3}+I_0\right)e^2+I_0e^4}\right\} = -ve$$

Hence, in the second equilibrium position λ is (–) ve. But Hooke's modulus of elasticity can not be (–) ve.

We conclude that second equilibrium position is untenable.

In this way, we conclude that only the first position of equilibrium provides meaningful value of λ and rest position give meaningless value of λ.

Therefore, we shall establish the stability for the system in the first equilibrium position $(a_1, 0)$ only.

1.5) STABILITY OF THE SYSTEM

We shall study the stability of the first equilibrium position of the system in the Liapunov's sense. The first equilibrium position is given by

$$x = a_1, \qquad y = 0$$

Let us suppose that there are small variations in the co-ordinates at the given equilibrium position.

Let σ_1, σ_2 be small variations in x, y co-ordinates respectively for a given position of equilibrium.

$$x = a_1 + \sigma_1\,, \qquad y = \sigma_2$$

$$x' = \sigma_1'\,, \qquad y' = \sigma_2'$$

$$x'' = \sigma_1''\,, \qquad y'' = \sigma_2'' \quad \ldots \quad [1.5.1]$$

Substituting these values in the set of equations [1.2.4], we have a system of variational equations.

$$\sigma_1'' - 2\sigma_2' - \left\{\frac{3(a_1+\sigma_1)}{(1-e^2)^{1/2}}\right\} - 4\beta(a_1+\sigma_2) = -\bar{\lambda}_\alpha\left[\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\} - \left\{\left(\frac{I_0}{2r_2}\right)\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}\right](a_1+\sigma_1) + A\cos i$$

And

$$\sigma_2'' + 2\sigma_1' + \beta\sigma_2 = -\bar{\lambda}_\alpha\left[\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\} - \left\{\left(\frac{I_0}{2r_2}\right)\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}\right]\sigma_2 \quad \ldots \quad [1.5.2]$$

$$\text{Where } r_2{}^2 = (a_1+\sigma_2)^2 + \sigma_2{}^2 \quad \ldots \quad [1.5.3]$$

We have already obtained the existence of Jacobi's integral in the original set of equations, the variational equations [1.5.2] must have the same.

We can easily obtain the Jacobi's Integral for the system of equations [1.5.2]. for this, multiplying the equations [1.5.2] by $(a_1\ \sigma_2)', 2\sigma_2'$ respectively and adding them together, we shall get after integration.

$$\sigma_1'^2 + \sigma_2'^2 - [\left\{\frac{3}{(1-e^2)^{1/2}}\right\} + 4\beta](a_1+\sigma_2)^2 + \beta\sigma_2^2 + \bar{\lambda}_\alpha\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\}\{(a_1+\sigma_2)^2+\sigma_2^2\} - \bar{\lambda}_\alpha I_0\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}\{(a_1+\sigma_1)^2 + \sigma_2^2\}^{\frac{1}{2}} + 2A(a_1+\sigma_2)cosi = h_2 \quad \ldots \quad [1.5.4]$$

Where h_2 is the constant of integration. In this way, we have obtained the equation [1.5.4] as Jacobi's Integral for the system of variation equation.

Expanding the terms, the equation [1.5.4] can be written as,

$$V_e(\sigma_1', \sigma_2', \sigma_1, \sigma_2) = \sigma_1'^2 + \sigma_2'^2 + \sigma_1{}^2\left[-\left\{\frac{3}{(1-e^2)^{1/2}}\right\} - 4\beta + \bar{\lambda}_\alpha\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\}\right] + \sigma_2{}^2\left[\beta + \bar{\lambda}_\alpha\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\} - \left(\frac{\bar{\lambda}_\alpha I_0}{2a_1}\right)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}\right] + \sigma_1\left[\left\{\frac{\bar{\lambda}_\alpha a_1(1+3e^2)}{(1-e^2)^{7/2}}\right\} - \left\{\frac{6a_1}{(1-e^2)^{1/2}}\right\} - 8\beta a_1 - (\bar{\lambda}_\alpha I_0)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\} - 2A\ cosi\right] + \left[-4\beta\ a_1{}^2 - \left\{\frac{3\ a_1{}^2}{(1-e^2)^{1/2}}\right\} + (\bar{\lambda}_\alpha)\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\}a_1{}^2 - (\bar{\lambda}_\alpha I_0)\left\{\frac{(2+e^2)\ a_1}{(1-e^2)^{5/2}}\right\} - 2\ a_1 Acosi\right] + 0(3) = h_2 \quad \ldots \quad [1.5.5]$$

Where 0(3) denotes the third and higher order terms in the small quantities σ_1 *and* σ_2 .

Similar to the circular orbit, we shall obtain the sufficient conditions for the stability with the help of Liapunov's theorem on stability. The Jacobian Integral V_e is the integral of the system for variational equations [1.5.2], its differential equation taken along the trajectory of the system must vanish identically.

Hence, the only condition that the unilateral position be stable in the Liapunov's senses the V_e must be positive definite. For making the function a ositive definite function it is necessary that the function [1.5.5] does not have the term of the first order in the variables shown in its argument and the terms of the second order must satisfy the Sylvestor's condition for positive definiteness of the quadratic form. The third and higher order terms will have no effect on the sign of the function V_e.

Hence, we conclude that the sufficient conditions for stability of the system at the said equilibrium position in the Liapunov's sense are

i) $(\bar{\lambda}_\alpha a_1)\left\{\frac{(2+3e^2)}{(1-e^2)^{7/2}}\right\} - \left\{\frac{6a_1}{(1-e^2)^{1/2}}\right\} - 8\beta a_1 - (\bar{\lambda}_\alpha I_0)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\} - 2Acosi = 0$

ii) $\begin{vmatrix} B_1 & 0 \\ 0 & B_2 \end{vmatrix} = +ve\ i.e;\ B_1 B_2 = +ve$

iii) $B_2 = +ve$ … [1.5.6]

Where $B_1 = \left\{\frac{-3}{(1-e^2)^{1/2}}\right\} - 4\beta + (\bar{\lambda}_\alpha)\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\}$

$$B_2 = \beta + (\bar{\lambda}_\alpha)\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\} - \left(\frac{\bar{\lambda}_\alpha I_0}{2a_1}\right)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}$$

Comparing conditions (ii) & (iii) of [1.5.6]. we shall have the sufficient conditions for stability of the system in the form

i) $(\bar{\lambda}_\alpha a_1)\left\{\frac{(2+3e^2)}{(1-e^2)^{7/2}}\right\} - \left\{\frac{6a_1}{(1-e^2)^{1/2}}\right\} - 8\beta a_1 - (\bar{\lambda}_\alpha I_0)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\} - 2Acosi = 0$

ii) $-\left\{\frac{3}{(1-e^2)^{1/2}}\right\} - 4\beta + (\bar{\lambda}_\alpha)\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\} > 0$

iii) $\beta + (\bar{\lambda}_\alpha)\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\} - \left(\frac{\bar{\lambda}_\alpha I_0}{2a_1}\right)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\} > 0$ … [1.5.7]

Let us now analyse the several condition of [1.5.7] for stability of the system at the given equilibrium position separately.

Condition (i)

$$LHS = a_1\left[(\bar{\lambda}_\alpha)\left\{\frac{(2+3e^2)}{(1-e^2)^{7/2}}\right\}-\left\{\frac{6}{(1-e^2)^{1/2}}\right\}-8\beta\right]-(\bar{\lambda}_\alpha I_0)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}+2Acosi$$

$$=a_1\left[\left\{\frac{\bar{\lambda}_\alpha(2+3e^2)-6(1-e^2)^3-8\beta(1-e^2)^{7/2}}{(1-e^2)^{7/2}}\right\}\right]-(\bar{\lambda}_\alpha I_0)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}+2Acosi$$

$$=\left[\left\{\frac{\bar{\lambda}_\alpha I_0(1-e^2)(2+e^2)+2Acosi(1-e^2)^{7/2}}{(1-e^2)^{7/2}}\right\}\right]-(\bar{\lambda}_\alpha I_0)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}+2Acosi$$

$$=\left[\left\{\frac{\bar{\lambda}_\alpha I_0(2+e^2)}{(1-e^2)^{5/2}}\right\}\right]+2Acosi-(\bar{\lambda}_\alpha I_0)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}+2Acosi=0$$

The first condition is satisfied identically.

Condition (ii)

$$LHS = (\bar{\lambda}_\alpha)\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\}-4\beta-\left\{\frac{3}{(1-e^2)^{1/2}}\right\}$$

$$=\left\{\frac{1}{2(1-e^2)^{7/2}}\right\}\left[\bar{\lambda}_\alpha(2+3e^2)-8\beta(1-e^2)^{7/2}-6(1-e^2)^3\right]$$

$$=\left\{\frac{1}{2(1-e^2)^{7/2}}\right\}\left[\frac{I_0\bar{\lambda}_\alpha(1-e^2)(2+e^2)+2Acosi(1-e^2)}{a_1}\right]$$

$$\left[\ a_1=\frac{I_0\bar{\lambda}_\alpha(1-e^2)(2+e^2)+2Acosi(1-e^2)}{\{\bar{\lambda}_\alpha(2+3e^2)-2(1-e^2)^3\}\left\{3+4\beta(1-e^2)^{1/2}\right\}}=+ve\right.$$

The second condition is also satisfied identically.

Condition (iii)

$$LHS = \beta+(\bar{\lambda}_\alpha)\left\{\frac{(2+3e^2)}{2(1-e^2)^{7/2}}\right\}-\left(\frac{\bar{\lambda}_\alpha I_0}{2a_1}\right)\left\{\frac{(2+e^2)}{(1-e^2)^{5/2}}\right\}$$

$$=\beta+\left(\frac{\bar{\lambda}_\alpha}{2a_1(1-e^2)^{7/2}}\right)[\{a_1(2+3e^2)\}-I_0(2+e^2)(1-e^2)]$$

$$=\beta+\left(\frac{\bar{\lambda}_\alpha}{2a_1(1-e^2)^{7/2}}\right)\left[(2+3e^2)\left\{\frac{I_0\bar{\lambda}_\alpha(1-e^2)(2+e^2)+2Acosi(1-e^2)^{\frac{7}{2}}}{\bar{\lambda}_\alpha(2+3e^2)-2(1-e^2)^3\left\{3+4\beta(1-e^2)^{1/2}\right\}}\right\}-I_0(1-e^2)(2+e^2)\right]$$

$$=\beta+\left(\frac{\bar{\lambda}_\alpha}{2a_1(1-e^2)^{7/2}}\right)\left[\left\{\frac{I_0\bar{\lambda}_\alpha(2+3e^2)(1-e^2)(2+e^2)+2Acosi(2+3e^2)(1-e^2)^{\frac{7}{2}}}{\bar{\lambda}_\alpha(2+3e^2)-2(1-e^2)^3\left\{3+4\beta(1-e^2)^{1/2}\right\}}\right.\right.$$

$$\left.\left.-\frac{I_0\bar{\lambda}_\alpha(2+3e^2)(1-e^2)(2+e^2)+2I_0(2+e^2)(1-e^2)^4\{3+4\beta(1-e^2)^{1/2}\}}{\bar{\lambda}_\alpha(2+3e^2)-2(1-e^2)^3\left\{3+4\beta(1-e^2)^{1/2}\right\}}\right\}\right]$$

$$=\beta+\left(\frac{\bar{\lambda}_\alpha}{2a_1(1-e^2)^{7/2}}\right)\left[\left\{\frac{6I_0(2+e^2)(1-e^2)^4+8I_0\beta(2+e^2)(1-e^2)^{9/2}+2Acosi(2+3e^2)(1-e^2)^{\frac{7}{2}}}{\bar{\lambda}_\alpha(2+3e^2)-2(1-e^2)^3\left\{3+4\beta(1-e^2)^{1/2}\right\}}\right\}\right]$$

$$=\beta+\left(\frac{\bar{\lambda}_\alpha}{2a_1(1-e^2)^{7/2}}\right)\left[\left\{\frac{6I_0(2+e^2)(1-e^2)^4+8I_0\beta(2+e^2)(1-e^2)^{9/2}+2Acosi(2+3e^2)(1-e^2)^{\frac{7}{2}}}{I_0\bar{\lambda}_\alpha(2+e^2)(1-e^2)+2Acosi(1-e^2)^{7/2}}\right\}\right]$$

= (+) ve

The third condition is also satisfied.

Thus, we see that the three conditions of [1.5.7] for stability are satisfied identically.

Therefore, we conclude that the equilibrium is stable at $(a_1, 0)$ in the Liapunov's sense, where β, λ_α and Acosi have usual meanings.

If we compare the condition for the stability of circular orbit with, in the present case of elliptic orbit, we observe that both the conditions are similar. If we put e = 0 in [1.5.7], we shall obtain the same result as in stability of the system of circular orbit.

1.6) CONCLUSION

The equilibrium solution of the problem and their stability in case of the elliptic orbit of the centre of mass of the system, on the basis of the analysis of the free motion of the system it has been proved that all the motions of the system are bound to be converted into constrained one and hence the Jacobian Integral of the averaged equations of motion of the system has been obtained. Two equilibrium solutions of the problem have been obtained and after obtaining Hooke's modulus of elasticity it has been shown that only one equilibrium solution is stable in the Liapunov's sense. Moreover, it has been concluded that only this equilibrium position is stable in the Liapunov's sense.

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