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Approximation of Poisson integrals by r-repeated de la Vallee Poussin sums

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ARTICLE INFO	ABSTRACT
Published online: 18 October 2024 **Corresponding Author:** Olga Rovenska	The paper investigates the approximation properties of linear means of Fourier series generated by repeated application of the de la Vallée Poussin summation method. Asymptotic formulas for the exact upper bounds of deviations of the r-repeated de la Vallée Poussin means on the classes of Poisson integrals of functions with bounded generalized derivatives are obtained. The derived relations, under certain conditions, represent asymptotically exact equalities.

KEYWORDS: de la Vallee Poussin sums, Poisson integral, asymptotic equality

I. INTRODUCTION

The work concerns the questions of approximation of periodic (ψ,β)-differentiable functions of high smoothness by repeated arithmetic means of Fourier sums. One of the classifications of periodic functions nowadays is the classification suggested by A. Stepanets [1] which is based on the concept of (ψ,β)-differentiation. The given classification allows to distinguish all classes of summable periodic functions from the functions where the Fourier series can deviate to infinitely differentiable functions including analytical and entire ones. When choosing the parameters properly, classes of (ψ,β)-differentiable functions exactly coincide with the well-known classes of Vail differentiable functions, Sobolev classes W_p^l and classes of convolutions with integral kernels.

Sets of (ψ,β) -differentiable functions are defined in the following way [1, p. 120].

Let f be a summable, 2π-periodic function,

$$S(f)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k coskx+b_k sinkx)=\sum_{k=0}^{\infty}A_k(f;x)$$

be its Fourier series. Let $\psi(k)$ be an arbitrary numerical sequence and $\beta\in R$. Then, if the series

$$\sum_{k=m+1}^{n}\frac{1}{\psi(k)}\left(a_k cos(kx+\frac{\beta\pi}{2})+b_k sin(kx+\frac{\beta\pi}{2})\right)$$

is Fourier series of some summable function, this function is called (ψ,β)-derivative of function f and is denoted by f_β^ψ. The set of continuous functions having (ψ,β)-derivative is denoted by C_β^ψ. Besides, if (ψ,β)-derivative is almost everywhere bounded by unity, the set of such functions is denoted by $C_{\beta,\infty}^\psi$. We consider the case when sequence $\psi(k)$ is defined by relationship $\psi(k)=q^k$, $q\in(0;1)$. In doing so classes $C_{\beta,\infty}^\psi$ consist of analytical functions which can be regularly extended in the corresponding strip.

Numerical sequence $\psi(k)$, giving the class is possible to select only from the set of all positive convex downwards and disappearing on the infinity sequences. In this case approximative properties of classes C_β^ψ are to characterized by the rate of functions $\psi(k)$ tending to zero.

We consider the case when sequence $\psi(k)$ is defined by relationship $\psi(k)=q^k$, $q\in(0;1)$. In doing so classes $C_{\beta,\infty}^\psi$ consist of analytical functions which can be regularly extended in the corresponding strip $|\operatorname{Im} z|<\ln\frac{1}{q}$. In this case classes $C_{\beta,\infty}^\psi$ are denoted by $C_{\beta,\infty}^q$. The functions from classes $C_{\beta,\infty}^q$ in each point can be presented in the form of Poisson integrals with an accuracy to the constant component

$$f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f_\beta^\psi(x+t)\sum_{k=1}^{\infty}q^k\cos(kt+\frac{\beta\pi}{2})dt.$$

Let $\Lambda = \| \lambda_k^{(n)} \|$, $k,\ n = 1,2,\ldots$ is an infinite numerical matrix $\lambda_k^{(n)} = 0$, $k \ge n$. Each matrix of such kind, on the basis of the Fourier series, gives a certain sequence of linear polynomial operators

$$U_n(f;x;\Lambda) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx).$$

For arbitrary natural $p < n$ the polynomials that are given by relationship

$$\lambda_k^{(n)} = \begin{cases} 1, & 1 \le k \le n-p-1, \\ 1 - \dfrac{k-n+p}{p}, & n-p \le k \le n-1 \end{cases}$$

are called de la Vallee Poussin sums.

De la Vallee Poussin sums are also arithmetic means of the last p Fourier sums [2, 3]

$$V_{n,p}(f;x) = \frac{1}{p} \sum_{k=n-p}^{n-1} S_k(f;x)$$

For arbitrary natural $p_1,\ p_2, \ldots,\ p_r$ and $\Sigma_p = \sum_{i=1}^{r} p_i < n$ the polynomials that are given by relationship

$$V^{(r)}_{n,p_1,p_2,\ldots,p_r}(f;x) = \frac{1}{p_1} \sum_{k_1=n-p_1}^{n-1} \frac{1}{p_2} \sum_{k_2=k_1-p_2+1}^{k_1} \ldots \frac{1}{p_r} \sum_{k_r=k_{r-1}-p_r+1}^{k_{r-1}} S_{k_r}(f;x)$$

are called r-repeated de la Vallee Poussin sums [7,8, 9].

If $p_1 = 1$ or $p_2 = 1$ these polynomials are de la Vallee Poussin sums, if $p_1 = p_2 = 1$, they are Fourier sums.

For upper bounds of deviations of Fourier sums on the classes of analytical functions S. Nikol'skiy [7] obtained the asymptotic equality:

$$\varepsilon\left(C^q_{\beta,\infty}; S_n\right) = \sup_{f \in C^q_{\beta,\infty}} \| f(x) - S_n(f;x) \|_C = \frac{8q^n}{\pi^2} K(q) + O\left(\frac{1}{n}\right), n \to \infty$$

In [8] S. Stechkin proposed another proof of this result which made it possible to refine the remainder in this formula. Asymptotic equalities for upper bounds of the deviations of de la Vallee Poussin sums on the classes $C^q_{\beta,\infty}$ may be found in [9] (look also [10, 11]).

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II. RESULT

For upper bounds of deviations of the r-repeated de la Vallee Poussin sums on the classes of analytical functions $C^q_{\beta,\infty}$ the following statement was obtained.

Theorem. *Suppose that* $q \in (0;1)$, $\beta \in R$ *and* $p_i = p_i(n)$, $i = 1,2,\ldots,r$ *are arbitrary natural numbers,* $\Sigma_p = \sum_{i=1}^{r} p_i < n$.

Then the following relations hold as $n \to \infty$, $n - \Sigma_p \to \infty$

$$\varepsilon\left(C^q_{\beta,\infty}; V^{(r)}_{n,\overline{p}}(f;x)\right) = \frac{4q^{n-\Sigma_p+r}}{\pi^2 \prod_{i=1}^{r} p_i} \int_0^{\pi} Z_q^{r+1}(x)dx + O(1)\frac{1}{\prod_{i=1}^{r} p_i}\left(\frac{q^{n-\Sigma_p+r}}{(n-\Sigma_p+r)(1-q)^{r+2}} + \sum_{\alpha_{r-1} \subset \overline{r}} \frac{q^{n-\sum_{j\in\alpha_{r-1}} p_j + r}}{(1-q)^{r+1}}\right), \quad (1)$$

where $Z_q(x) = (1 - 2q\cos x + q^2)^{-1/2}$, α_{r-1} *is an arbitrary* $r-1$*-element's subset of set* $\overline{r} = \{1,2,\ldots,r\}$, $|\alpha_r|$ *is number of elements of set* α_r, $O(1)$ *is quantity uniformly bounded with respect to* n, q, β, p_i, $i = 1,2,\ldots,r$.

Proof. The statement of the theorem are proved using the procedure proposed by A. Stepanets in [1, p. 294]. While constructing the proof, first the convenient integral representations for quantities $\delta_n(f;x;V^{(r)}_{n,\overline{p}})$ were found

$$\delta_n(f;x;V^{(r)}_{n,\overline{p}}) = f(x) - \frac{1}{p_1} \sum_{k_1=n-p_1}^{n-1} \frac{1}{p_2} \sum_{k_2=k_1-p_2+1}^{k_1} \ldots \frac{1}{p_r} \sum_{k_r=k_{r-1}-p_r+1}^{k_{r-1}} S_{k_r}(f;x)$$

$$= \frac{1}{\prod_{i=1}^{r} p_i} \sum_{k_1=n-p_1}^{n-1} \sum_{k_2=k_1-p_2+1}^{k_1} \ldots \frac{1}{p_r} \sum_{k_r=k_{r-1}-p_r+1}^{k_{r-1}} (f(x) - S_{k_r}(f;x))$$

$$= \frac{1}{\pi \prod_{i=1}^{r} p_i} \int_{-\pi}^{\pi} \frac{f_\beta^\psi(x+t)}{(1-2qcost+q^2)^{r+1}} \sum_{\alpha \in \overline{r}} (-1)^{r-|\alpha|} \times \sum_{\nu=0}^{r+1} (-1)^\nu C^\nu_{r+1} q^{n-\sum_{j\in\alpha} p_j + r - \nu} cos\left((n - \sum_{j\in\alpha} p_j + r - \nu)t + \frac{\beta\pi}{2}\right)dt, \quad (2)$$

where $|\alpha|$ is number of elements of set α, $\overline{r} = \{1,2,\ldots,r\}$.

Further let

$$b_m^\beta(t) = (1 - 2qcost + q^2)^{-\frac{r+1}{2}} \cos\left(mt + \frac{\beta\pi}{2} + (r+1)\frac{sint}{1-qcost}\right).$$

Then the quantity $\delta_n(f;x;V^{(r)}_{n,\overline{p}})$ may be represented as follows

$$\delta_n(f;x;V^{(r)}_{n,\overline{p}}) = \frac{q^{n-\Sigma_p+r}}{\pi \prod_{i=1}^{r} p_i} \int_{-\pi}^{\pi} f_\beta^\psi(x+t) b^\beta_{n-\Sigma_p+r}(t)dt + O(1)\frac{1}{\prod_{i=1}^{r} p_i} \sum_{\alpha_{r-1} \in \overline{r}} q^{n-\sum_{j\in\alpha_{r-1}} p_j + r} \int_{-\pi}^{\pi} f_\beta^\psi(x+t) b^\beta_{n-\sum_{j\in\alpha_{r-1}} p_j + r}(t)dt. \quad (3)$$

Taking into account that $f(x) \in C^q_{\beta,\infty}$, also (3) one can find the upper value for quantity $\varepsilon(C^q_{\beta,\infty}; V^{(r)}_{n,\overline{p}})$

$$\varepsilon(C^{\psi}_{\beta,\infty};V^{(r)}_{n,\overline{p}}(f;x)) \le \frac{4q^{n-\Sigma_p+r}}{\pi^2\prod_{i=1}^{r}p_i}\int_{-\pi}^{\pi}|b^{\beta}_{n-\Sigma_p+r}(t)|dt$$

$$+O(1)\frac{1}{\prod_{i=1}^{r}p_i}\sum_{\alpha_{r-1}\in\overline{r}}q^{n-\sum_{j\in\alpha_{r-1}}p_j+r}\int_{-\pi}^{\pi}|b^{\beta}_{n-\sum_{j\in\alpha_{r-1}}p_j+r}(t)|dt, \quad (4)$$

Using work [1, p. 294] (also [6, p. 235–238]), further we find function $f_0(x)\in C^{q}_{\beta,\infty}$ for which this value cannot be improved. As

$$\int_{-\pi}^{\pi}|b^{\beta}_{m}(t)|dt = O(1)\int_{-\pi}^{\pi}\frac{dt}{(\sqrt{1-qcost+q^2})^{r+1}}$$

$$=O(1)\frac{1}{(1-q)^{r+1}},$$

the quantity $\delta_n(f;x;V^{(r)}_{n,\overline{p}})$ can be rewritten as follows

$$\delta_n(f;x;V^{(r)}_{n,\overline{p}}) = \frac{q^{n-\Sigma_p+r}}{\pi\prod_{i=1}^{r}p_i}\int_{-\pi}^{\pi}f^{\psi}_{\beta}(x+t)b^{\beta}_{n-\Sigma_p+r}(t)dt$$

$$+O(1)\frac{1}{\prod_{i=1}^{r}p_i}\sum_{\alpha_{r-1}\in\overline{r}}\frac{q^{n-\sum_{j\in\alpha_{r-1}}p_j+r}}{(1-q)^{r+1}}. \quad (5)$$

Based on formula (5) for any function $f\in C^{q}_{\beta,\infty}$ the following equality is true

$$\delta_n(f;0;V^{(r)}_{n,\overline{p}}) = \frac{q^{n-\Sigma_p+r}}{\pi\prod_{i=1}^{r}p_i}\int_{-\pi}^{\pi}f^{\psi}_{\beta}(x)b^{\beta}_{n-\Sigma_p+r}(t)dt$$

$$+O(1)\frac{1}{\prod_{i=1}^{r}p_i}\sum_{\alpha_{r-1}\in\overline{r}}\frac{q^{n-\sum_{j\in\alpha_{r-1}}p_j+r}}{(1-q)^{r+1}}. \quad (6)$$

Functions $b^{\beta}_{n-\Sigma_p+r}(t)$ may be refined on the set, the measure of which is less than K$(n-\Sigma_p+r)^{-1}q(1-q)^{-1}$, so that the following condition for new functions $b^{\beta,1}_{n-\Sigma_p+r}(t)$ will be fulfilled [9, p. 235–238] $\int_{-\pi}^{\pi} sign\, b^{\beta,1}_{n-\Sigma_p+r}(t)dt = 0.$

For the found function the equality is true

$$\delta_n(f_0;0;V^{(r)}_{n,\overline{p}}) = \frac{q^{n-\Sigma_p+r}}{\pi\prod_{i=1}^{r}p_i}\int_{-\pi}^{\pi}|b^{\beta}_{n-\Sigma_p+r}(t)|dt$$

$$+O(1)\frac{1}{\prod_{i=1}^{r}p_i}\left(\sum_{\alpha_{r-1}\in\overline{r}}\frac{q^{n-\sum_{j\in\alpha_{r-1}}p_j+r}}{(1-q)^{r+1}}+\frac{q^{n-\Sigma_p+r}}{(n-\Sigma_p+r)(1-q)^{r+2}}\right). \quad (7)$$

Comparing relationships (4) and (7) we get asymptotic formula

$$\varepsilon(C^{q}_{\beta,\infty};V^{(r)}_{n,\overline{p}}(f;x)) = \sup_{f\in C^{q}_{\beta,\infty}} \| f(x)-V^{(r)}_{n,\overline{p}}(f;x)\|_C$$

$$=\frac{q^{n-\Sigma_p+r}}{\pi\prod_{i=1}^{r}p_i}\int_{-\pi}^{\pi}|b^{\beta}_{n-\Sigma_p+r}(t)|dt$$

$$+O(1)\frac{1}{\prod_{i=1}^{r}p_i}\left(\sum_{\alpha_{r-1}\in\overline{r}}\frac{q^{n-\sum_{j\in\alpha_{r-1}}p_j+r}}{(1-q)^{r+1}}+\frac{q^{n-\Sigma_p+r}}{(n-\Sigma_p+r)(1-q)^{r+2}}\right).$$

According to [6, p. 239–241] and counting the integral in the first component

$$\int_{-\pi}^{\pi}\left|b^{\beta}_{n-\Sigma_p+r}(t)\right|dt = O(1)\int_{-\pi}^{\pi}\frac{1}{(\sqrt{1-qcost+q^2})^{r+1}}$$

$$\times cos\left((n-\Sigma_p+r)t+\frac{\beta\pi}{2}+(r+1)\frac{qsint}{1-cost}\right)dt$$

$$=\frac{4}{\pi}\int_{0}^{\pi}Z^{r+1}_{q}(x)dx+O(1)\frac{1}{n-\Sigma_p+r},$$

we obtain the equality (1). The theorem is proved.

III. CONCLUSION

The problem connected with the search for upper bounds of approximation errors with respect to a class of Poisson integrals and for repeated de la Vallee Poussin sums is considered. Our approach turned out to be effective for obtaining exact asymptotic. The key point in this approach is to construct the function $f_0(x)\epsilon C^{q}_{\beta,\infty}$ that implements the upper bound.

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