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Generalized * - Higher Derivation on Prime * - Rings

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ABSTRACT:
In this paper, first we prove that, let R be a prime *- ring .If R admits a generalized *- higher derivation f with an associated non zero reverse *- higher derivation d then either $(d(x),d(z)=0)$(or) f is a left *multiplier. And next we prove that, let R be a prime ring, if R admits a generalized * left higher derivation associated with * left higher derivation d, then either $(d(y),d(z)=0)$ (or) f is a right * multiplier.

Key words: Higher derivation, Generalized higher derivation, *- higher derivation *- generalized higher derivation, reverse *- derivation, commutator.

Introduction:
Let R be an associative ring not necessarily with an identity element .A derivation (resp.Jordan derivation) ‘d’ of R is an additive mapping $d: R \to R$ Such that $d(xy) = d(x)y + x(dy)$, for every $x, y \in R$ (resp. $d(x^2) = d(x)x + xd(x)$, for every $x, y \in R$).As its is well known, every derivation is a Jordan derivation and the converse is, in general not true .If R is a 2-torsion free semi prime ring, then by the results of I.N .Herstein and M. Bresar ,every Jordan derivation of R is a derivation ((1),(2),(3).
Following B.Hvala page (4) an additive mapping $F: R \to R$ is called a generalized derivation if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.we call an additive mapping $F: R \to R$ a Jordan generalized derivation if there exists a derivation $d: R \to R$ such that $F(x^2) = F(x)x + xd(x)$ holds for all $x \in R$ ([1]).
On the other hand, higher derivations have been studied in many papers mainly in commutative rings, but also in non- commutative rings. M. Ferrero and C.Haetinger extended some of the above results to the higher derivations, in particular ,they pointed out that every Jordan higher derivation in a 2-torsion –free semi prime ring is a higher derivation ([6]), .Thus, it is natural to ask whether every Jordan generalized higher derivation on a ring R is a generalized higher derivation.
Now we give the Corresponding definitions.
As usual, $[x, y]$ will denote the commutator $xy - yx$ and N is the set of natural numbers including 0.

1. Definitions

Definition 1.1 Let $D = (d_i)_{i \in N}$ is a family of additive mappings of R such that $d_0 = id_R$.D is said to be a higher derivation if every $n \in N$ we have $d_n(xy) = \sum_{i+j=n} d_i(x) d_j(y)$ for all $x, y \in R$.

A Jordan higher derivation If for every $n \in N$ we have $d_n(x^2) = \sum_{i+j=n} d_i(x) d_j(x)$ for all $x \in R$.

Definition 1.2: Let $F=(f_i)_{i\in N}$ be a family of additive mappings of R such that $f_0=id_R$. F is said to be a generalized higher derivation if there exists a higher derivation $D=(d_i)_{i\in N}$ of R such that for every $n\in N$ we have $f_n(xy)=\sum_{i+j=n} f_i(x)d_j(y)$ for all $x,y\in R$.

Definition 1.3: Let $D=(d_i)_{i\in N}$ be a family of additive mappings of R such that $d_0=id_R$ D is said to be higher *-derivation, if for every $n\in N$ we have $d(xy)=\sum_{i+j=n} d_i(x)d_j(y^*)^i$ $x,y\in R$.

A Jordan higher *- derivation of R, if for each $n\in N$, $d_n(x^2)=\sum_{i+j=n} d_i(x)d_j(x^*)^i$ for all $x\in R$.

A Generalized higher * derivation of R, if for each $n\in N$ $f_n(xy)=\sum_{i+j=n} f_i(x)d_j(y^*)^i$ for all $x,y\in R$.

A mapping $d:R\to R$ is called centralizer if $[d(X),X]\in Z(R)$ for all $x\in R$.

Main Results:

Theorem 1.4: Let R be a prime *- ring .if R admits a generalized *- higher derivation f with an associated non-zero reverse *- higher derivation then either $(d(x),d(z)=0)$(or) f is a left * multiplier.

Proof. We are given that f is a generalized reverse * - higher derivation with an associated non – zero reverse * - higher derivation, we have

$$f_n(xy)=\sum_{i+j=n} f_i(y)d_j(x^*)^i \qquad (1)$$

Replace x by xz in equation (1) we have

$$f_n(xzy)=\sum_{i+j=n} f_i(y)d_j((xz)^*)^i$$

$$=\sum_{i+j=n} f_i(y)d_j(z^*x^*)^i$$

$$=\sum_{i+j+l=n} f_i(y)d_j(x^*)^i d_l(z)^{i+j} \qquad (2)$$

On the other hand

$$f_n(xzy)=\sum_{i+j=n} f_i(zy)d_j(x^*)^i$$

$$f_n(xzy)=\sum_{i+j+l=n} f_i(y)d_l(z^*)^{l+j} d_j(x^*)^i \qquad (3)$$

Replacing x^* by x and z^* by z ,reordering the indices and comparing equations (2) and (3), we have

$$\sum_{i+j+l=n} f_i(y)(d_j(x),d_l(z))=0$$

Then either $(d(x),d(z))=0$ (or) f is a left * multiplier.

Theorem 1.5: Let R be a prime ring, If R admits a generalized * left higher derivation f associated with * left higher derivation d then either R is $(d(y),d(z))=0$ (or) f is a right *- multiplier.

Proof. By the definition of generalized * - higher left derivation

$$f_n(xy)=\sum_{i++j=n} d_j(y^*)^i f_i(x)$$

Replacing y by yz we have

$$f_n(xyz)=\sum_{i++j=n} d_j(yz^*)^i f_i(x)$$

$$= \sum_{i+j+l=n} d_j\left(z^* y^*\right)^i f_i(x)$$

$$f_n(xyz) = \sum_{i+j+l=n} d_j\left(y^*\right)^i d_l\left(z^*\right)^{l+j} f_i(x) \quad (4)$$

On the other hand

$$f_n(xyz) = \sum_{i+j=n} d_j\left(z^*\right)^i f_i(xy)$$

$$f_n(xyz) = \sum_{i+j+l=n} d_j\left(z^*\right)^i d_l\left(y^*\right)^{l+j} f_i(x) \quad (5)$$

Reordering the indices of equation (4) and (5) ,replacing y^* by y and z^* by z and comparing the equations (4) and (5) ,we have

$$\sum_{i+j+l=n} [d(y), d(z)] f_i(x) = 0$$

Then either $(d(y), d(z)) = 0$ (or) f is a right * -multiplier.

Results .1.6: Let R be a 2-torsion free non-commutative prime * -ring and let $f: R \to R$ be a generalized Jordan higher *- derivation which satisfies $\sum_{i+j=n} d_i(h) d_j\left(h^*\right)^i \in Z(R)$, then

$$[f_n(hg+gh), y] = \sum_{i+j=n} (f_i(h) d_j\left(g^*\right)^i, y) + \sum_{i+j=n} (f_i(g) d_j\left(h^*\right)^i, y)$$

Proof. For any $r \in R$

$$f_n\left(h^2\right) = \sum_{i+j=n} f_i(h) d_j\left(h^*\right)^i, \qquad \text{for all } h \in H(R)$$

Now $f_n\left((h+g)^2\right) = \sum_{i+j=n} f_i(h+g).d_j\left((h+g)^*\right)^i$. (6)

RHS of equation (6) is

$$\sum_{i+j=n} f_i(h+g) d_j\left((h+g)^*\right)^i = \sum_{i+j=n} f_i(h) d_j\left(h^*\right)^i + \sum_{i+j=n} f_i(h).d_j\left(g^*\right)^i + \sum_{i+j=n} f_i(g) d_j\left(h^*\right)^i + \sum_{i+j=n} f_i(g).d_j\left(g^*\right)^i$$

$$= f_n\left(h^2\right) + f_n\left(g^2\right) + \sum_{i+j=n} f_i(h) d_j\left(g^*\right)^i + \sum_{i+j=n} f_i(g) d_j\left(h^*\right)^i \quad .$$

Commuting with y on both sides

$$\left(\sum_{i+j=n} f_i(h+g) d_j\left((h+g)^*\right)^i, y\right) = \left(\sum_{i+j=n} f_n\left(h^2\right), y\right) + \sum_{i+j=n} (f_n\left(g^2\right), y) + \sum_{i+j=n} (f_i(h) d_j\left(g^*\right)^i, y) + \sum_{i+j=n} (f_i(g) d_j\left(h^*\right)^i, y)$$

(7)

LHS of equation (6) is

$$f_n(h+g)^2 = f_n\left(h^2 + g^2 + hg + gh\right)$$

$$= f_n\left(h^2\right) + f_n\left(g^2\right) + f_n(hg+gh)$$

Commute with y on both sides we have

$$\left(f_n(h+g)^2, y\right) = \left(f_n\left(h^2\right), y\right) + \left(f_n\left(g^2\right), y\right) + \left(f_n(hg+gh), y\right) \quad . \quad (8)$$

Comparing equations (7) and (8)

$$f_n((hg+gy),y)=\sum_{i+j=n}(f_i(h)d_j(g^*)^i,y)+\sum_{i+j=n}(f_i(g)d_j(h^*)^i,y) \quad (9)$$

Result 1.7: Let R be a 2-torsion free non commutative prime * -ring and $d:R\to R$ be a Jordan * higher derivation which satisfies $\sum_{i+j=n} d_i(h)d_j(h^*)^i \in Z(R)$, for all $h\in H(R)$,then

$$\sum_{i+j=n} d_n(hg+gh,y)=[\sum_{i+j=n} d_i(h)d_j(g^*)^i+d_i(g)d_j(h^*)^i,y\ ]$$

Proof: By Jordan higher * -derivation, we have

$$d_n(h^2)=\sum_{i+j=n} d_i(h)d_j(h^*)^i \text{ for all } h\in H(R). \quad (10)$$

Replacing 'h' by (h+g) in equation (10) we get

$$d_n(h+g)^2=\sum_{i+j=n} d_i(h+g)d_j((h+g)^*)^i. \quad (11)$$

RHS of equation (11) is

$$\sum_{i+j=n} d_i(h+g)d_j((h+g)^*)^i=\sum_{i+j=n} d_i(h)d_j(h^*)^i+\sum_{i+j=n} d_i(h)d_j(g^*)^i+\sum_{i+j=n} d_i(g)d_j(h^*)^i+\sum_{i+j=n} d_i(g)d_j(g^*)^i$$

$$\sum_{i+j=n} d_i(h+g)d_j((h+g)^*)^i=d_n(h^2)+\sum_{i+j=n} d_i(h)d_j(g^*)^i+\sum_{i+j=n} d_i(g)d_j(h^*)^i+d_n(g^2) \quad (12)$$

Commute with y on both sides, we have

$$\sum_{i+j=n}(d_i(h+g)d_j((h+g)^*)^i,y)=((d_n(h^2),y)+\left(\sum_{i+j=n} d_i(h)d_j(g^*)^i+\sum_{i+j=n} d_i(g)d_j(h^*)^i,y\right)+(d_n(g^2),y)$$ LHS of equation (11) is

$$d_n(h+g)^2=d_n(h^2+g^2+hg+gh)$$
$$=d_n(h^2)+d_n(g^2)+d_n(hg+gh).$$

Commute with ' y ' on both sides we have

$$d_n((h+g)^2,y)=d_n((h^2),y)+d_n((g^2),y)+d_n((hg+gh),y). \quad (13)$$

Comparing equations (12) and (13), we have

$$[d_n(hg+gh),y]=\left(\sum_{i+j=n}(d_i(h)d_j(g^*)^i)+d_i(g)d_j(h^*)^i,y\right), \quad (14)$$

Result: 1.8 Let R be 2-torsion free non-Commutative prime *- ring, and $d:R\to R$ be a Jordan *-higher derivation which satisfies for all $h\in H(R)$ if $\sum_{i+j=n} f_i(h)d_j(h^*)^i\in Z(R)$ then

$$(2f_n(hgh),y)=\left(\sum_{i+j=n}\sum_{j+l=n} f_i(h)d_j(g^*)^i d_l(h^*)^{i+j},y\right)+\left(\sum_{i+j=n}\sum_{j+l=n} f_i(h)d_j(g^*)^i d_l(h^*)^{i+j},y\right)$$

Proof. $d_n(x^2)=\sum_{i+j=n} d_i(x)d_j(x^*)^i$ is a Jordan higher *-derivation

Replacing g by (hg+gh) in equation (9) we have

$$f_n\left(h(hg+gh)+(hg+gh)h,y\right)=\sum_{i+j=n}\left(f_i(h)d_j((hg+gh)^*)^i,y\right)+\sum_{i+j=n}f_i\left((hg+gh)d_j\left(h^*\right)^i,y\right) \quad (15)$$

$$f_n\left(\left(h^2g+hgh+hgh+gh^2\right),y\right)=\sum_{i+j=n}\left(f_i(h)d_j((hg+gh)^*)^i,y\right)+\sum_{i+j=n}(f_i(hg+gh)d_j\left(h^*\right)^i,y)$$

$$f_n\left((h^2g+2hgh+gh^2),y\right)=\sum_{i=j=n}(f_i(h)d_j((hg+gh)^*)^i,y)+\sum_{i+j=n}f_i(hg+gh)d_j\left(h^*\right)^i,y) \quad (16)$$

The RHS of equation (16) is

$$=\sum_{i+j=n}f_i(h)\left(\sum_{l+j=n}d_j(h)d_l\left(g^*\right)^j+\sum_{l+j=n}d_j(g)d_l\left(h^*\right)^j,y\right)+$$

$$\left(\sum_{i+j=n}\sum_{l+j=n}(f_i(h)d_j\left(g^*\right)^i d_l\left(h^*\right)^{i+j},y)+\sum_{i+j=n}\sum_{i+j=n}(f_i(g)d_j\left(h^*\right)^i)d_l\left(h^*\right)^{i+j},y)\right)$$

$$=\sum_{i+j=n}\sum_{j+l=n}(f_i(h)d_j(h)d_l\left(g^*\right)^{i+j},y)+\sum_{i+j=n}\sum_{l+j=n}\left(f_i(h)d_j(g)d_l\left(h^*\right)^{i+j},y\right)$$

$$+\sum_{i+j=n}\sum_{l+j=n}\left(f_i(h)d_j\left(g^*\right)^i d_l\left(h^*\right)^{i+j},y\right)+\sum_{i+j=n}\sum_{l+j=n}\left(f_i(g)d_j\left(h^*\right)^i d_l\left(h^*\right)^{i+j},y\right)$$

$$=(\sum_{i+j=n}f_n\left(h^2\right)d_j\left(g^*\right)^{i+j},y)+\sum_{i+j=n}\sum_{i++=n}\left(f_i(h)d_j(g)d_l\left(h^*\right)^{i+j},y\right)+\sum_{i+j=n}\sum_{i+=n}\left(f_i(h)d_j\left(g^*\right)^i d_l\left(h^*\right)^{i+j},y\right)+$$

$$\sum_{i+j=n}f_i(g)d_n(\left(h^2\right)^*)^i \quad (17)$$

Now LHS of equation (16) is

$$f_n(\left(h^2g+gh^2,y\right)+2f_n(hgh),y)=\sum_{i+j=n}\left(f_i\left(h^2\right)d_j\left(g^*\right)^i+\sum_{i+j=n}f_i(g)d_j\left(h^{2^*}\right)^i,y\right)+2(f_n(hgh),y) \quad (18)$$

By comparing equations (17) and (18) and by reordering indices, we have

$$(2f_n(hgh),y)=\left(\sum_{i+j=n}\sum_{j+l=n}f_i(h)d_j\left(g^*\right)^i d_l\left(h^*\right)^{i+j},y\right)+\left(\sum_{i+j=n}\sum_{j+l=n}f_i(h)d_j\left(g^*\right)^i d_l\left(h^*\right)^{i+j},y\right).$$

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