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Majoration of the modulus of the Schwarzian of certain quasi-conformal maps

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Abstract:

In this work, we have proposed a majoration of the modulus of the Schwarzian of certain quasi-conformal maps, defined in the open unit disk $\Delta = z \in \mathbb{C} : |z| < 1\}$. Using the Schwarzian, we have found some relations linking the Taylor coefficients of these maps.

Keywords:

Quasi-conformal map, class S, complex dilation, Schwarzian, Nehari's theorem, harmonic map.

1. Introduction

In this article, we have proposed, on the one hand, a majoration of the Schwarzian modulus of certain quasi-conformal maps and, on the other hand, some relations on the Taylor coefficients of these maps, defined in the open unit disk using the variational method. This method is proving effective in solving several problems that undermine the geometric theory of complex-variable functions.

The various aspects of using this method are significantly mentioned in a large number of works (for example [3], [4], [7]).

Recall that the Schwarzian of harmonic maps is defined by :

$$\left\{f, z\right\} = 2\left[\left[\ln \lambda(z)\right]_{zz} - \left[(\ln \lambda(z))_z\right]^2\right] \text{avec } \lambda(z) = |h'(z)| + |g'(z)| \tag{1}$$

We will adapt this Schwarzian to a quasi-conformal map, defined in the open unit disk.

2. Theorem

Let $f(z) = h(z) + \overline{g(z)}$ be a quasi-conformal map, defined in $\Delta = \left\{z \in \mathbb{C} : |z| < 1\right\}$, with h and g two univalent and holomorphic maps, defined in Δ, with values in $\mathbb{C}$ such that $h'(0) = 1$. Then for all $z = re^{i\theta} \in \Delta$ we have :

$$\left|\left\{f, z\right\}\right| \leq \frac{2}{(1-r^2)^2} + \left(\frac{4+2r}{1-r^2}\right)^2,$$

Demonstration

- **Expression of $\left\{f,z\right\}$ as a function of $\left\{h,z\right\}$ and $\left\{g,z\right\}$.**

$$\left\{f,z\right\} = 2\left[\left[\ln\lambda(z)\right]_{zz} - \left[(\ln\lambda(z))_z\right]^2\right]$$

Thus

$$\left\{f,z\right\} = \frac{2\lambda''}{\lambda} - 4\left(\frac{\lambda'}{\lambda}\right)^2 \tag{2}$$

Looking for $\lambda'(z)$ and $\lambda''(z)$

$$\lambda'(z) = \left[|h'(z)|\right]' + \left[|g'(z)|\right]'$$

$$|h'(z)| = \sqrt{h'(z).\overline{h'(z)}} \Longrightarrow \left[|h'(z)|\right]' = \frac{h''(z)\overline{h'(z)}}{2|h'(z)|}$$

In the same way :

$$|g'(z)| = \sqrt{g'(z).\overline{g'(z)}} \Longrightarrow \left[|g'(z)|\right]' = \frac{g''(z)\overline{g'(z)}}{2|g'(z)|}$$

The result is $\lambda'(z) = \dfrac{h''(z)\overline{h'(z)}}{2|h'(z)|} + \dfrac{g''(z)\overline{g'(z)}}{2|g'(z)|}$

$$\lambda''(z) = \frac{\overline{h'(z)}}{2}\left[\frac{h''(z)}{|h'(z)|}\right]' + \frac{\overline{g'(z)}}{2}\left[\frac{g''(z)}{|g'(z)|}\right]'$$

$$\lambda''(z) = \frac{h'''(z)\overline{h'(z)}}{2|h'(z)|} - \frac{[h''(z)]^2\left[\overline{h'(z)}\right]^2}{4|h'(z)|^3} + \frac{g'''(z)\overline{g'(z)}}{2|g'(z)|} - \frac{[g''(z)]^2\left[\overline{g'(z)}\right]^2}{4|g'(z)|^3}$$

$$\frac{\lambda''(z)}{\lambda(z)} = \frac{h'''(z)\overline{h'(z)}}{2|h'(z)|\left[|h'(z)|+|g'(z)|\right]} - \frac{[h''(z)]^2\left[\overline{h'(z)}\right]^2}{4|h'(z)|^3\left[|h'(z)|+|g'(z)|\right]} + \frac{g'''(z)\overline{g'(z)}}{2|g'(z)|\left[|h'(z)|+|g'(z)|\right]} - \frac{[g''(z)]^2\left[\overline{g'(z)}\right]^2}{4|g'(z)|^3\left[|h'(z)|+|g'(z)|\right]}$$

Or

$$\mu_{f(z)} = \frac{f_{\overline{z}(z)}}{f_z(z)} = \frac{g'(z)}{h'(z)}$$

Therefore:

$$\frac{\lambda''(z)}{\lambda(z)} = \frac{h'''(z)\overline{h'(z)}}{2|h'(z)|^2\Big(1+|\mu_{f(z)}|\Big)} - \frac{\Big[h''(z)\Big]^2\Big[\overline{h'(z)}\Big]^2}{4|h'(z)|^4\Big(1+|\mu_{f(z)}|\Big)} + \frac{g'''(z)\overline{g'(z)}|\mu_{f(z)}|}{2|g'(z)|^2\Big(1+|\mu_{f(z)}|\Big)} - \frac{\Big[g''(z)\Big]^2\Big[\overline{g'(z)}\Big]^2|\mu_{f(z)}|}{4|g'(z)|^4\Big(1+|\mu_{f(z)}|\Big)}$$

$$\frac{2\lambda''(z)}{\lambda(z)} = \frac{h'''(z)}{h'(z)(1+|\mu_{f(z)}|)} - \frac{[h''(z)]^2}{2(h'(z))^2(1+|\mu_{f(z)}|)} + \frac{g'''(z)|\mu_{f(z)}|}{g'(z)(1+|\mu_{f(z)}|)} - \frac{[g''(z)]^2|\mu_{f(z)}|}{2(g'(z))^2(1+|\mu_{f(z)}|)} \tag{3}$$

$$\frac{\lambda'}{\lambda} = \frac{h''(z)\overline{h'(z)}}{2|h'(z)|\Big[|h'(z)|+|g'(z)|\Big]} + \frac{g''(z)\overline{g'(z)}}{2|g'(z)|\Big[|h'(z)|+|g'(z)|\Big]}$$

$$\frac{\lambda'}{\lambda} = \frac{h''(z)}{2h'(z)(1+|\mu_{f(z)})|} + \frac{g''(z)|\mu_{f(z)}|}{2g'(z)(1+|\mu_{f(z)})|}$$

$$4\left(\frac{\lambda'}{\lambda}\right)^2 = \frac{[h''(z)]^2}{[h'(z)]^2(1+|\mu_{f(z)}|)^2} + \frac{[g''(z)]^2|\mu_{f(z)}|^2}{[g'(z)]^2(1+|\mu_{f(z)}|)^2} + \frac{2g''(z)h''(z)|\mu_{f(z)}|}{g'(z)h'(z)(1+|\mu_{f(z)}|)^2} \tag{4}$$

By replacing the relations (3) and (4) in the relation (2), we have :

$$\Big\{f,z\Big\} = \frac{h'''(z)}{h'(z)(1+|\mu_{f(z)}|)}\frac{[h''(z)]^2}{2(h'(z))^2(1+|\mu_{f(z)}|)} + \frac{g'''(z)|\mu_{f(z)}|}{g'(z)(1+|\mu_{f(z)}|)} - \frac{[g''(z)]^2|\mu_{f(z)}|}{2(g'(z))^2(1+|\mu_{f(z)}|)}$$
$$- \frac{[h''(z)]^2}{[h'(z)]^2(1+|\mu_{f(z)}|)^2} - \frac{[g''(z)]^2|\mu_{f(z)}|^2}{[g'(z)]^2(1+|\mu_{f(z)}|)^2} - \frac{2g''(z)h''(z)|\mu_{f(z)}|}{g'(z)h'(z)(1+|\mu_{f(z)}|)^2}$$

$$\Big\{f,z\Big\} = \frac{1}{(1+|\mu_{f(z)}|)^2}\{h,z\} + \frac{|\mu_{f(z)}|^2}{(1+|\mu_{f(z)}|)^2}\{g,z\} + \frac{|\mu_{f(z)}|}{(1+|\mu_{f(z)}|)^2}\{h,z\} + \frac{|\mu_{f(z)}|}{(1+|\mu_{f(z)}|)^2}\{g,z\} + \frac{|\mu_{f(z)}|}{(1+|\mu_{f(z)}|)^2}\left[\left(\frac{h''(z)}{h'(z)}\right)^2 + \left(\frac{g''(z)}{g'(z)}\right)^2 - 2\left(\frac{h''(z)}{h'(z)}\right)\left(\frac{g''(z)}{g'(z)}\right)\right]$$

$$\Big\{f,z\Big\} = \frac{1}{1+|\mu_{f(z)}|}\{h,z\} + \frac{|\mu_{f(z)}|}{1+|\mu_{f(z)}|}\{g,z\} + \left[\frac{h''(z)}{h'(z)} - \frac{g''(z)}{g'(z)}\right]^2\frac{|\mu_{f(z)}|}{(1+|\mu_{f(z)}|)^2} \tag{5}$$

Let's put : $\tilde{h}(z) = \dfrac{h(z)-h(0)}{h'(0)}$ et $\tilde{g}(z) = \dfrac{g(z)-g(0)}{g'(0)}$

It is easy to check that $\tilde{h}$ and $\tilde{g}$ are functions of class S.

So $$\left\{\tilde{h},z\right\}=\frac{\tilde{h}'''(z)}{\tilde{h}'(z)}-\frac{3}{2}\left[\frac{\tilde{h}''(z)}{\tilde{h}'(z)}\right]^2$$

$$\left\{\tilde{h},z\right\}=\frac{h'''(z)}{h'(z)}-\frac{3}{2}\left[\frac{h''(z)}{h'(z)}\right]^2$$

Then $$\left\{\tilde{h},z\right\}=\left\{h,z\right\}$$

In the same way

$$\left\{\tilde{g},z\right\}=\frac{\tilde{g}'''(z)}{\tilde{g}'(z)}-\frac{3}{2}\left[\frac{\tilde{g}''(z)}{\tilde{g}'(z)}\right]^2$$

$$\left\{\tilde{g},z\right\}=\frac{g'''(z)}{g'(z)}-\frac{3}{2}\left[\frac{g''(z)}{g'(z)}\right]^2$$

$$\left\{\tilde{g},z\right\}=\left\{g,z\right\}$$

The relation (5) becomes:

$$\{f,z\}=\frac{1}{1+|\mu_{f(z)}|}\{\tilde{h},z\}+\frac{|\mu_{f(z)}|}{1+|\mu_{f(z)}|}\{\tilde{g},z\}+\left[\frac{\tilde{h}''(z)}{\tilde{h}'(z)}-\frac{\tilde{g}''(z)}{\tilde{g}'(z)}\right]^2\frac{|\mu_{f(z)}|}{(1+|\mu_{f(z)}|)^2} \tag{6}$$

Taking the modulus of both and applying the triangle inequality in the relation (6), we have :

$$\left|\{f,z\}\right|\leq\frac{1}{1+|\mu_{f(z)}|}\left|\{\tilde{h},z\}\right|+\frac{|\mu_{f(z)}|}{1+|\mu_{f(z)}|}\left|\{\tilde{g},z\}\right|+\left|\frac{\tilde{h}''(z)}{\tilde{h}'(z)}-\frac{\tilde{g}''(z)}{\tilde{g}'(z)}\right|^2\frac{|\mu_{f(z)}|}{(1+|\mu_{f(z)}|)^2} \tag{7}$$

Based on Nehari's theorem, $\left|\left\{\tilde{h},z\right\}\right|\leq\frac{2}{(1-|z|^2)^2}$ and $\left|\left\{\tilde{g},z\right\}\right|\leq\frac{2}{(1-|z|^2)^2}$

The relation (7) becomes :

$$\left|\left\{f,z\right\}\right|\leq\frac{1}{1+|\mu_{f(z)}|}\frac{2}{(1-|z|^2)^2}+\frac{|\mu_{f(z)}|}{1+|\mu_{f(z)}|}\frac{2}{(1-|z|^2)^2}+\left|\frac{\tilde{h}''(z)}{\tilde{h}'(z)}-\frac{\tilde{g}''(z)}{\tilde{g}'(z)}\right|^2\frac{|\mu_{f(z)}|}{(1+|\mu_{f(z)}|)^2} \tag{8}$$

Or $$\left|\frac{\tilde{h}''(z)}{\tilde{h}'(z)}-\frac{\tilde{g}''(z)}{\tilde{g}'(z)}\right|\leq\left|\frac{\tilde{h}''(z)}{\tilde{h}'(z)}\right|+\left|\frac{\tilde{g}''(z)}{\tilde{g}'(z)}\right|$$

The relation (8) becomes:

$$\left|\left\{f,z\right\}\right|\leq\frac{2}{(1-|z|^2)^2}+\left[\left|\frac{\tilde{h}''(z)}{\tilde{h}'(z)}\right|+\left|\frac{\tilde{g}''(z)}{\tilde{g}'(z)}\right|\right]^2\frac{|\mu_{f(z)}|}{(1+|\mu_{f(z)}|)^2} \tag{9}$$

$\tilde{g}$ and $\tilde{h}$ being functions of S, then the applications $I(\tilde{h}(z)) = \dfrac{z\tilde{h}''}{\tilde{h}'(z)}$ and $I(\tilde{g}(z)) = \dfrac{z\tilde{g}''}{\tilde{g}'(z)}$ have the values of the disks

$\Delta'(z) = \left\{z : \left|\dfrac{z\tilde{h}''(z)}{\tilde{h}'(z)} - \dfrac{2r^2}{1-r^2}\right| \le \dfrac{4r}{1-r^2}\right\}$ and $\Delta''(z) = \left\{z : \left|\dfrac{z\tilde{g}''(z)}{\tilde{g}'(z)} - \dfrac{2r^2}{1-r^2}\right| \le \dfrac{4r}{1-r^2}\right\}$

$\left|\dfrac{z\tilde{h}''(z)}{\tilde{h}'(z)} - \dfrac{2r^2}{1-r^2}\right| \le \dfrac{4r}{1-r^2} \Longrightarrow \left|\dfrac{\tilde{h}''(z)}{\tilde{h}'(z)}\right| \le \dfrac{4+2r}{1-r^2}$ with $|z| = r$

$\left|\dfrac{z\tilde{g}''(z)}{\tilde{g}'(z)} - \dfrac{2r^2}{1-r^2}\right| \le \dfrac{4r}{1-r^2} \Longrightarrow \left|\dfrac{\tilde{g}''(z)}{\tilde{g}'(z)}\right| \le \dfrac{4+2r}{1-r^2}$

The relation (9) becomes:

$$\left|\left\{f, z\right\}\right| \le \frac{2}{(1-r^2)^2} + 16\left(\frac{2+r}{1-r^2}\right)^2 \frac{|\mu_{f(z)}|}{(1+|\mu_{f(z)}|)^2} \tag{10}$$

Let $t = |\mu_{f(z)}|$ with $t \in]0;1[$

We have :

$$\left|\left\{f, z\right\}\right| \le \frac{2}{(1-r^2)^2} + 16\left(\frac{2+r}{1-r^2}\right)^2 \frac{t}{(1+t)^2} \tag{11}$$

Let's put $\varphi(t) = \dfrac{t}{(1+t)^2}$

Thus

$$\left|\left\{f, z\right\}\right| \le \frac{2}{(1-r^2)^2} + 16\left(\frac{2+r}{1-r^2}\right)^2 \varphi(t) \tag{12}$$

<u>Let's study the φ function for all $t \in]0;1[$</u>

$\forall t \in]0;1[$, $\varphi'(t) = \dfrac{1-t}{(1+t)^3} > 0$, then φ is increasing on $]0;1[$

t	0		1
$\varphi'(t)$		$+$	
$\varphi(t)$	0	$\nearrow$	$\frac{1}{4}$

According to the study by φ, for all $t \in]0;1[$, we have $\varphi(t) < \dfrac{1}{4}$

So the relation (12) becomes :

$$\left|\left\{f,z\right\}\right| \leq \frac{2}{(1-r^2)^2} + 16\left(\frac{2+r}{1-r^2}\right)^2 \times \frac{1}{4} \tag{13}$$

From this

$$\boxed{\left|\left\{f,z\right\}\right| \leq \frac{2}{(1-r^2)^2} + \left(\frac{4+2r}{1-r^2}\right)^2} \quad \blacksquare \tag{14}$$

3. Consequence

Let $f(z) = h(z) + \overline{g(z)}$ be a quasi-conformal map, defined in $\Delta = \left\{z \in \mathbb{C}; |z| < 1\right\}$, with $h(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$ two univalent and holomorphic maps, defined in Δ with values in $\mathbb{C}$ such that $h'(0) = 1$, then :

a) we have either :

- $|b_2| - |a_2| \leq 4$

or either

- $4|b_1| \leq |a_2| + |b_2|$

b) $$\left|b_2\right|^2 - \left|b_3\right| \leq \frac{1}{3}$$

Demonstration

From the relation (6), we have:

$$\left\{f,0\right\} = \frac{1}{1+|\mu_{f(0)}|}\left\{h,0\right\} + \frac{|\mu_{f(0)}|}{1+|\mu_{f(0)}|}\left\{g,0\right\} + \left[\frac{h''(0)}{h'(0)} - \frac{g''(0)}{g'(0)}\right]^2 \frac{|\mu_{f(0)}|}{\left(1+|\mu_{f(0)}|\right)^2} \tag{15}$$

Taking the modulus on both sides and applying the triangle inequality in the relation (15), we have :

$$\left|\{f,0\}\right| \leq \frac{1}{1+|\mu_{f(0)}|}\left|\{h,0\}\right| + \frac{|\mu_{f(0)}|}{1+|\mu_{f(0)}|}\left|\{g,0\}\right| + \left|\frac{h''(0)}{h'(0)} - \frac{g''(0)}{g'(0)}\right|^2 \frac{|\mu_{f(0)}|}{\left(1+|\mu_{f(0)}|\right)^2} \tag{16}$$

On the one hand $\left|\left\{h,0\right\}\right| \leq 2$ and $\left|\left\{g,0\right\}\right| \leq 2$ [8]

The relation (16) becomes :

$$\left|\left\{f,0\right\}\right| \leq 2+\left|\frac{h''(0)}{h'(0)}-\frac{g''(0)}{g'(0)}\right|^{2}\frac{|\mu_{f(0)}|}{\left(1+|\mu_{f(0)}|\right)^{2}} \tag{17}$$

On the other hand :

$$\frac{h''(0)}{h'(0)}-\frac{g''(0)}{g'(0)}=2a_2-\frac{2b_2}{b_1} \tag{18}$$

Note that

$$|\mu_{f(0)}|=\left|\frac{g'(0)}{h'(0)}\right|=|b_1| \tag{19}$$

Then by replacing the relations (18) and (19) in the relation (17) , we obtain :

$$\left|\left\{f,0\right\}\right| \leq 2+\left|2a_2-\frac{2b_2}{b_1}\right|^{2}\frac{|b_1|}{(1+|b_1|)^{2}} \tag{20}$$

From the previous theorem, $\left|\left\{f,0\right\}\right| \leq 18$

First case :

Suppose that $2+\left|2a_2-\frac{2b_2}{b_1}\right|^{2}\frac{|b_1|}{(1+|b_1|)^{2}} \leq 18$

We have :

$$2+4\left|a_2-\frac{b_2}{b_1}\right|^{2}\frac{|b_1|}{(1+|b_1|)^{2}} \leq 18$$

$$\left|a_2-\frac{b_2}{b_1}\right|^{2}\frac{|b_1|}{(1+|b_1|)^{2}} \leq 4$$

$$\left|a_2-\frac{b_2}{b_1}\right|^{2} \leq \frac{4(1+|b_1|)^{2}}{|b_1|} \tag{21}$$

By definition of μ_f, we have :

$$|\mu_{f(z)}|<1 \Longrightarrow |b_1|<1$$
$$\Longrightarrow (1+|b_1|)^2<4$$

The relation (21) becomes:

$$\left|a_2b_1-b_2\right|^{2} \leq 16\left|b_1\right|$$

$$\left|a_2b_1 - b_2\right|^2 \leq 16$$

$$\left|b_2 - a_2b_1\right|^2 \leq 16$$

$$\left|b_2 - a_2b_1\right| \leq 4$$

$$\left|b_2\right| - \left|a_2\right|\left|b_1\right| \leq 4$$

Or $$\left|b_1\right| < 1$$

Thus

$$\boxed{\left|b_2\right| - \left|a_2\right| \leq 4} \quad \blacksquare \tag{22}$$

Second case :

Suppose that $$2 + \left|2a_2 - \frac{2b_2}{b_1}\right|^2 \frac{|b_1|}{(1+|b_1|)^2} \geq 18$$

This inequality holds true for any value of $|b_1| \in]0;1[$, so

$$2 + \left|2a_2 - \frac{2b_2}{b_1}\right|^2 \max_{|b_1|\in]0;1[} \frac{|b_1|}{(1+|b_1|)^2} \geq 18 \tag{23}$$

Or

$$\max_{|b_1|\in]0;1[} \frac{|b_1|}{(1+|b_1|)^2} = \frac{1}{4}$$

So the relation (23) becomes :

$$\left|a_2 - \frac{b_2}{b_1}\right|^2 \times \frac{1}{4} \geq 4 \tag{24}$$

$$\left|a_2b_1 - b_2\right|^2 \geq 16|b_1|^2$$

$$\left|a_2\right|\left|b_1\right| + \left|b_2\right| \geq 4\left|b_1\right|$$

Hence

$$\boxed{4\left|b_1\right| \leq \left|a_2\right| + \left|b_2\right|} \quad \blacksquare \tag{25}$$

We had established that $\{g, z\} = \{\tilde{g}, z\}$

So

$$\{g,0\} = \frac{g'''(0)}{g'(0)} - \frac{3}{2}\left(\frac{g''(0)}{g'(0)}\right)^2$$

Then

$$\left\{g,0\right\} = \frac{6b_3}{b_1} - 6\left(\frac{b_2}{b_1}\right)^2$$

Or

$$\left|\left\{g,0\right\}\right| \leq 2$$

Thus

$$\left|\frac{6b_3}{b_1} - 6\left(\frac{b_2}{b_1}\right)^2\right| \leq 2$$

$$\left|b_3 b_1 - b_2^2\right| \leq \frac{1}{3}\left|b_1\right|^2$$

$$\left|b_2\right|^2 - \left|b_3\right|\left|b_1\right| \leq \frac{1}{3}\left|b_1\right|^2$$

Or

$$\left|b_1\right| < 1$$

Thus

$$\boxed{\left|b_2\right|^2 - \left|b_3\right| \leq \frac{1}{3}} \quad \blacksquare \qquad (26)$$

4. Conclusion

In this paper, a majorization of the Schwarzian modulus and some relations on Taylor coefficients have been proposed in a class of quasi-conformal maps in the open unit disk.

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