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The Wavelet Transform and Its Spectral Analysis: A Modern Review and Some Results in Harmonic Discrete Analysis

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ABSTRACT

The wavelet transform and its multiple versions in many aspects of the spectral analysis, the wavelet transform, likewise its relation with other transforms and special functions, needs a special treatment due to that in last years the developments in quantum electronics and recognition of signals in the frequency domain, likewise exploration profuse on new research in advanced materials like special semiconductors or fine devices in medicine, where the research is more detailed and majorly more specific in different applications. Likewise we have the wavelet transform of special function where each one can establish of very useful way the creation and design of special signal filters used in the modern industry in electronics, data processing analysis or even in the configuration of the advanced electronic equipment. Also the interphase between reception-emission devices with sensorial parts of the human body, in biomedical engineering. A version is interesting like the quantum wavelet transform which is very useful in the spectral study of traces of particles, for example studied from atomic accelerator. Finally are given general results to the discrete study of signal, which is analyzed to the wavelet transform and its spectra.

KEYWORDS: Discrete Fourier Transforms, Discrete Wavelet Transforms, Fast Fourier Transforms, Gabor Transforms, Harmonic Discrete Analysis, Modern Spectral Analysis, Signal Analysis, Wavelet Transforms

2020 AMS Classification: 42B10, 43A25, 43A45, 43A77, 44A51, 55N20

I. INTRODUCTION

Let be a set of functions

$$\psi_{1k}, \psi_{2k}, \ldots, \psi_{nk}, \ldots \in L^2(\mathbb{R}), \qquad (1)$$

where a Hilbert basis of square integrable functions is defined [1]. For each $j, k \in \mathbb{Z}$, the elements ψ_{jk}, represents by the functions[1]:

$$\psi_{jk}(x) = 2^{\frac{j}{2}} \psi(2^j x - k), \qquad (2)$$

$\forall j, k \in \mathbb{Z}$. Then, for a function $\xi(x) \in L^2(\mathbb{R})$, considering the orthonormal functions family, and by completeness we have:

$$\xi(x) = \sum_{-\infty}^{+\infty} c_{jk} \psi_{jk}(x), \qquad (3)$$

Then its convergence of this series will understood in norm. Likewise, $\xi(x)$, is known as a wavelet series with wavelet coefficients c_{jk}. Then an orthonormal wavelet is self-dual. Therefore the wavelet integral transform is given for [2][2]:

$$W_\psi \xi = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} \overline{\psi\left(\frac{x-b}{a}\right)} \xi(x) dx, \qquad (4)$$

We consider $a = 2^{-j}$, which is the binary dilation or dyadic dilation, and $b = k2^{-j}$, the binary or dyadic position. Modifications of the wavelet transforms can be realized depending on the response treatments that are given.

For example in images compression through of impulse function $x(n) = \delta(n - n_i)$, for a discrete signal is required an

[1] Dyadics and dilations of ψ.

[2] To recover the original signal $w(t)$, the first inverse continuous wavelet transform can be used:

$$\xi(t) = \chi_\psi^{-1} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_\psi \xi(a,b) \frac{1}{\sqrt{|a|}} \tilde{\psi}\left(\frac{t-b}{a}\right) db \frac{da}{a^2}.$$

impulse response to evaluate the image compression-reconstruction system.

In different signal treatments, one fundamental problem in electronics is the obtaining of sufficiently clean signals in the different processes of communication, perception and manage of the signals in different continuous and discrete domains.

In the signal processing on several accelerometers today is necessary implement and apply a strong and robous programming in real time, as example in the drones or other many devices of another vehicle class, even in biomedical engineering to create interphases between human body parts with accelerations process in failure detections for design of low power pacemakers and also in ultra-wideband in wireless [3].

As transformation the wavelet transform allows only changes in time domain and extensions. Then a choosing of a suitable base can affect before mentioned. For example in numerical and simulation analysis, is common consider the scale factor $c_n = c_0^n$, with the discrete frequency $\eta_m = mLc_0^n$, having the wavelets (considering the discrete formula with the basis wavelet):

$$\Psi(k,n,m,) = \frac{1}{\sqrt{c_0^n}}\Psi\left[\frac{k - mc_0^n}{c_0^n}L\right] = \frac{1}{\sqrt{c_0^n}}\Psi\left[\left(\frac{k}{c_0^n} - m\right)L\right], \quad (5)$$

where these discrete wavelets can be used through the discrete wavelet transform version, that is:

$$W_D(n,m) = \frac{1}{\sqrt{c_0^n}}\sum_{k=0}^{K-1} w(k)\Psi\left[\left(\frac{k}{c_0^n} - m\right)L\right], \quad (6)$$

whose continuous (or analogic version) is the known standard wavelet transform:

$$W(c,\eta) = \frac{1}{\sqrt{c}}\int_{-\infty}^{+\infty} w(t)\Psi\left(\frac{t-\eta}{c}\right)dt, \quad (7)$$

here c, is scaling factor, and η, represents time shift factor. For (7) we can obtain the Fourier transformation of signal $w(k)$, with the FFT help. An adequate selection of a discrete scaling factor, is pertinent c_n, and changes in the time extension are expected agree to the corresponding analysis frequency of the basis function, considering the uncertainty principle of signal processing.

An example of it, is when the wavelet transform of Shannon function is very useful in the creation of windows $\Psi^{Sha}(\omega)$, through functions $Sha(t)$, and with gate functions $\Pi(x)$, which are very useful in the signal analysis by ideal band-pass filters that define a decomposition known as Shannon wavelets. Another example is when the Complex-valued Morlet wavelet is closely related to human perception, both hearing and the vision [4].

The "classical" wavelet transform (with some modifications accepted) to quantum wavelet transform, can be developed considering the classical operators of the transformation into direct sums, direct products, and dot products of unitary matrices. Likewise the permutation matrices are fundamental part [5].

II. STARTING OF THE SIGNAL RESOLUTION PROBLEMS UNTIL BIOLOGICAL-SENSORIAL-PERCEPTION

An important and fundamental wavelet transform property in the signal resolution problem is explained and studiedd simultaneously in time and frequency domains starting of the wavelet spectra:

$$W(\omega) = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{+\infty} w\left(\frac{t-b}{a}\right)e^{-j\omega t}dt = \sqrt{|a|}W(a\omega)e^{-j\omega t}, \quad (8)$$

where $W(\omega)$, is the Fourier transform of the basic wavelet $w(t)$. Normalizing wavelets in terms of amplitude, the Fourier wavelets transforms with different scales will have the same amplitude, that is suitable for implementation of the continuous wavelet transform using the frequency domain filtering. This property is fundamental in the samples of frequency pulses of signal spectra, since shows a dilatation $t/a \quad (a > 1)$, of a function in the time domain that produces a contraction $a\omega$, of its Fourier transform, which are corresponding "spectral wavelets". Then, the term t/a, has a frequency metrology, which is equivalent to ω. Although in the technical standard, this term is known as scale, since the term "frequency" is reserved for the Fourier transform. Then the design of signal filters in frequency obeys to the correlation between the signal and the wavelets, in the time domain that can be written as the inverse Fourier transform of the product of the conjugate Fourier transforms of the wavelets and the Fourier transform of the signal:

$$W_\xi(a,b) = \frac{\sqrt{|a|}}{2\pi}\int_{-\infty}^{+\infty} \Xi(\omega)W^*(j\omega a)e^{-j\omega t}d\omega \quad (9)$$

Fourier transforms of the wavelets are referred with precision as the wavelet transform filters. The impulse response of the wavelet transform filter $\sqrt{|a|}W(a\omega)$, is the scaled wavelet $\frac{1}{\sqrt{|a|}}w\left(\frac{t}{a}\right)$. Therefore, the wavelet transform can be viewed as a collection of wavelet transform filters with different scales, a. Then we have relations between the short-time Fourier transform STFT, [6] with the application of the wavelets to obtain the sinusoidal frequency and phase content of local sections of a signal considering as changes over time. Introducing the Gaussian function, which can be regarded as a window function, we can obtain the STFT, as the Gabor transform. Here are constructed and characterized the Gabor atoms (expression used to element basis to Gabor function) or functions used to construct from translations and modulations of generating function a function family.

Likewise, direct applications of the STFT, are had, to samples in real time of the complex processes which require a speed compute of data through of direct relation between machine and real time domains in the measured and perception of the phenomena for example in the signal analysis of certain particles phenomena, or certain phenomena of astrophysics like anti-matter). Likewise, the STFT is performed and developed on a computer using the fast Fourier transform FFT, so both variables are discrete and quantized.

From a point of view of the Morlet transform that is the Gabor transform where a wavelet composed of a complex exponential (carrier) multiplied by a Gaussian window (envelope), the perception is used to measure fine processes as many natural organic processes. This wavelet is closely related to human perception, both hearing [2] and vision. Results that the functions related with these bio-sensorial perceptions use a $Sha(t)$, functions as special Gabor functions to discriminate steps of signal spectra in the perception and create of a signal response audible or visible required to the eye organ, the iris of eye, in the case of the vision and to audition, we have the audiphones that amplify the sounds (commercially existent) to equilibrate the lack of the eardrum or other parts of middle and inner ear to perceive the sounds.

III. SOME RESULTS IN MODERN DISCRETE SIGNAL ANALYSIS

Let be S^N, the N-dimensional complex sphere and let be

$$\ldots, e^{-2\Omega j}, e^{-\Omega j}, 1, e^{\Omega j}, e^{-2\Omega j}, \ldots, \qquad (10)$$

a linear basis of signals space in $L^2[K]$, that generates the subspace $\mathcal{W}$, such that $\forall \mathcal{w} \in \mathcal{W}$, is had that

$$\mathcal{w} = \sum_{n=1}^{N} c_n e^{-n\Omega j}, \qquad (11)$$

Then is possible to define the nilpotent classes on $E_{[K]}$, (where $E_{[K]} = E_1 \oplus \ldots \oplus E_n$, [7] is the total discrete signal space) with the points set component:

$$N(E_{[K]}) = \{F \in D'(G^0 / K^\wedge) | F = 0\}, \qquad (12)$$

Proposition 3. 1 [8, 9]. If $z \in N(E_{[K]})$, and if $\beta \in H^i(\mathfrak{n}_0^\wedge, L^2[K])$, then

$$z\beta = e^{-n\Omega j} x[m] = p(z)x[n], \qquad (13)$$

where $x[n]$, is a Gabor discrete function[3].

Proof. We consider the isomorphism between nilpotent algebras

$$\mathfrak{n}_0^\wedge \cong \mathfrak{n}, \qquad \mathfrak{n}_0^\wedge \cong \mathfrak{p}_I,$$

here $\mathfrak{p}_I$, is the unitary sphere $\mathfrak{p} \cap \mathfrak{g}_I$, where clarly $\mathfrak{g}_I, = [\mathfrak{g}, \mathfrak{g}]$. Now, we demonstrate on the dimension i, of the cohomological space $H^i(\mathfrak{g}_I, L^2[K])$. If $i = n = \dim \mathfrak{n}_0^\wedge$, then

$$H^i(\mathfrak{g}_I, L^2[K]) = \wedge \mathfrak{n}_0^\wedge * \otimes L^2 / \mathfrak{n}_0^\wedge L^2, \qquad (14)$$

Then z, acts by $I \otimes^\wedge p(z)$. Therefore $p(z)$, acts for $(\mathbb{C} \otimes^\wedge \pi) p(z)$. Then

$$(I \otimes^\wedge p(z))\beta = p_{\mathbb{C}}(z)\beta, \qquad (15)$$

As a special note we have as a particular example a LTI system $L(e^{-n\Omega j}) = H(\Omega)e^{-n\Omega j}$, where $H(\Omega)$, a the system function projection is.

Likewise is verified to $i = r + 1 \leq n$, then demonstrate for $i = r$. We consider C, a periodic complex. Let $\alpha \in H^*(X_i, \mathbb{Q}_b / \mathbb{Z}_a)$, such that $\alpha(g \otimes^\wedge \upsilon) g\upsilon$. Then we can affirm that α, is the homeomorphism

$$\alpha : \mathrm{Hom}_K(\mathfrak{p}_I, C) \to \mathrm{Hom}_K(\mathfrak{p}_I, L^2[K]), \qquad (16)$$

We consider $X = \ker \alpha$, where specifically

$$X = \{\alpha \in \mathrm{Hom}_K(C, L^2[K]) | \alpha(g \otimes^\wedge \upsilon) = 0, \quad \forall g \in U(\mathfrak{g}^\wedge), \upsilon \in L^2[k]\}, (17)$$

Therefore

$$0 \to X_i \to C \to L^2 \to 0, \qquad (18)$$

Now $U(\mathfrak{g}^\wedge)$, is a $U(\mathfrak{p}_I)-$complex free under left translations. Then

$$H^i(\mathfrak{p}_I, C) = 0, \qquad (19)$$

$\forall C = E^*_{[K]} \otimes^\wedge \mathbb{Q}_b / \mathbb{Z}_a$, therefore

$$H^i(\mathfrak{p}_I, E^*_{[K]} \otimes^\wedge \mathbb{Q}_b / \mathbb{Z}_a) = 0, \qquad (20)$$

$\forall j < n$, and $b \equiv a \bmod j$. Then the long exact sequence of discrete cohomology for this case of periodic complexes and $N(E_{[K]})-$complex is:

$$0 \to H^i(\mathfrak{p}_I, L^2[K]) \to H^{i+1}(\mathfrak{p}_I, X) \to H^{i+1}(\mathfrak{p}_I, E^*_{[K]} \otimes^\wedge \mathbb{Q}_b / \mathbb{Z}_a) \to 0, (21)$$

which implies the result. ◆

In signal and system analysis applied to linear systems can be approximated in the time-frequency domain through the composition of an analysis filter-bank, a transfer matrix (sub-band model) and a synthesis filter-bank, which defines a method known as sub-band technique.

Through the corresponding integral equation, the time--frequency representations of LTV systems have connection

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[3] A discrete version of Gabor representation is

$$x(t) = \sum_{n=-\infty}^{\infty} \sum_{m=\infty}^{\infty} C_{nm} g_{nm}(t),$$

with $g_{nm}(t) = s(t - m\eta_0)e^{-n\Omega jt}$. Analogously to the DFT (discrete Fourier transformation) is had:

$$x(k) = \sum_{n=-\infty}^{\infty} \sum_{m=\infty}^{\infty} C_{nm} g_{nm}(k),$$

where Gabor basis-functions are $g_{nm}(k) = s(k - m\eta_0)e^{-n\Omega jk}$.

with the Gabor expansion of signal. Then we will have and integral equation with Gabor function. For example, the creation of 3D Gabor-frame based in a spatial spectral integral equation designed to solve the scattering from dielectric objects embedded in a multi-layer medium. Likewise this is based on the Gabor frame, as a new set of functions basis [10] together with a set of equidistant Dirac-delta test functions.

Proposition 3. 2. Exists an isomorphism defined by DFT that maps the proper nilpotent classes of the system controlled under transformations of $\mathfrak{p}_I$. Then DS-TFT is a the FFT.

Let DFT be the isomorphism of the discrete signals:

$$E_{[K]} \to \widetilde{E}_{[K]}, \qquad (22)$$

where the explicit rule for any $\forall \upsilon \in L^2[K]$, is

$$\mathrm{DFT}(\upsilon) = (1/N)\mathrm{DFT}(\upsilon) = \mathrm{DWT}(\upsilon), \qquad (23)$$

Therefore in each component of $E_{[K]}$, ($E_{[K]} = E_1 \oplus \ldots \oplus E_n$.) is had that:

$$F^{(k)} = 0, \qquad (24)$$

where the DS-TFT is satisfied in short-time interval. Of fact in each component we have:

$$U(\mathfrak{p}_I)\mathrm{Hom}_{K^\wedge}(F, L^2[K]) = \chi_\wedge p_\xi, \qquad (25)$$

which exists $\mathrm{FFT}^{\sim}(\upsilon)$, such that

$$\mathrm{DWT}(\upsilon) = \mathrm{FFT}(\upsilon), \qquad (26)$$

For more details on the demonstration can be consulted [11].

IV. CONCLUSIONS

Various and several advantages of the wavelet transform and its properties on the signal and system analysis have been showed, taking different specialized windows functions and the wavelet function basis. Likewise, wavelet analysis is well-known for its utility and successful approach to the resolution of the signal analysis problems in the time domain and frequency domain. Also, the analysis of the non-stationary signal generated by physical phenomena has a great challenge for various signal conversion techniques and require the regularity in metrologies of signal analysis [12], being even extended to physical systems dynamics outputs analysis. In inverse problems the wavelet transforms can give us the idea of origin conditions of phenomena so complex like a electromagnetic plasmas and some phenomena in the Universe, for certain observables, for example [12].

In different studies have been shown that the transformation techniques such as Fourier transform and Short Fourier Transform fail to analyze non-stationary signals. But wavelet transform methods arise as efficient methods to analyze both stable and unstable signals. In the Gabor transform the resolution analysis consider the uncertainty principles on nilpotent Lie groups and their corresponding algebras, which were established in the propositions given through modern spectral analysis given in the classes $H^{\mathrm{i}}(\mathfrak{p}_I, L^2[K]), H^{\mathrm{i}+1}(\mathfrak{p}_I, X)$, and $H^{\mathrm{i}+1}(\mathfrak{p}_I, E^*_{[K]} \otimes^\wedge \mathbb{Q}_b / \mathbb{Z}_a)$. This study highlights as application and spectral study of discrete cohomology on periodic complexes, group actions on harmonic spherical analysis in discrete Hilbert representations and discrete groups [13, 14].

Schemes with neural network as dynamical system components can be designed to show that neural networks and linear filters in cascade and/or feedback configurations, can construct and determine a preponderant rich class of models of signaling and systematization in wide perspective and prospective, having the different filters designed by the different wavelet transform versions in short-time resolution or conventional resolution improving the canonical Fourier transform resolution. The multi-resolution analysis or also known as multi-scale approximation can design a method very effective and practically relevant discrete wavelet transforms (DWT), where these can be considered as a fundamental set of special functions to realize approximations to solutions of different processes in time and the justification for the algorithm of the fast wavelet transform (FWT), for the computing methods develop started with good wavelet bases.

Technical Notation

STFT - Short-Time Fourier Transform.

FFT - Fast Fourier Transform.

DFT - Discrete Fourier Transform.

LTV - Linear-Time Varying System.

DS - TFT - Discrete Short-Time Fourier Transform.

$w(t)$ – Basic wavelet.

$W(\omega)$ – Fourier transform of the basic wavelet $w(t)$.

ψ_{jk} – Dyadic and dilatations of the wavelets.

$x[n]$ – Discrete signal. In the proposition 2. 1, is a discrete Gabor function.

$E_{[K]}$ – Discrete signal space. This space is a Hilbert space on the discrete domain K. Its orthogonal decomposing is $E_{[K]} = E_1 \oplus \ldots \oplus E_n$. In the case of wavelets each ones of the components $E_j\,(j = 1,2,\ldots,n)$ are dyadic translations and dilations of wavelet $w(k)$.

DWT - Discrete Wavelet Transform.

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