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All Known Perfect Numbers other than 6 Satisfy N=4+6n

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ARTICLE INFO	ABSTRACT
Published Online: **20 March 2024** **Corresponding Author:** **Segura J. J.**	For all 51 known perfect numbers ranging from (p=2 to p= 82589933) and with the only exception of N=6, all perfect numbers belong to the group of natural numbers formed by N=4+6n. If this observation can be proven valid for all existing even perfect numbers, that would automatically exclude 2/3 of all even numbers out of the possibility of being perfect. If this can be proven a necessary condition for all perfect numbers, then it would rule out the possibility of having any odd perfect numbers.

KEYWORDS: Perfect numbers, prime numbers, composite numbers, number theory, even, odd

I. INTRODUCTION

In number theory, a perfect number is a positive integer that is equal to the sum of its positive divisors, excluding the number itself. For instance, 6 has divisors 1, 2 and 3 (excluding itself), and 1+2+3=6, so 6 is a perfect number. The next perfect number is 28, since 1+2+4+7+14=28.

The first known recorded mathematical study concerning perfect numbers is 300 BC's Elements written by Euclid. Euclid found a way to generate perfect numbers: a number would be perfect if it fulfilled the following 2 conditions:

$$Perfect\ Number = (2^p - 1) \times 2^{p-1}$$

Where $2^p - 1$ is prime number.

While Euclid's algorithm generates perfect numbers, it was not proven that it would generate all existing perfect numbers. For example, Euclid's algorithm only generates even perfect numbers, and thus one question that arose already in Euclid's time was if it was possible to find an odd perfect number.

Circa year 100, Nicomachus of Gerassa published Introduction to Arithmetic. There he stated 5 conjectures that he did not demonstrated at the time. One of Nicomachus conjectures was that the n^{th} perfect number has n digits. This conjecture was proven false *circa* 1230 by Ibn Fallus when he found the that 5^{th} perfect number, (p=13 in Euclid's algorithm), has not 5 digits but 8 digits. Another conjecture of Nicomachus was that all perfect numbers end alternatively in 6 and in 8. This was disproven also by Ibn Fallus when he found that the 5^{th} and 6^{th} perfect numbers (p=13 and p=17 in Euclid's algorithm) end both in 6. Mersenne work in prime numbers furthered the horizon of finding new perfect numbers in the centuries to come by using the Mersenne prime numbers to find valid values of p.

The next Nicomachus conjecture was to be confirmed. It stated that Euclid's algorithm produces every even perfect number. This was proven by Euler in the XVIII century thanks to introducing the σ function, where the number itself is added to the sum of all its divisors, resulting in twice the value of the perfect number. The use of the σ function also allowed for further advance on the study of perfect number theory by allowing to work with the prime powers composing the perfect number. This allowed to conclude that for an odd perfect number, only one single power prime could have an odd power, while all the remaining prime powers forming had even powers. Despite all the contributions made by Euler, he could not conclude if odd perfect numbers existed or not. Two conjectures of Nicomachus remain to be confirmed or disproven: that there is an infinite number of perfect numbers and that all perfect numbers are even.

There are certain conditions that have been proven to be necessary for an even perfect number: they end always in 6 or in 28, their iterative digital sum is 1, except for perfect number 6. The only square free perfect number is 6 [1]. A perfect number is an Ore harmonic number [2].

But the inability to have generated a single odd perfect number until now has originated a search for conditions that must apply to any existing such number, they include among others: $N>10^{1500}$ [3], N is not divisible by 105 [4], N is of the form N ≡ 1 (mod 12) or N ≡ 117 (mod 468) or N ≡ 81 (mod 324) [5], the largest prime factor of N is greater than 108 [6]… and the list goes on. To this day no odd perfect number has been found. This complex net of conditions is sometimes used as a reason to claim that the probability of any number to be a perfect odd number makes it almost impossible to exist and in case it exists, it helps limiting the number of possible options where to look for it.

On the other hand and to the best of the authors research, there seems to be not a parallel advance restricting the conditions required for a perfect number to be even other than the ones presented already above. The authors of these paper believe they may have found an indication of such a condition.

II. CLASSIFICATION OF KNOWN PERFECT NUMBERS

In their work published in 2023, Segura and Barchino presented 5 equations that, when together, generate all and only composite numbers [7]. They sorted all natural numbers N in 6 groups, A to F according to N=y+6n where y is a natural number from 1 to 6:

A contains all natural numbers generated by N=1+6n where n={0, 1, 2, 3, 4…}

B contains all natural numbers generated by N=2+6n where n={0, 1, 2, 3, 4…}

C contains all natural numbers generated by N=3+6n where n={0, 1, 2, 3, 4…}

D contains all natural numbers generated by N=4+6n where n={0, 1, 2, 3, 4…}

E contains all natural numbers generated by N=5+6n where n={0, 1, 2, 3, 4…}

F contains all natural numbers generated by N=6+6n where n={0, 1, 2, 3, 4…}

By using this classification, each natural number N belongs to one and only one of the 6 groups. All even natural numbers belong to groups B, D and F, where no odd numbers are found. All odd natural numbers belong to groups A, C and E, where no even natural numbers are found.
In order to find the group to which a given natural number N belongs, it only requires to solve

$$f(N) = \frac{N - y}{6}$$

For y = 1, 2, 3, 4, 5 and 6.

For a given natural number N, the only y returning an integer result, is the one of the group where x belongs. The other 5 values of y will return a fractional result.

In attempting to find the distribution of all known perfect numbers in groups B, D and F, the authors found that, with the exception of the perfect number 6 all the remaining known 50 perfect numbers belonged to group D, and therefore, were of the form N=4+6n. The only exception being 6 a natural number belonging to N=6+6n group F.

The authors found this observation striking. As described in the introduction, the perfect number 6 is the exception to several known properties of even perfect numbers, this seems to be yet another case. This indicates that other than 6, all even perfect numbers could very well belong to D with no exception.

III. COMPUTER CALCULATION

To find the group to which each perfect number belongs, we used a computer to run a Rust Code: Multithreaded algorithm, where each known perfect number was checked for the existence or not of decimals as follows:

$$\frac{[(2^{p-1}) \times (2^p - 1)] - y}{6}$$

For y=1, 2, 3, 4, 5 and 6.

Where all tested p are the numbers known as per today to originate perfect numbers in Euclid’s algorithm and y is the natural number associated to each group from A to F as described in the previous section.
For a given p and y, if the result is an integer number, then $(2^{p-1}) \times (2^p - 1)$ belongs to the group of natural numbers formed by N=y+6n. If the result is a fractional number, then $(2^{p-1}) \times (2^p - 1)$ does not belong to the group N=y+6N.

An example of such an algorithm is shown next, in this case for testing all known perfect numbers for their belonging to group D (y=4):

```
use rayon::prelude::*;
use rug::ops::Pow;
use rug::Float;

fn check_decimals(p_values: Vec<u32>) {
  p_values.par_iter().enumerate().for_each(|(idx, &p)| {
    // This is to fine tune calculations precision
    let precision = match p {
      n if n < 86243 => 100000,
      n if n >= 86243 && n <= 1398269 => 10000000,
      n if n > 1398269 && n <= 6972593 => 120000000,
      _ => 175000000,
    };
    let two = Float::with_val(precision, 2 as u32);
    let two_2 = two.clone();
    let one = Float::with_val(precision, 1 as u32);
    let one_2 = one.clone();
    let four = Float::with_val(precision, 4 as u32);
    let six = Float::with_val(precision, 6 as u32);

    let result = two.pow(p - one) * (two_2.pow(p) - one_2)
- four;
    let final_result = result / six;
    if final_result.is_integer() {
      println!("index={} p={} integer", idx + 1, p);
```

```
    } else {
      println!("index={} p={} fractional", idx + 1, p);
    }
  });
}

fn main() {
  let p_values = vec![
      2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607,
1279, 2203, 2281, 3217, 4253, 4423,
      9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243,
110503, 132049, 216091, 756839,
      859433, 1257787, 1398269, 2976221, 3021377,
6972593, 13466917, 20996011, 24036583,
      25964951, 30402457, 32582657, 37156667, 42643801,
43112609, 57885161, 74207281, 77232917, 82589933,
  ];

  check_decimals(p_values);
}
```

The results obtained on the perfect numbers belonging or not to group D are:

```
index=1 p=2 fractional
index=2 p=3 integer
index=3 p=5 integer
index=4 p=7 integer
index=5 p=13 integer
index=6 p=17 integer
index=7 p=19 integer
index=8 p=31 integer
index=9 p=61 integer
index=10 p=89 integer
index=11 p=107 integer
index=12 p=127 integer
index=13 p=521 integer
index=14 p=607 integer
index=15 p=1279 integer
index=16 p=2203 integer
index=17 p=2281 integer
index=18 p=3217 integer
index=19 p=4253 integer
index=20 p=4423 integer
index=21 p=9689 integer
index=22 p=9941 integer
index=23 p=11213 integer
index=24 p=19937 integer
index=25 p=21701 integer
index=26 p=23209 integer
index=27 p=44497 integer
index=28 p=86243 integer
index=29 p=110503 integer
index=30 p=132049 integer
index=31 p=216091 integer
index=32 p=756839 integer
index=33 p=859433 integer
index=34 p=1257787 integer
index=35 p=1398269 integer
index=36 p=2976221 integer
index=37 p=3021377 integer
index=38 p=6972593 integer
index=39 p=13466917 integer
index=40 p=20996011 integer
index=41 p=24036583 integer
index=42 p=25964951 integer
index=43 p=30402457 integer
index=44 p=32582657 integer
index=45 p=37156667 integer
index=46 p=42643801 integer
index=47 p=43112609 integer
index=48 p=57885161 integer
index=49 p=74207281 integer
index=50 p=77232917 integer
index=51 p=82589933 integer
```

When replacing y by 6, p=2 returns “integer” and all the other perfect numbers return “fractional”. When replacing y by 1, 2, 3 or 5, all of them return the “fractional” result.

IV. RESULTS

For all known perfect numbers up to now and with the only exception of the first perfect number, N=6, we have demonstrated that all perfect numbers belong to the family of natural numbers N=4+6n.

V. DISCUSSION

The authors believe that such observation has not been reported previously and that this contribution could be of value to the field of number theory.

It is not the aim of this paper to demonstrate if this observation will apply to other perfect numbers unknown to us today. While this work does not predict if other even perfect numbers will belong or not to N=4+6n, it shows a clear indication that if we remove all N belonging to groups A, B, C, E and F, it is likely that the probability of finding perfect numbers will increase, maybe to the point of being the only valid option. If this was the case, it would rule out all odd numbers from the perfect numbers. The validity or not of this condition for all existing perfect numbers remains to be proven.

VI. CONTRIBUTIONS

Segura J. J.: Mathematical reasoning and writing.

Ortega S.: Computation.

The authors do not incur into any conflict of interests by publishing these results.

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