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SUBDIRECTLY IRREDUCIBLE PSEUDO-COMPLEMENTED ADL'S

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Abstract

We introduce the concept of congruence on pseudo-complemented ADL's and study on certain properties of these. Mainly, in this paper all subdirectly irreducible pseudo-complemented ADL's are characterized.

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1 Introduction

In [4], W.H. Cornish defined a congruence Θ on a pseudo-complemented distributive lattice $(L, \vee, \wedge, *, 0, 1)$ as a congruence on the lattice $(L, \vee, \wedge, 0, 1)$ which is also compatible with the unary operation $*$ and called such a congruence as a $*-$congruence. The concept of an Almost Distributive Lattice

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(ADL) was introduced by Swamy and Rao [8] as a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebra and Boolean rings. Swamy, Rao and Rao [9] have introduced the notion of pseudo-complementation on an ADL and proved that the class of pseudo-complemented ADL's is equationally definable. It was observed that an ADL can have more than one pseudo-complementation. In this paper we prove that the compatible property for a pseudo-complementation $*$ on an ADL A is independent of any pseudo-complementation on A. In the sense that if Θ is a congruence on an ADL A and $*, +$ are pseudo-complementations on A, then $(a^*, b^*) \in \Theta$ if and only if $(a^+, b^+) \in \Theta$ for all $a, b \in A$. With is motivation, we introduce the concept of congruence on a pseudo-complemented ADL A. The main purpose of this paper is to characterize subdirectly irreducible pseudo-complemented ADL's. In particular we characterize subdirectly irreducible discrete ADL.

2 Preliminaries

We first recall certain elementary definitions and results concerning Almost Distributive Lattices. These are collected from [8] and [9].

Definition 2.1. An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following identities

(1). $0 \wedge a = 0$

(2). $a \vee 0 = a$

(3). $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(4). $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(5). $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(6). $(a \vee b) \wedge b = b$.

Any distributive lattice bounded below is an ADL, where 0 is the smallest element. Also, a commutative regular ring $(R, +, ., 0, 1)$ with unity can be made into an ADL by defining the operations $\wedge$ and $\vee$ on R by

$$a \wedge b = a_0 b \text{ and } a \vee b = a + b - a_0 b,$$

where, for any $a \in R$, a_0 is the unique idempotent in R such that $aR = a_0 R$ and 0 is the additive identity in R. Further any nonempty set X can be made into an ADL by fixing an arbitrarily choosen element 0 in X and by defining the operations $\wedge$ and $\vee$ on X by

$$a \wedge b = \begin{cases} 0, & \text{if } a = 0 \\ b, & \text{if } a \neq 0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } a \neq 0. \end{cases}$$

This ADL $(X, \wedge, \vee, 0)$ is called a discrete ADL. An ADL A is said to be associative ADL if the operation $\vee$ on A is associative. Through out this paper, by an ADL we always mean an associative ADL only.

Definition 2.2. Let A be an ADL. For any a and $b \in A$, define

$$a \leq b \text{ if and only if } a = a \wedge b \text{ (this is equivalent to } a \vee b = b).$$

Then $\leq$ is a partial order on A.

Theorem 2.3. *The following hold for any a, b and c in an ADL A.*

(1). $a \wedge 0 = 0 = 0 \wedge a$ *and* $a \vee 0 = a = 0 \vee a$

(2). $a \wedge a = a = a \vee a$

(3). $a \wedge b \leq b \leq b \vee a$

(4). $a \wedge b = a \Leftrightarrow a \vee b = b$

(5). $a \wedge b = b \Leftrightarrow a \vee b = a$

(6). $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *(i.e., $\wedge$ is associative)*

(7). $a \vee (b \vee a) = a \vee b$

(8). $a \leq b \Rightarrow a \wedge b = a = b \wedge a \ \Leftrightarrow a \vee b = b = b \vee a$

(9). $(a \wedge b) \wedge c = (b \wedge a) \wedge c$

(10). $(a \vee b) \wedge c = (b \vee a) \wedge c$

(11). $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$

(12). $a \wedge b = \inf\{a, b\} \Leftrightarrow a \wedge b = b \wedge a \Leftrightarrow a \vee b = \sup\{a, b\}$.

An element $m \in A$ is said to be maximal if $m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$ for all $x \in A$. A non-empty subset I of A is called an ideal (filter) of A if $a \vee b \in I$ $(a \wedge b \in I)$ and $a \wedge x \in I$ $(x \vee a \in I)$ whenever $a, b \in I$ and $x \in A$. For any $a \in A, (a] = \{a \wedge x : x \in A\}$ is the principal ideal generated by a and $[a) = \{x \vee a : x \in A\}$ is the principal filter generated by a.

Definition 2.4. An equivalence relation θ on an ADL A is called a congruence if θ is compatible with $\wedge$ and $\vee$, in the sense that, for any $a, b, c, d \in A$, (a, b) and $(c, d) \in \theta$ implies $(a \wedge c, b \wedge d) \in \theta$ and $(a \vee c, b \vee d) \in \theta$. We denote the zero congruence on A by Δ_A. That is $\Delta_A = \{(x, y) \in A \times A : x = y\}$

Definition 2.5. Let A be an ADL. A mapping $a \mapsto a^*$ of A into itself is called a pseudo-complementation on A if the following conditions are satisfied for any a and $b \in A$.

(1) $a \wedge a^* = 0$

(2) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$

(3) $(a \vee b)^* = a^* \wedge b^*$

An ADL with a pseudo-complementation is called a pseudo-complemented ADL.

Example 2.6. Let X be a discrete ADL and, for any arbitrarily fixed $x \neq 0$ in X, define the unary operation $*$ on X by

$$a^* = \begin{cases} 0, & \text{if } a \neq 0 \\ x, & \text{if } a = 0 \end{cases}$$

Then $*$ is a pseudo complementation on X. Here, with each $x \neq 0$ in X, we obtain a pseudo complementation on X.

Theorem 2.7. *Let $*$ be a pseudo complementation on an ADL A. Then the following hold for any a and $b \in A$.*

(1) 0^* *is a maximal element in A.*

(2) $m^* = o$ *for all maximal elements m.*

(3) $0^{**} = 0$

(4) $a^* \leq 0^*$

(5) $a^* \wedge a = 0$ *(In fact, $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$)*

(6) $a^{**} \wedge a = a$

(7) $a^{***} = a^*$

(8) $a \leq b \Rightarrow b^* \leq a^*$

(9) $a^* \wedge b^* = b^* \wedge a^*$ *and* $a^* \vee b^* = b^* \vee a^*$

(10) $a \wedge b = 0 \Leftrightarrow a^{**} \wedge b = 0 \Leftrightarrow a \wedge b^{**} = 0 \Leftrightarrow a^{**} \wedge b^{**} = 0$

(11) $(a \wedge b)^* = (b \wedge a)^*$ *and* $(a \vee b)^* = (b \vee a)^*$

(12) $a^* \wedge b = (a \wedge b)^* \wedge b$

(13) $(a \wedge b)^{**} = a^{**} \wedge b^{**} = b^{**} \wedge a^{**}$.

Theorem 2.8. *Let* $*$ *and* $+$ *be two pseudo-complementations on an ADL* A. *Then the following hold for any* a *and* $b \in A$.

(1) $a^* \wedge a^+ = a^+$

(2) $a^{*+} = a^{++}$

(3) $a^* \wedge 0^+ = a^+$

(4) $a^* = b^* \Leftrightarrow a^+ = b^+$

(5) $a^* = 0 \Leftrightarrow a^+ = 0$

(6) $a^* \vee a^{**} = 0^* \Leftrightarrow a^+ \vee a^{++} = 0^+$

3 Congruences on pseudo-complemented ADLs

In this section, we introduce the concept of congruence on a pseudo-complemented ADL A by considering any pseudo- complementation on A as one of the fundamental operations on the algebra A.

Before going to the main text we prove an important lemma which shows that the compatible property for a pseudo-complementation $*$ on A is independent of any pseudo-complementations on A.

Lemma 3.1. *Let* θ *be a congruence relation on an ADL* A *and* $*,+$ *be pseudo-complementations on* A. *Then* θ *is compatible with* $*$ *if and only if it is so with* $+$.

Proof. For any $a \in A$ we have that

$$a^* = a^+ \wedge 0^* \text{ and } a^+ = a^* \wedge 0^+ \text{(from Theorem 2.8(3))}$$

Suppose θ is compatible with $*$. Then,

$$\begin{aligned} (a,b) \in \theta &\Rightarrow (a^*, b^*) \in \theta \\ &\Rightarrow (a^* \wedge 0^+, b^* \wedge 0^+) \in \theta \\ &\Rightarrow (a^+, b^+) \in \theta \end{aligned} .$$

Therefore θ is compatible with $+$. Converse is similar. □

Definition 3.2. An equivalence relation θ is said to be a congruence on a pseudo-complemented ADL A if θ is compatible with $\vee, \wedge$ and any pseudo-complementation on A.

In the following we give different examples of congruences on pseudo-complemented ADL's.

Example 3.3. Let D be a discrete ADL with more than two elements and define

$$\Phi = \{(a,b) \in D \times D : a = 0 = b \text{ or both } \ a \neq 0 \text{ and } b \neq 0\}.$$

Then Φ is a non-zero congruence on the pseudo-complemented ADL D.

Example 3.4. Let D be discrete ADL with more then three elements.
Let $0 \neq x \in D$. Define

$$\Theta = \{(a,b) \in D \times D : a = b \text{ or } a, b \in D - \{0, x\}\}$$

Then Θ is a non-zero congruence on the pseudo-complemented ADL D.

Theorem 3.5. *Let A be an ADL and $*$ a pseudo-complementation on A. Define*

$$\sim = \{(a,b) \in A \times A : \ a \wedge b = b \text{ and } b \wedge a = a\}.$$

Then $\sim$ is a congruence on the pseudo-complemented ADL A.

Proof. It is easy to prove that $\sim$ is reflexive and symmetric.
Let $(a,b), (b,c) \in \sim$. Then $a \wedge b = b$ and $b \wedge a = a$, $b \wedge c = c$ and $c \wedge b = b$.
Now, $a \wedge c = b \wedge a \wedge c = a \wedge b \wedge c = b \wedge c = c$ and
$c \wedge a = b \wedge c \wedge a = c \wedge b \wedge a = b \wedge a = a$.
Therefore $(a,c) \in \sim$ and hence $\sim$ is an equivalence relation on A.
Let $(a,b), (c,d) \in \sim$. Then $a \wedge b = b, b \wedge a = a, c \wedge d = d$ and $d \wedge c = c$.
Now, $(a \wedge c) \wedge (b \wedge d) = a \wedge c \wedge b \wedge d = a \wedge b \wedge c \wedge d = b \wedge d$.
Similarly, $(b \wedge d) \wedge (a \wedge c) = a \wedge c$. Therefore $(a \wedge c, b \wedge d) \in \sim$.

$$\begin{aligned} \text{Now, } \ (a \vee c) \wedge (b \vee d) &= ((a \vee c) \wedge b) \vee ((a \vee c) \wedge d) \\ &= ((a \wedge b) \vee (c \wedge b)) \vee ((a \wedge d) \vee (c \wedge d)) \\ &= (b \vee (c \wedge b)) \vee ((a \wedge d) \vee d) \\ &= b \vee d \end{aligned}$$

and similarly, $(b \vee d) \wedge (a \vee c) = a \vee c$. Therefore $(a \vee c, b \vee d) \in \sim$.
Let $(a,b) \in \sim$. Then $a \wedge b = b$ and $b \wedge a = a$ and hence $a \vee b = a$ and $b \vee a = b$.
Now, $a^* \wedge b^* = (a \vee b)^* = (b \vee a)^* = b^*$ and $b^* \wedge a^* = (b \vee a)^* = (a \vee b)^* = a^*$.
Therefore $(a^*, b^*) \in \sim$. Hence $\sim$ is a congruence on the pseudo-complemented ADL A. $\square$

Theorem 3.6. *Let A be an ADL and $*$ a pseudo-complementation on A. For any nonempty subset I of A such that $x \vee y \in I$ for all $x, y \in I$, define*

$$\Theta(I) = \{(a, b) \in A \times A : a \wedge x^* = b \wedge x^* \text{ for some } x \in I\}$$

Then, $\Theta(I)$ is a congruence on the pseudo-complemented ADL A.

Proof. Clearly $\Theta(I)$ is reflexive and symmetric. Let $(a, b), (b, c) \in \Theta(I)$. Then $a \wedge x^* = b \wedge x^*$ and $b \wedge y^* = c \wedge y^*$ for some $x, y \in I$. Now, $a \wedge (x \vee y)^* = a \wedge x^* \wedge y^* = b \wedge x^* \wedge y^* = x^* \wedge b \wedge y^* = x^* \wedge c \wedge y^* = c \wedge x^* \wedge y^* = c \wedge (x \vee y)^*$ and $x \vee y \in I$. Therefore $(a, c) \in \Theta(I)$ and hence $\Theta(I)$ is an equivalence relation on A. Let (a, b) and $(c, d) \in \Theta(I)$. Then $a \wedge x^* = b \wedge x^*$ and $c \wedge y^* = d \wedge y^*$ for some $x, y \in I$.

$$\begin{aligned}
\text{Now } (a \wedge c) \wedge (x \vee y)^* &= (a \wedge c) \wedge x^* \wedge y^* \\
&= a \wedge x^* \wedge c \wedge y^* \\
&= b \wedge x^* \wedge d \wedge y^* \\
&= b \wedge d \wedge x^* \wedge y^* \\
&= (b \wedge d) \wedge (x \vee y)^* \\
\text{and } (a \vee c) \wedge (x \vee y)^* &= (a \vee c) \wedge (x^* \wedge y^*) \\
&= (a \wedge x^* \wedge y^*) \vee (c \wedge x^* \wedge y^*) \\
&= (b \wedge x^* \wedge y^*) \vee (x^* \wedge c \wedge y^*) \\
&= (b \wedge x^* \wedge y^*) \vee (x^* \wedge d \wedge y^*) \\
&= (b \wedge x^* \wedge y^*) \vee (d \wedge x^* \wedge y^*) \\
&= (b \vee d) \wedge (x^* \wedge y^*) \\
&= (b \vee d) \wedge (x \vee y)^*.
\end{aligned}$$

Therefore $(a \wedge c, b \wedge d)$ and $(a \vee c, b \vee d) \in \Theta(I)$. Thus $\Theta(I)$ is a congruence relation on the ADL A. Again let $(a, b) \in \Theta(I)$. Then $a \wedge x^* = b \wedge x^*$ for some $x \in I$.
Now, $a^* \wedge x^* = (a \wedge x^*)^* \wedge x^*$ (by 2.7(12)) $= (b \wedge x^*)^* \wedge x^* = b^* \wedge x^*$.
Therefore $(a^*, b^*) \in \Theta(I)$ and hence $\Theta(I)$ is a congruence on pseudo-complemented ADL A. □

Theorem 3.7. *Let A be an ADL and $*$ a pseudo-complementation on A. For any nonempty subset F of A such that $x \wedge y \in F$ for all $x, y \in F$, define*

$$\psi(F) = \{(x, y) \in A \times A : x \wedge a^{**} = y \wedge a^{**} \text{ for some } a \in F\}$$

Then, $\psi(F)$ is a congruence on the pseudo-complemented ADL A.

Proof. This is similar to above. □

Theorem 3.8. *Let A be an ADL and $*$ a psuedo- complimentation on A and F be a filter of A, define*

$$\theta_F = \{(a, b) \in A \times A : x \wedge a = x \wedge b \text{ for some } x \in F\}.$$

Then θ_F is a congruence on the pseudo-complemented ADL A.

Proof. It is easy to prove that θ_F is reflexive and symmetric.
Let $(a, b), (b, c) \in \theta_F$. Then $x \wedge a = x \wedge b$ and $y \wedge b = y \wedge c$, for some $x, y \in F$. Then $x \wedge y \in F$ and $x \wedge y \wedge a = y \wedge x \wedge a = y \wedge x \wedge b = x \wedge y \wedge b = x \wedge y \wedge c$. Therefore $(a, c) \in \theta_F$ and hence θ_F is an equivalence relation on A. Let $(a, b), (c, d) \in \theta_F$. Then $x \wedge a = x \wedge b$ and $y \wedge c = y \wedge d$ for some $x, y \in F$. Now, $x \wedge y \wedge a \wedge c = x \wedge a \wedge y \wedge c = x \wedge b \wedge y \wedge d = x \wedge y \wedge b \wedge d$ and

$$\begin{aligned} x \wedge y \wedge (a \vee c) &= (x \wedge y \wedge a) \vee (x \wedge y \wedge c) \\ &= (y \wedge x \wedge a) \vee (x \wedge y \wedge d) \\ &= (y \wedge x \wedge b) \vee (x \wedge y \wedge d) \\ &= ((x \wedge y) \wedge b) \vee ((x \wedge y) \wedge d) \\ &= (x \wedge y) \wedge (b \vee d). \end{aligned}$$

Therefore $(a \wedge c, b \wedge d)$ and $(a \vee c, b \vee d) \in \theta_F$. Finally
$(a, b) \in \theta_F \Rightarrow x \wedge a = x \wedge b$ for some $x \in F \Rightarrow x \wedge b \wedge a^* = 0 \Rightarrow b \wedge x \wedge a^* = 0 \Rightarrow b^* \wedge x \wedge a^* = x \wedge a^* \Rightarrow x \wedge b^* \wedge a^* = x \wedge a^* \wedge b^* = x \wedge a^* (\because a^*, b^* \leq 0^*)$
Similarly, we can obtain that $x \wedge a^* \wedge b^* = x \wedge b^*$ and hence $x \wedge a^* = x \wedge b^*$. Therefore $(a^*, b^*) \in \theta_F$. Thus θ_F is a congruence on the pseudo-complemented ADL A. □

Corollary 3.9. *Let A be an ADL and $*$ a psuedo- complimentation on A. For any $a \in A$, define*

$$\theta_a = \{(x, y) \in A \times A : a \wedge x = a \wedge y\}$$

Then θ_a is a congruence on the pseudo-complemented ADL A. Moreover $\theta_a = \Delta_A$ if and only if a is maximal.

Proof. This follows from the fact that $\theta_a = \theta_{[a)}$. □

Definition 3.10. Let A be an ADL and $a \in A$. Define

$$\theta^a = \{(x, y) \in A \times A : a \vee x = a \vee y\}$$

From [8], it follows that θ^a is a congruence on the ADL A for any $a \in A$. However, if A is pseudo-complemented, then θ^a may not be a congruence on the pseudo-complemented ADL A, that is, θ^a may not be compatible with a pseudo-complementation $*$. For, consider the following.

Example 3.11. Let A be the set of all open subsets of the real number system with the usual topology. Then A is an ADL (infact, it is a lattice) under the set operations $\cap$ and $\cup$. Also, A is pseudo-complemented, where, for any X in A, X^* is the interior of the complement of X. Now, consider

$$P = \mathbb{R} - \mathbb{Z},\ Q = (0,1) \text{ and } S = (1,2).$$

Then P, Q and $S \in A$ and $P \cup Q = P \cup S$ and therefore $(Q,S) \in \theta^P$. However, $(Q^*, S^*) \notin \theta^P$, since

$$Q^* = (-\infty, 0) \cup (1, \infty) \text{ and } S^* = (-\infty, 1) \cup (2, \infty)$$

$$\text{and } \ P \cup Q^* \neq P \cup S^* \text{ (for } 2 \in P \cup Q^* \text{ and } 2 \notin P \cup S^*)$$

Theorem 3.12. *Let A be an ADL with a pseudo- complimentation $*$. If θ^a is compatible with $*$ for all $a \in A$, then A is a Stone ADL (that is, $a^* \vee a^{**} = 0^*$ for all $a \in A$ [10]).*

Proof. Let $a \in A$. Now $a^{**} \vee 0 = a^{**} \vee a^{**}$ and hence $(0, a^{**}) \in \theta^{a^{**}}$. If $\theta^{a^{**}}$ is compatible with $*$, then $(0^*, a^{***}) \in \theta^{a^{**}}$; therefore,

$$a^{**} \vee 0^* = a^{**} \vee a^{***} = a^{**} \vee a^*$$

and hence $a^* \vee a^{**} = 0^*$ (note that $x^* \leq 0^*$ for all $x \in A$). □

Remark 3.13. The converse of the above theorem is false. Let $A = [0,1]$, the closed unit interval of real numbers where $\wedge$ and $\vee$ are the minimum and maximum operations. Then A is a Stone ADL (infact a lattice), where $0^* = 1$ and $a^* = 0$ for all $a \neq 0$. Take $a = 0.5$. Then $(0, a) \in \theta^a$; but $(0^*, a^*) \notin \theta^a$ (since $1 = a \vee 1 = a \vee 0^*$ and $a \vee a^* = a \vee 0 = a \neq a \vee 0^*$).

Theorem 3.14. *Let A be an ADL and $*$ a pseudo- complimentation on A. Then for any $a \in A$,*

$$\sim_a = \{(x,y) \in A \times A : x \wedge y = y,\ y \wedge x = x \ \textit{and} \ \ a \vee x = a \vee y\}$$

is a congruence on the pseudo-complemented ADL A. Moreover $\sim_a = \Delta_A$ if and only if a is the zero element of A.

Proof. Clearly $\sim_a = \sim \cap \theta^a$ and hence $\sim_a$ is compatible with $\vee$ and $\wedge$ on A. Further we prove that $\sim_a$ is compatible with $*$. Let $(x,y) \in \sim_a$. Then $x \wedge y = y,\ y \wedge x = x$ and $a \vee x = a \vee y$. Then $x^* = (x \vee y)^* = (y \vee x)^* = y^*$ and hence $(x^*, y^*) \in \sim_a$. Thus $\sim_a$ is a congruence on the pseudo-complemented ADL A. □

4 Subdirectly irreducible pseudo-complemented ADLs

Let us recall that a non-trivial algebra L (containing more than one element) is called subdirectly irreducible if the intersection of any family of non-zero congruences is again non-zero; or equivalently $\mathcal{C}(L)$, the lattice of all congruence relations on A has smallest non-zero congruence. Here we characterize subdirectly irreducible pseudo-complemented ADL's.

We first characterize subdirectly irreducible discrete ADL.

Theorem 4.1. *Let A be a discrete ADL. Suppose that A is subdirectly irreducible as a pseudo-complemented ADL. Then $|A| \leq 3$.*

Proof. Let A be a discrete ADL with more than three elements. Let x, y, z be three distinct elements in $A - \{0\}$. Define

$$\theta = \{(a,b) \in A \times A : \text{ either } a = b \text{ or } \ a, b \in \{x, y\}\}$$

$$\text{and } \phi = \{(a,b) \in A \times A : \text{ either } a = b \text{ or } a, b \in \{x, z\}\}$$

Then θ and ϕ are congruence relations on the pseudo-complemented ADL A. Since $(x, y) \in \theta$ and $(x, z) \in \phi$, we have $\theta \neq \Delta_A$ and $\phi \neq \Delta_A$ and also $\theta \cap \phi = \Delta_A$. Therefore A is not subdirectly irreducible, which is a contradiction. Thus A has atmost three elements. □

Finally in the following theorem we find all the subdirectly irreducible pseudo-complemented ADL's.

Theorem 4.2. *Let $(A, \vee, \wedge, 0)$ be any subdirectly irreducible pseudo-complemented ADL. Then A is discrete .*

Proof. Suppose that A is a subdirectly irreducible pseudo-complemented ADL. Then there exists smallest non-zero congruence on the pseudo-complemented ADL A, say φ. Choose $x, y \in A$ such that $x \neq y$ and $(x, y) \in \varphi$. Then we prove that atleast one of x and y is maximal. Assume that both x and y are not maximal. Then by the Corollary 3.9, $\theta_x \neq \Delta_A \neq \theta_y$, so that $(x, y) \in \theta_x \cap \theta_y$. Hence $x = x \wedge x = x \wedge y$ and $y = y \wedge y = y \wedge x$ This implies that $x = y$ which is a contradiction. Therefore without loss of generality we may assume that x is maximal. Now, we prove that every non-zero element in A is maximal. Let $0 \neq a \in A$. Suppose if possible a is not maximal. Since x is maximal, $x \wedge a = a$ so that $a \wedge x$ is a non zero element in A (otherwise, $a \wedge x = 0 \Rightarrow x \wedge a = 0 \Rightarrow a = 0$). Therefore, by the Theorem 3.14, $\sim_{a \wedge x} \neq \Delta_A$. Also, since a is not maximal, by the Corollary 3.9, $\theta_a \neq \Delta_A$ and hence $\theta_a \cap \sim_{a\wedge x} \neq \Delta_A$. Therefore $\varphi \subseteq \theta_a \cap \sim_{a\wedge x}$ and hence $(x, y) \in \theta_a \cap \sim_{a\wedge x}$. Now, $x = (a \wedge x) \vee x = (a \wedge x) \vee y = (a \wedge y) \vee y = y$ which is a contradiction. Thus a is maximal and hence A is discrete. □

The following is an immediate consequence of the above two results.

Theorem 4.3. *A pseudo-complemented ADL is subdirectly irreducible if and only if it is discrete and has atmost three elements.*

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