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PBIB-Designs Arising From BI-Connected Dominating Sets of Cubic Graphs of Order at Most 12

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ABSTRACT

A $(v,b,\gamma_{bc},r,\lambda_i)$-design over regular graph $G = (V,E)$ of degree k is an ordered pair $D = (V,B)$, where $|V| = v$ and B is the set of bi-connected dominating sets of G called blocks such that two vertices α and β which are i^{th} associates occur together in λ_i blocks, the numbers λ_i being independent of the choice of the pair α and β. In this paper, we obtain Partially Balanced Incomplete Block (PBIB)-designs arising from bi-connected dominating sets in cubic graphs. Also, we give a complete list of PBIB-designs with respect to the bi-connected dominating sets for cubic graphs of order at most 12. The discussion of non-existence of some designs corresponding to bi-connected dominating sets from certain graphs concludes the article.

KEYWORDS: Partially balanced incomplete block designs, Bi-connected dominating sets, Cubic graphs Mathematics Subject Classification[2010] 05C51, 53C22, 58E10

1. INTRODUCTION

In combinatorial mathematics, a block design is a particular kind of hyper graph or set system which has applications to finite geometry, cryptography and algebraic geometry. In the class of incomplete block designs, a balanced incomplete block design (here after called BIBD) is the simplest one. The origin of design theory can be traced back in experiments of crop cultivation in agriculture. Today we find vast growth of the subject both in theoretical and practical applications, a 1000+ pages Handbook of Combinatorial Designs [5], is a testimony. A BIBD is a set of v elements called vertices and a collection of $b > 0$ subsets, called blocks, such that each block consists of exactly k vertices, $v > k > 0$, each vertex appears in exactly r blocks, $r > 0$, each pair of vertices appear simultaneously in exactly λ blocks, $\lambda > 0$. The combinatorial configuration so obtained is called a (v,b,r,k,λ)-design. Graph theoretically, a BIBD is an edge-disjoint decomposition of a complete multigraph into k-cliques. In these designs, $v > k$, that is, the block size k, is strictly less than the number of vertices, so that no block contains all the vertices, justifying the phrase “incomplete” in its name. Although BIBDs have many optimal properties, they do not fit well into most experimental situations as their repetition number is too large. To overcome this a class of binary, equireplicate designs were introduced viz. Partially Balanced Incomplete Block Designs (PBIBDs).

The interplay between graphs and combinatorial designs exists by interpreting one in terms of the other. Generating new sets to be realizable design parameters, different graph theoretic notions are used viz., connected dominating sets[3], bi-connected dominating sets [4], diametral paths [7], geodetic sets [8], maximum indpendent sets [12], etc. These graph invariants and subsets help in construction, enumeration of many designs if exist, else non-existence can also be given through graph parameters.

In this paper, we construct PBIB-designs through bi-connected dominating sets corresponding to cubic graphs of order at most 12, to add to the ever incresing set of graph invariants used in design theory. We have given a complete list of PBIB-designs with respect to bi-connected dominating sets as cubic graphs hold a special consideration among graph theorists.

2. PRELIMINARIES

Definition 2.1. [11] *Given a set* $\{1,2,...,v\}$ *a relation satisfying the following conditions is said to be an association scheme with m classes.*

(1) *Any two symbols* α *and* β *are* i^{th} *associates for some i, with* $1 \le i \le m$ *and this relation of being* i^{th} *associates is symmetric;*

(2) *The number of* i^{th} *associates of each symbol is* n_i*;*

(3) *If* α *and* β *are two symbols which are* i^{th} *associates, then the number of symbols which are* j^{th} *associates of* α *and* k^{th} *associates of* β *is* p^i_{jk} *and is independent of the pair of* i^{th} *associates* α *and* β.

Definition 2.2. [11] *Consider a set of symbols* $V = \{1,2,...,v\}$ *and an association scheme with m classes on* V *. A partially balanced incomplete block design (PBIBD) is a collection of b subsets of x called blocks, each of them containing k symbols* $(k < v)$*, such that every symbol occurs in r blocks and two symbols* α *and* β *which are* i^{th} *associates occur together in* λ_i *blocks, the numbers* λ_i *being independent of the choice of the pair* α *and* β.

The numbers v, b, r, k, λ_i $(1,2,...,m)$ *are called the parameters of first kind and the numbers* n_i*'s and* p^i_{jk} *are called the parameters of second kind.*

A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set D is a connected dominating set, if the induced subgraph $< D >$ is connected and a co-connected dominating set, if the induced subgraph $< V - D >$ is connected. The minimum cardinality of a connected dominating set is called connected domination number γ_c and minimum cardinality of co-connected dominating set is co-connected domination number γ_{cc}. A dominating set is said to be a bi-connected dominating set, if both the induced subgraphs $< D >$ and $< V - D >$ are connected. The minimum cardinality of bi-connected dominating set is called bi-connected domination number γ_{bc} [6].

3. CUBIC GRAPHS OF ORDER AT MOST 12

Cubic graphs are of quite interest. We know that there exists only one cubic graph of order four, namely, the complete graph K_4, 2 cubic graphs of order six, 5 cubic graphs of order eight, 21 cubic graphs of order ten and 85 cubic graphs of order 12 from [2]. Here we give a list of those cubic graphs that give PBIBdesigns arising from bi-connected dominating sets in them, and also we give the cubic graphs which do not give PBIB-designs from bi-connected dominating sets. For ready reference we have listed all these graphs in Appendix 1.

3.1. Cubic graphs with 6 vertices. There exist 2 cubic graphs on six vertices [2]. There exists no PBIBdesign, whose blocks are bi-connected dominating sets, which is clear from the Table 1.

TABLE 1. Graphs of order 6

Graphs	$\gamma(G)$	γ_{bc}	b	r	λ_1	λ_2
G_1	2	2	9	2	0,1	0,1
G_2	2	2	9	2	0,1	1

Remark 1. *For the graphs where repetition number is not unique,* λ_i *values are not calculated. Since we are using distance based association scheme,* λ_i'*s can be calculated upto the diameter of that particular graph.*

3.2. Cubic graphs with 8 vertices.

There are 5 cubic graphs on eight vertices [2]. The following results establish existence and non-existence of designs.

Theorem 3.1. *For* G_5, G_7 *there exist PBIB-designs with parameters* (8,16,3,6,2,3) *and* (8,24,3,9,2,2,6), *respectively.*

Proof. By simple calculation we can see that, for G_5, we get $v = 8$, $b = 16$, $\gamma_{bc}(G_5) = 3$, $r = 6$, $\lambda_1 = 2$, $\lambda_2 = 3$. Similarly, for G_7, $v = 8$, $b = 24$, $\gamma_{bc}(G_7) = 3$, $r = 9$, $\lambda_1 = 2$, $\lambda_2 = 2$ and $\lambda_3 = 6$.

Hence, we can conclude that graphs G_5 and G_7 form PBIB-design, whose blocks are bi-connected dominating sets. □

Corollary 3.1. *For* G_3, G_4, G_6 *PBIB-designs do not exist.*

Proof. Proof follows from the Table 2 below as repetition number is not unique.

TABLE 2. Graphs of order 8

Graphs	$\gamma(G)$	γ_{bc}	b	r	λ_1	λ_2	λ_3
G_3	2	3	16	not unique	not unique	not unique	not unique
G_4	3	3	18	not unique	not unique	not unique	not unique
G_5	3	3	16	6	2	3	...
G_6	2	3	20	not unique	not unique	not unique	not unique
G_7	2	3	24	9	2	2	6

Hence we can conclude that, there does not exist PBIB-design for the graphs G_3, G_4 and G_6. □

3.3. Cubic graphs with 10 vertices. There are 21 cubic graphs on 10 vertices [2]. We observe that out of 21 graphs only one graph form PBIB-design, whose blocks are bi-connected dominating sets of graphs.

Theorem 3.2. *The graph* G_{28} *forms a PBIB-design with parameters* (10,10,4,4,2,1), *respectively.*

Proof. Clearly the graph G_{28} is isomorphic to Petersen graph. In [9], given that the value of γ_{bc} of Petersen graph is 4, the corresponding bi-connected dominating sets are {1,2,3,9},{1,2,5,10},{1,4,5,6},{1,7,8,10},{2,3,4,8},{2,6,7,9},{3,4,5,7},{3,6,8,10},{4,7,9,10},{5,6,8,9} . Hence we get (10,10,4,4,2,1)design.

TABLE 3. Graphs of order 10

Graphs	$\gamma(G)$	γ_{bc}	b	r	λ_1	λ_2	Λ^1
G_8	3	7	16	not unique	..	..	..
G_9	3	3	6	not unique	..	..	..
G_{10}	3	3	8	not unique	..	..	..
G_{11}	3	3	8	not unique	..	..	..
G_{12}	3	3	4	not unique	..	..	..
G_{13}	3	3	6	not unique	..	..	..
G_{14}	3	3	4	not unique	..	..	..
G_{15}	3	4	40	not unique	..	..	..
G_{16}	3	3	12	not unique	..	..	..
G_{17}	3	3	12	not unique	..	..	..
G_{18}	3	3	12	not unique	..	..	..
G_{19}	3	3	4	not unique	..	..	..
G_{20}	3	4	32	not unique	..	..	..
G_{21}	3	4	36	not unique	..	..	..
G_{22}	3	4	60	24	not unique	not unique	not unique
G_{23}	3	4	40	not unique	..	..	..
G_{24}	3	3	4	not unique	..	..	..
G_{25}	3	4	60	24	not unique	not unique	not unique

G_{26}	3	3	4	not unique	..	..	..
G_{27}	3	4	40	16	not unique	not unique	not unique
G_{28}	3	4	10	4	2	1	..

3.4. Cubic graphs with 12 vertices.
There are 85 cubic graphs on 12 vertices [2] as listed in Appendix
1. Here we prove for which of these there exist PBIB-designs.
Proof. For all the graphs listed in the statement we give details as follows in Table 4, 5, 6.

TABLE 4. Graphs of order 12

Graphs	$\gamma(G)$	$\gamma_{bc}(G)$	b	r	λ_i
G_{29}	3	7	4	not unique	..
G_{30}	3	7	2	not unique	..
G_{31}	4	7	4	not unique	..
G_{32}	3	4	4	not unique	..
G_{33}	4	4	6	2	not unique
G_{34}	3	4	12	4	not unique
G_{35}	4	8	17	not unique	..
G_{36}	4	4	8	not unique	..
G_{37}	4	4	8	not unique	..
G_{38}	4	4	16	not unique	..
G_{39}	4	5	64	not unique	..
G_{40}	3	4	8	not unique	..
G_{41}	3	4	16	not unique	..
G_{42}	3	4	16	not unique	..
G_{43}	3	4	16	not unique	..
G_{44}	3	4	8	not unique	..
G_{45}	4	4	4	not unique	..
G_{46}	3	6	12	not unique	..
G_{47}	3	4	8	not unique	..
G_{48}	4	4	24	8	not unique
G_{49}	3	4	16	not unique	..
G_{50}	3	4	13	not unique	..
G_{51}	4	4	16	not unique	..
G_{52}	4	4	8	not unique	..

G_{53}	3	4	12	not unique	..
G_{54}	4	4	8	not unique	..
G_{55}	3	4	13	not unique	..
G_{56}	3	4	13	not unique	..
G_{57}	4	4	12	not unique	..
G_{58}	3	4	14	not unique	..

TABLE 5. Graphs of order 12

Graphs	$\gamma(G)$	$\gamma_{bc}(G)$	b	r	λ_i
G_{59}	3	4	8	not unique	..
G_{60}	3	4	8	not unique	..
G_{61}	4	5	80	not unique	..
G_{62}	4	4	11	not unique	..
G_{63}	3	4	14	not unique	..
G_{64}	3	4	20	not unique	..
G_{65}	4	4	14	not unique	..
G_{66}	3	4	12	4	not unique
G_{67}	4	4	24	4	not unique
G_{68}	3	4	17	not unique	..
G_{69}	4	4	6	2	not unique
G_{70}	4	4	12	not unique	..
G_{71}	4	4	8	not unique	..
G_{72}	3	4	16	not unique	..
G_{73}	4	4	18	not unique	..
G_{74}	4	4	12	not unique	..
G_{75}	3	4	24	not unique	..
G_{76}	3	4	8	not unique	..
G_{77}	3	4	14	not unique	..
G_{78}	3	4	13	not unique	..
G_{79}	4	4	16	not unique	..
G_{80}	3	4	20	not unique	..
G_{81}	4	4	20	not unique	..
G_{82}	4	4	8	not unique	..
G_{83}	3	4	16	not unique	..

G_{84}	3	4	22	not unique	..
G_{85}	3	4	4	not unique	..
G_{86}	4	4	14	not unique	..
G_{87}	4	4	14	not unique	..
G_{88}	4	4	12	4	not unique
G_{89}	4	4	20	not unique	..
G_{90}	4	4	6	2	not unique
G_{91}	4	4	14	not unique	..
G_{92}	3	4	36	12	not unique

TABLE 6. Graphs of order 12

Graphs	$\gamma(G)$	$\gamma_{bc}(G)$	b	r	λ_i
G_{93}	3	4	22	not unique	..
G_{94}	4	4	10	not unique	..
G_{95}	3	4	18	6	not unique
G_{96}	4	4	12	4	not unique
G_{97}	3	4	12	4	not unique
G_{98}	4	4	8	not unique	..
G_{99}	3	4	18	not unique	..
$G100$	4	4	12	not unique	..
$G101$	3	4	18	not unique	..
$G102$	4	5	72	not unique	..
$G103$	4	4	20	not unique	..
$G104$	3	5	42	not unique	..
$G105$	4	4	10	not unique	..
$G106$	4	4	12	not unique	..
$G107$	4	4	20	not unique	..
$G108$	3	4	17	not unique	..
$G109$	3	4	9	not unique	..
$G110$	4	4	24	8	not unique
$G111$	4	4	8	not unique	..
$G112$	4	4	8	not unique	..
$G113$	3	4	9	not unique	..

From the above Tables 4, 5, 6, we can see that the repetition number r and λ_i's are not unique. Hence we conclude that, there does not exist PBIB-design for the graphs G_{29} to G_{113}, whose blocks are bi-connected dominating sets.

4. CONCLUSION

In this paper we have determined PBIB-designs for cubic graphs of order upto 12. Also non-existence of these two types of designs are addressed.

5. CONFLICT OF INTEREST

The authors have no conflict of interests related to this publication.

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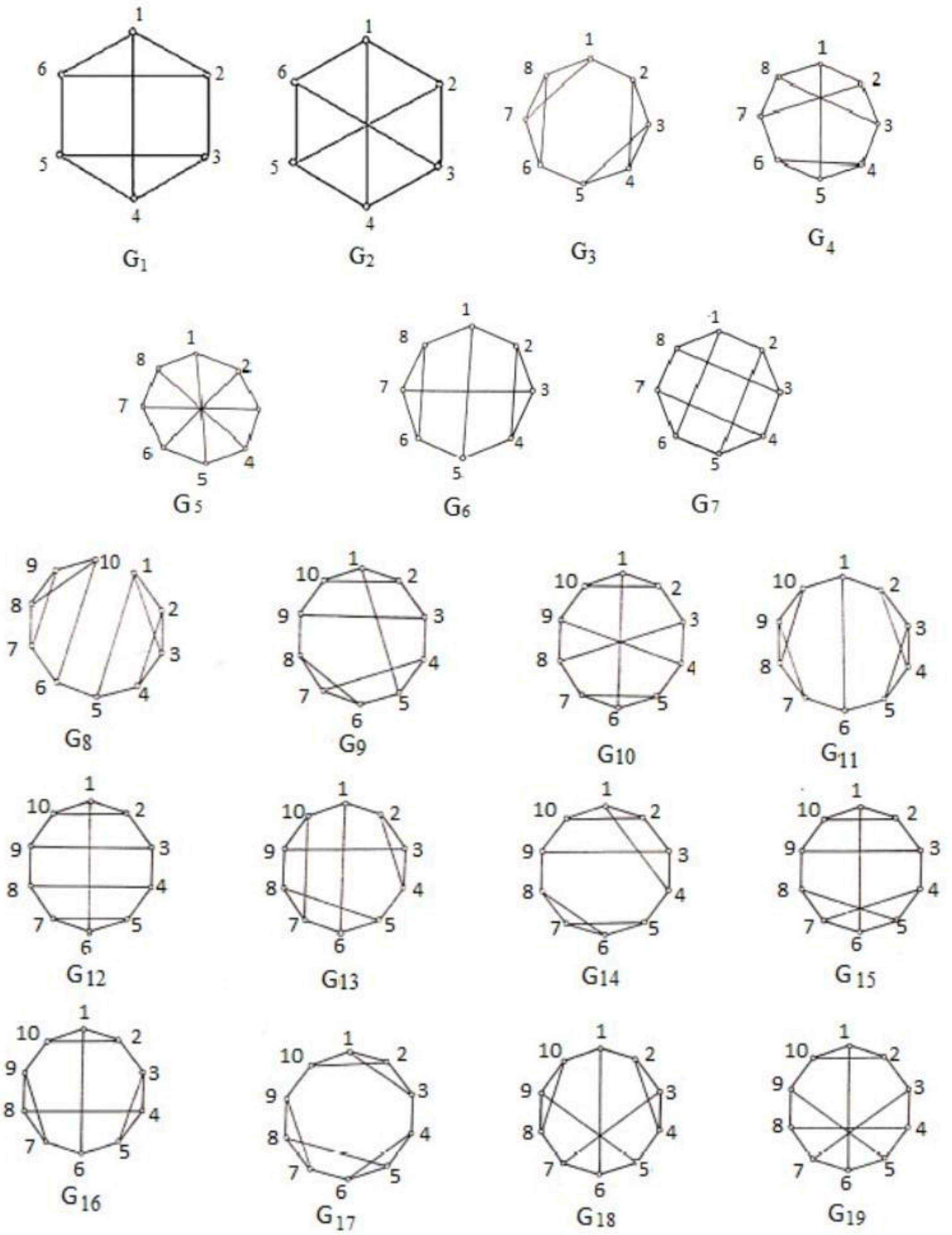
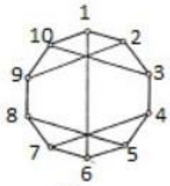
APPENDIX 1-LIST OF ALL CUBIC GRAPHS OF ORDER AT MOST



G_{20}



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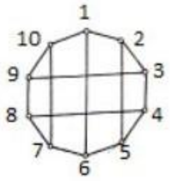

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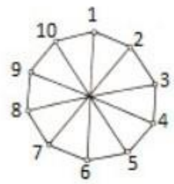
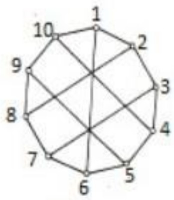
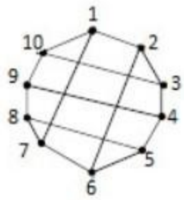


G_{23}



G_{24}



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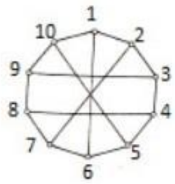
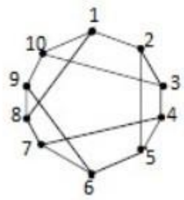


G_{26}



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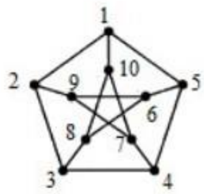
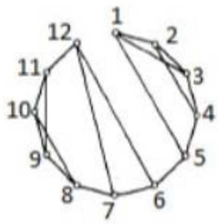
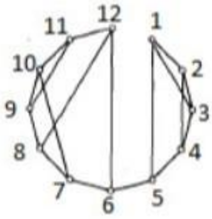


G_{28}



G_{29}



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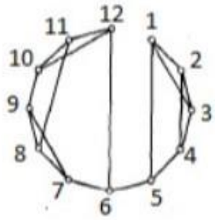

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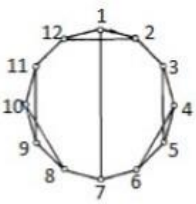

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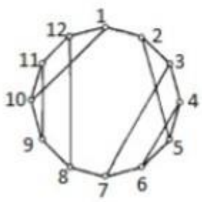

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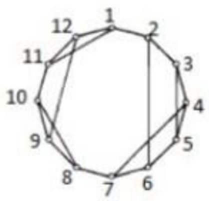

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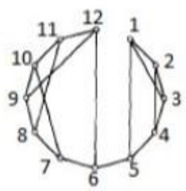

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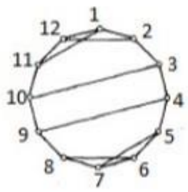

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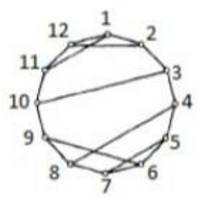

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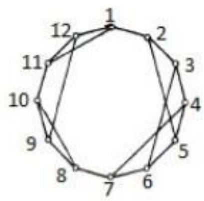

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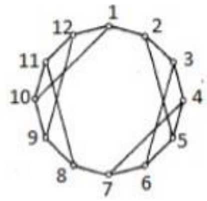
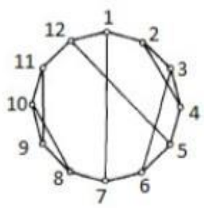


G_{39}



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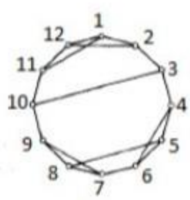

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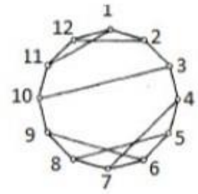

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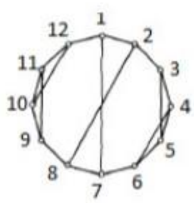
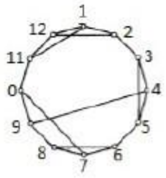
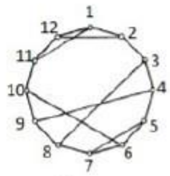


G_{43}



G_{44}



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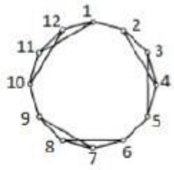

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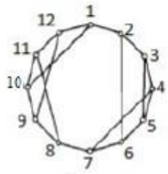

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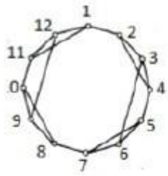

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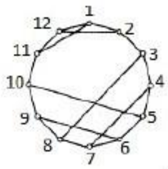

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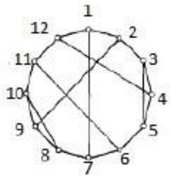

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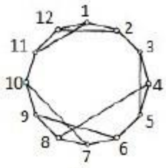

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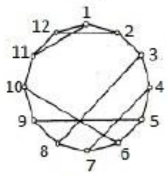

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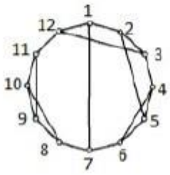

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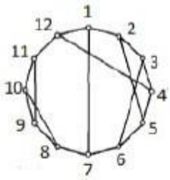

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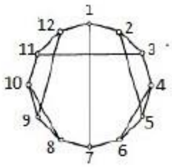

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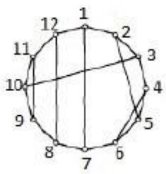

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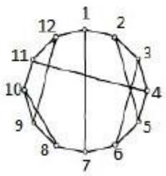

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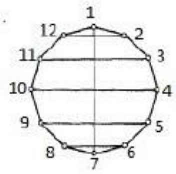

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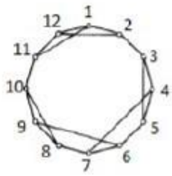

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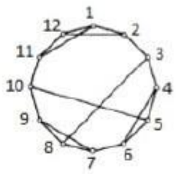

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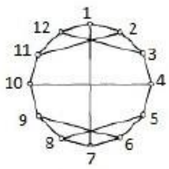

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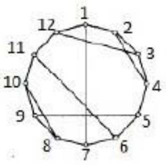

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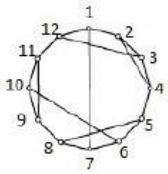

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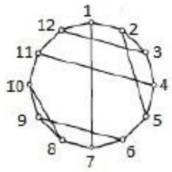

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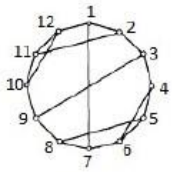

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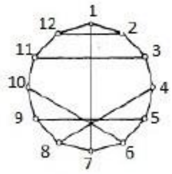

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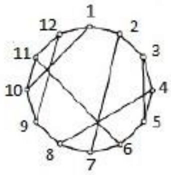

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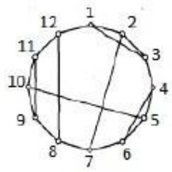

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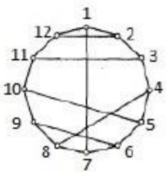

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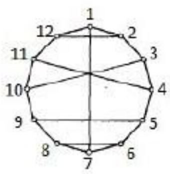

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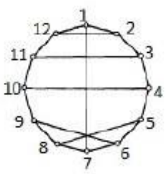

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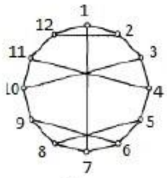
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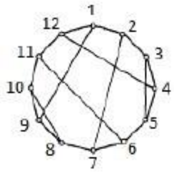
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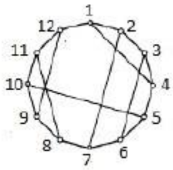
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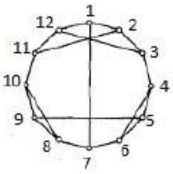
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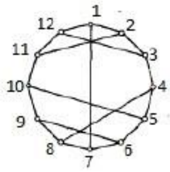
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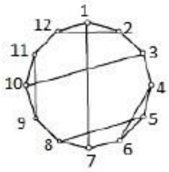
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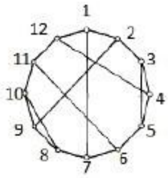
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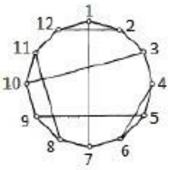
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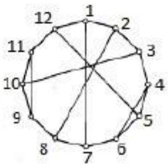
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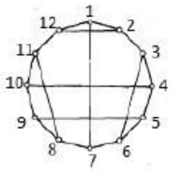
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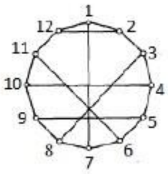
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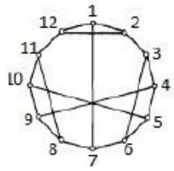
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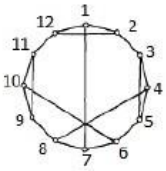
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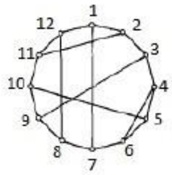
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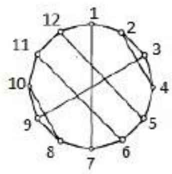
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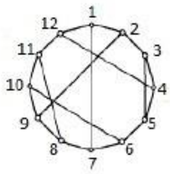
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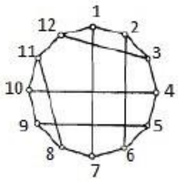
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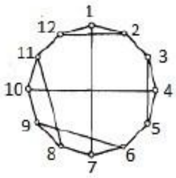
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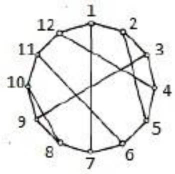
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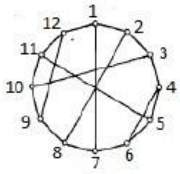
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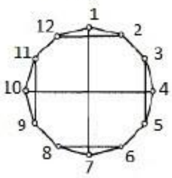
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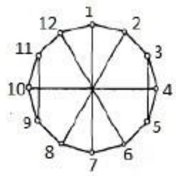
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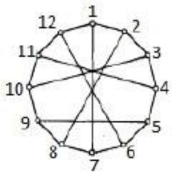
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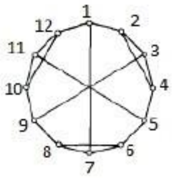
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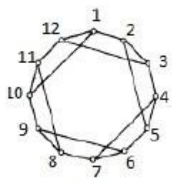
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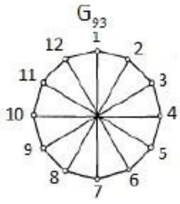
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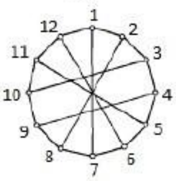
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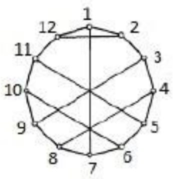
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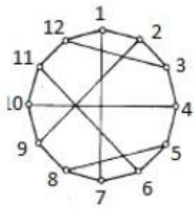

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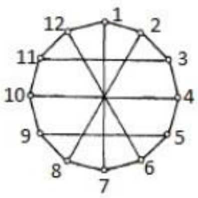

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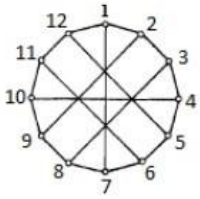

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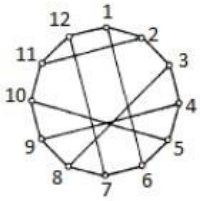

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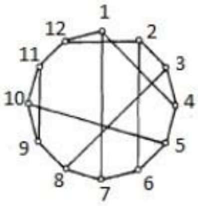

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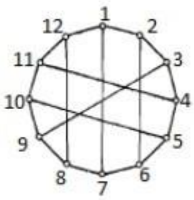

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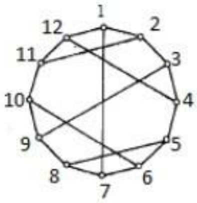

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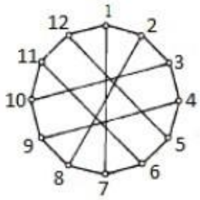

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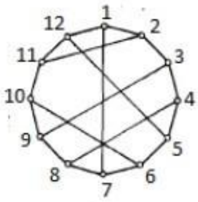

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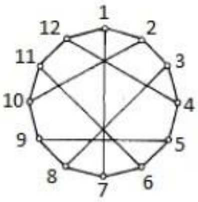

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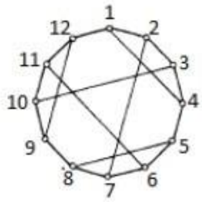

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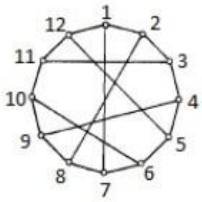

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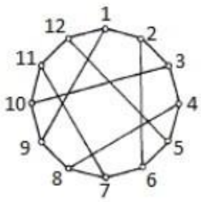

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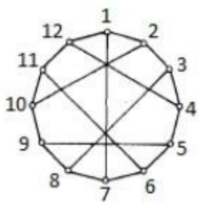

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