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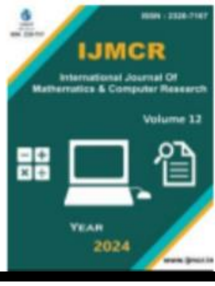

A Stochastic Model for Length Biased Loai Distribution with Properties and Its Applications

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ABSTRACT
One of the most important applications of statistical analysis is in health research and application. Cancer studies are mostly required special statistical considerations in order to find the appropriate model for fitting the survival data. In this paper, we examine some general models leading to a weighted distribution. The developed distribution, also known as the length biased model, is referred to as the length biased Loai distribution. This distribution is a specific type of basic distribution, the Loai distribution. The length biased distribution has been compared with the original distribution. Some statistical properties of this distribution are derived, such as moments, the moment generating function, the reliability analysis, and the included functions. Also, the distribution of order statistics, quantile function, and likelihood ratio test are presented. The Bonferroni and Lorenz carvers, as well as the Rényi and Tsallis entropies, are derived. The method of maximum likelihood estimation is used to estimate the distribution parameters. A simulation study is performed to investigate the performance of the estimation. The real data applications demonstrate that the proposed distribution can provide better results than several well-known distributions.

KEYWORDS: Length biased Model, Reliability Analysis, Statistical Properties, Entropies, Likelihood Ratio, Estimation of the Parameter.

1. INTRODUCTION

Medical scientists are mostly interested in studying the survival of patients with cancer in their applied research. Many statistical distributions have been extensively utilized for analyzing time -to-event data, also referred to as survival of reliability data, in different areas of applicability, including the medical field. These studies are most often requiring special statistical attention and adjustment in the context of finding and choosing the appropriate model that accurately
Determines and estimates the survival data and yields reliable results and valid inferences. The concepts of size biased sampling and weighted distribution pertaining to observational studies and surveys of research related to forestry, ecology, bio-medicine, reliability, and several other areas have been widely studied in the literature. Adding extra parameter to an existing distribution brings the classical distribution in a more flexible situation and the distribution becomes useful for data analysis purpose. As a result, weighted distributions Aries naturally generated from a stochastic process and are recorded with some weighted function. When the weight function depends on the length of the unit of, interest, the resulting distribution is called length biased. Rao identified various situations that can be modeled by weighted distribution. An investigator records an observation by nature according to a certain stochastic model. The observation will not have the original distribution unless every observation is given an equal chance of being recorded. Suppose that the original observation X has $f(x)$ as the probability density function pdf (which may be probability density when X is continuous) and that the probability of recording the observation x is $0 \leq w(x) \leq 1$, then the pdf X_w, the observation is

$$f_w(x) = \frac{w(x)f(x)}{\omega}, x > 0$$

Where ω is the normalizing factor obtained to make the total probability equal to unity. With an arbitrary non-negative weight function $w(x)$ which may exceed unity, where $w(x) = x$ or $x^{\alpha}, \alpha > 0$ when $\alpha = 1$, are he called distributions with arbitrary $w(x)$ of is a special case. The weighted distribution with $w(x) = x$ is called the (length biased) or sized biased distribution. When the probability of observing a positive-valued random variable is proportional to the value of the variable the resultant is length biased distribution.

Weighted distribution was firstly introduced by Fisher, [12] developed a new concept of distribution the weighted distribution, to model the ascertainment bias. The concept of weighted functions was first introduced by Rao, [28] on discrete distributions arising out of ascertainment, then identified various situations that can be modelled by weighted distributions. A sampling plan that gives unequal probabilities to the various units by Patil and Rao, [23] a weighted distributions a survey of their applications. Patil and Rao, [17] weighted distributions and size-biased sampling with application to wildlife populations and human families. We study the properties of the weighted distributions in comparison with those of the original distributions for positive-valued random variables. Such random variables and distributions arise naturally in life testing, reliability, and economics. Blumental, [7] proportional sampling in life length studies. Cox, [8] some sampling problems in technology. Schaeffer, [31] size-biased sampling. Mahfoud and Patil, [21] on weighted distributions. Gupta, [9] relations for reliability measures under length-biased sampling. Kochar and Gupta, [6] some result on weighted distributions for positive-valued random variables, the weighted distributions have compared with the original distributions with the partial orderings of probability distributions. Also find out finally moments of the weighted distribution have been obtained. Oluyede, [22] on inequalities and selection of experiments for length biased distribution occurs naturally for some sampling plans in reliability, and survival analysis. Also, length biased distributions are proved for monotone hazard functions and mean residual life functions. Finally, entropy measures are also investigated. Blumenthal (1967) and Scheaffer (1972) the sampling mechanism selects units with probability proportional to some measure of the unit, the relating distribution is called a size biased.

Size biased and length biased distributions have been used in etiological studies Simon, [20].

Cnaan, [10] on survival models with two phases and length biased sampling.

Recently, different authors have reviewed and studied the various length biased probability models illustrated their applications in different fields. Ahad and Ahmad, [2] discussed the Characterization and estimation of the length biased Nakagami distribution. Al-Omari and Alsmairam, [3] obtained the Length-biased Suja distribution and Its application. Rather and Subramanian, [30] discussed the length-biased erlang–truncated exponential distribution with life time data. Ekhosuehi et. al, [11] presented The Weibull length biased exponential distribution. Klinjan and Aryuyuen, [16] the author discussed by the length-biased power Garima distribution and its application to model lifetime data. Ben Ghorbal, [4] introduced the On Properties of Length-Biased Exponential Model. Benchettah et. al, [5] discussed the on composite length-biased exponential-Pareto distribution: Properties, simulation, and application in actuarial science.

In this research, we adopt the idea of a propose a new two parameter distribution. The proposed distribution such that, length biased Loai distribution. The Loai distribution introduced by Loai Alzoui et. al, [19] is a newly proposed two parameter lifetime model for various medical science applications. The proposed distribution (Length biased Loai distribution) shows its flexibility and superiority to fit some real lifetime data sets compared to some competing distributions.

The present paper is organized, as a in section 2, we derived the probability density function (pdf) and cumulative distribution function (cdf) of the length biased Loai distribution. In section 3, we discussed in reliability analysis of length biased Loai distribution. In section 4, we have some statistical properties, including moment generating function, r^{th} moment, mean, variance, coefficient of variance, and hormonic mean. In section 5, we derived mean deviation. In section 6, mean deviation from median. In section 6, we explore the distribution of order statistics and, quantile function. In section 7, likelihood ratio test. In section 8, we present Bonferroni and Lorenz curves and Gini index. Section 9, provides the stochastic ordering of the distribution. Entropies is derived in section 10. Maximum likelihood estimates and Fisher's information matrix we have derived section 11. Finally, different applications of the length biased Loai distribution to complete and censored datasets are presented section 12. All computations throughout this paper were performed using the statistical programming language R. Conclusion is presented in section 13.

2. LENGTH BIASED LOAI DISTRIBUTION

In this section, we define the probability density function (pdf) and cumulative distribution function (cdf) of the length biased Loai distribution.

A new two parameter life time distribution name as Loai distribution. The probability density function (pdf) of the Loai distribution is given by

$$f(x) = \frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\,\alpha\theta\, x^2 + \frac{1}{\theta+1}(1+x)\right] e^{-\theta x} \quad ; x > 0, \alpha > 0, \theta > 0 \qquad (1)$$

We have considered a random variable x with a probability density function $f(x)$. Let $w(x)$ be a non-negative weight function. Denote a new probability density function.

$$f_w(x) = \frac{w(x) f(x)}{E[w(x)]} \quad ; x > 0$$

Where $w(x)$ be the non-negative weight function and $E[w(x)] = \int w(x)f(x)dx < \infty$
and corresponding random variable by X_w, which is called the weighted random variable corresponding to x. When $w(x) = x^c$, $c > 0$, we say that X_w size-biased of order c. such a selection procedure is called size-biased sampling of order c. when $c = 1,2$, X_w is simply called size-biased (or length biased) and has a probability density function

$$f_w(x) = \frac{xf(x)}{E(X)} \quad ; x > 0$$

Gupta [14] has obtained some relations between the reliability measures of the original distribution and those of the length biased distribution.
The weighted distribution is obtained by applying the weighted function as $w(x) = x^c$, in weights we use $c = 1$, $w(x) = x$ to the Loai distribution in order to obtain the length biased Loai distribution. The probability density function of the length biased Loai distribution given by

$$f_w(x;\alpha,\theta) = \frac{x\, f(x;\theta)}{E(X)} \qquad ; x > 0, \theta > 0, \alpha > 0 \qquad (3)$$

Where,

$$E(X) = \int_0^\infty x\; f(x;\theta)\, dx$$

After simplification we get, gamma function is given by

$$\Gamma(Z) = \int_0^\infty t^{z-1}\; e^{-t} dt$$

$$E(X) = \left[\frac{3\,\alpha\,(\theta+1)+\theta+2}{(\alpha+1)(\theta+1)\,\theta}\right] \qquad (4)$$

Substituting the value of equation (1) and (4) in equation (3), we get the probability density function (pdf) of length biased Loai distribution.

$$f_l(x;\alpha,\theta) = \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\; x \left[\frac{1}{2}\,\alpha\theta\, x^2 + \frac{1}{\theta+1}(1+x)\right] e^{-\theta x} \qquad (5)$$

After simplification, using a lower incomplete gamma function is given by

$$\gamma\big((z+1),\theta x\big) = \int_0^{\theta x} t^{(z+1)-1} e^{-t} dt$$

We will get Cumulative distribution function of length biased Loai distribution is given by

$$F_l(x;\alpha,\theta) = \int_0^x f_l(x;\alpha,\theta)\; dx$$

$$= \int_0^x \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\; x \left[\frac{1}{2}\,\alpha\theta\, x^2 + \frac{1}{\theta+1}(1+x)\right] e^{-\theta x}\, dx$$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2} \int_0^x \left[\frac{1}{2}\,\alpha\theta\, x^3 + \frac{1}{\theta+1}(x+x^2)\right] e^{-\theta x}\, dx$$

Put $\theta x = t$, $x = \frac{t}{\theta}$, $dx = \frac{1}{\theta}\, dt$

When $x \to 0, t \to 0$ and $x \to x, t \to \theta x$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\left[\int_0^{\theta x} \frac{1}{2}\,\alpha\theta \left(\frac{t}{\theta}\right)^3\; e^{-t}\frac{1}{\theta}\, dt + \frac{1}{\theta+1}\int_0^{\theta x}\left(\frac{t}{\theta}+\left(\frac{t}{\theta}\right)^2\right) e^{-t}\frac{1}{\theta}\, dt\right]\right.$$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta\int_0^{\theta x}\frac{t^3}{\theta^3}\; e^{-t}\frac{1}{\theta}\, dt + \frac{1}{\theta+1}\int_0^{\theta x}\left(\frac{t}{\theta}+\frac{t^2}{\theta^2}\right) e^{-t}\frac{1}{\theta}\, dt\right]$$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta\int_0^{\theta x}\frac{t^3}{\theta^3}\; e^{-t}\frac{1}{\theta}\, dt + \frac{1}{\theta+1}\int_0^{\theta x}\left(\frac{\theta t+t^2}{\theta^2}\right) e^{-t}\frac{1}{\theta}\, dt\right]$$

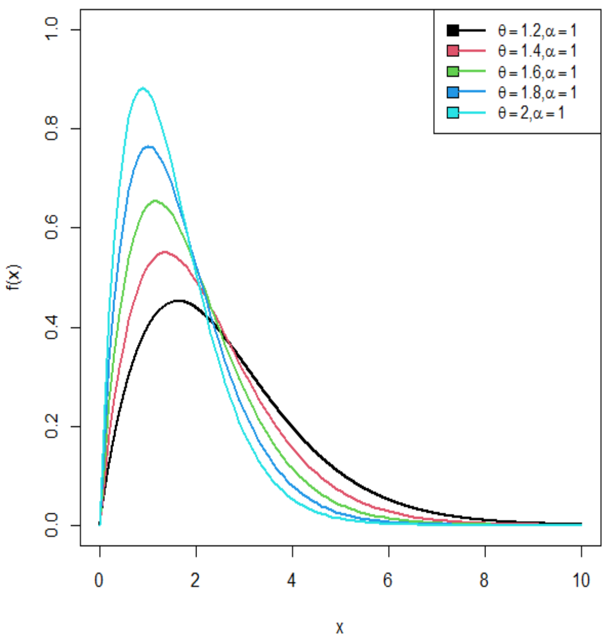
$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\;\frac{1}{\theta^3}\int_0^{\theta x} t^3\, e^{-t}\, dt + \frac{1}{\theta+1}\int_0^{\theta x}\frac{1}{\theta^3}(\theta t+t^2)e^{-t}\, dt\right]$$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2} \times \frac{1}{\theta^3}\left[\frac{\alpha}{2}\int_0^{\theta x} t^3\, e^{-t}\, dt + \frac{1}{\theta+1}\int_0^{\theta x}(\theta t+t^2)e^{-t}\, dt\right]$$

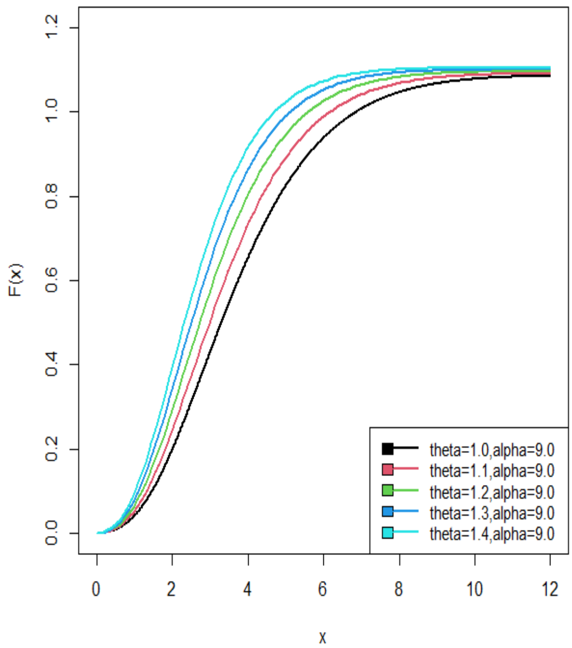
$$= \frac{(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{\alpha}{2}\int_0^{\theta x} t^3\, e^{-t}\, dt + \frac{1}{\theta+1}\left(\int_0^{\theta x}\theta t\, e^{-t}\, dt + \int_0^{\theta x} t^2\, e^{-t}\, dt +\right)\right] \qquad (6)$$

After simplification of equation (6), we obtain the cumulative distribution function of length biased Loai distribution

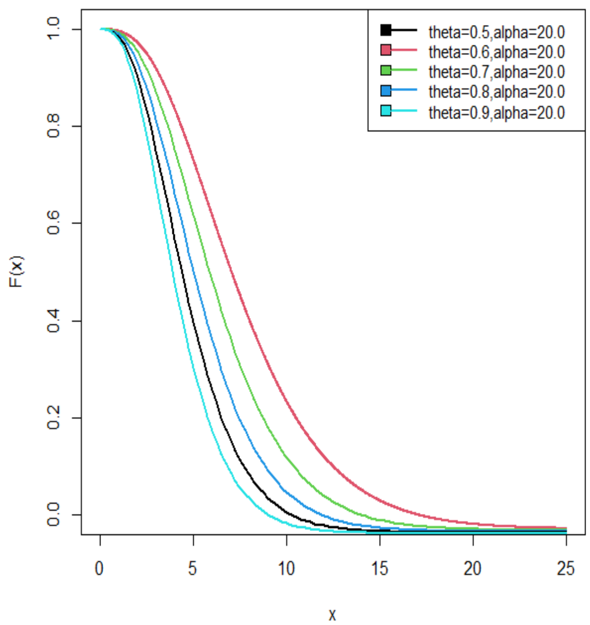
$$F_l(x;\alpha,\theta) = \frac{(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(2,\theta x) + \gamma(3,\theta x)\big)\right] \qquad (7)$$



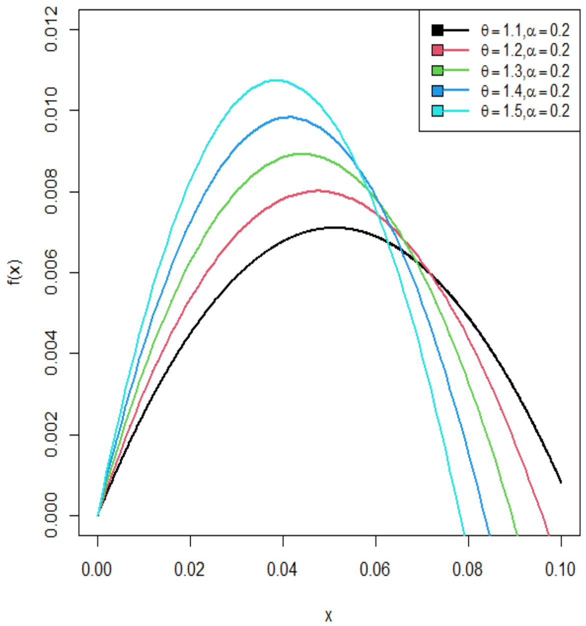
Figures.1:Pdf plot of length biased Loai distribution



Figures.2 Cdf plot of length biased Loai distribution



Figures.3 survival function of length biased loai distribution



Figures.4:Hazard function of length biased Loai distribution

3. RELIABILITY ANALYSIS

In this section, we will discuss the reliability function, hazard function, reverse hazard function, cumulative hazard function, Odds rate, Mills ratio and, Mean Residual function for the proposed length biased Loai distribution.

3.1 Reliability function

The reliability function is also known as survival function. It can be computed as complement of the cumulative distribution function. The survival function of length biased Loai distribution is given by

$$S(x) = 1 - F_l(x;\alpha,\theta)$$

$$S(x) = 1 - \left(\frac{(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta x) + \frac{1}{\theta+1}\left(\theta\,\gamma(2,\theta x)+\gamma(3,\theta x)\right)\right]\right)$$

3.2 Hazard function

The Hazard function is also known as hazard rate, instantaneous failure rate or force mortality of length biased Loai distribution is given by

$$h(x) = \frac{f_l(x;\alpha,\theta)}{1 - F_l(x;\alpha,\theta)}$$

$$h(x) = \left(\frac{\theta^3\,(\theta+1)\,x\,\left[\frac{1}{2}\,\alpha\theta\,x^2 + \frac{1}{\theta+1}(1+x)\right] e^{-\theta x}}{3\,\alpha\,(\theta+1) + \theta + 2 - (\theta+1)\left[\frac{\alpha}{2}\gamma(4,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(2,\theta x) + \gamma(3,\theta x)\big)\right]} \right)$$

3.3 Revers hazard rate

Reverse hazard function of length biased Loai distribution is given by

$$h_r(x) = \frac{f_l(x;\alpha,\theta)}{F_l(x;\alpha,\theta)}$$

$$h_r(x) = \left(\frac{\theta^3\,(\theta+1)}{(\theta+1)\left[\frac{\alpha}{2}\gamma(4,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(2,\theta x) + \gamma(3,\theta x)\big)\right]} x \times \left[\frac{1}{2}\,\alpha\theta\,x^2 + \frac{1}{\theta+1}(1+x)\right] e^{-\theta x} \right)$$

3.4 Odds rate function

Odds Rate function of length biased Loai distribution is given by

$$O(x) = \frac{F_l(x;\alpha,\theta)}{1 - F_l(x;\alpha,\theta)}$$

$$O(x) = \left(\frac{\frac{(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(2,\theta x) + \gamma(3,\theta x)\big)\right]}{1 - \frac{(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(2,\theta x) + \gamma(3,\theta x)\big)\right]} \right)$$

$$O(x) = \left(\frac{(\theta+1)\,\left[\frac{\alpha}{2}\gamma(4,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(2,\theta x) + \gamma(3,\theta x)\big)\right]}{3\,\alpha\,(\theta+1) + \theta + 2 - \;(\theta+1)\left[\frac{\alpha}{2}\gamma(4,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(2,\theta x) + \gamma(3,\theta x)\big)\right]} \right)$$

3.5 Cumulative hazard function

Cumulative hazard function of length biased Loai distribution is given by

$$H(x) = -\ln\big(1 - F_l(x;\alpha,\theta)\big)$$

$$H(x) = \ln\left(\frac{(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(2,\theta x) + \gamma(3,\theta x)\big)\right] - 1 \right)$$

3.6 Mills Ratio

Mills Ratio $= \frac{1}{h_r(x)}$

$$\text{Mills Ratio} = \left(\frac{(\theta+1)\left[\frac{\alpha}{2}\gamma(4,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(2,\theta x) + \gamma(3,\theta x)\big)\right]}{\theta^3\,(\theta+1)\,x\,\left[\frac{1}{2}\,\alpha\theta\,x^2 + \frac{1}{\theta+1}(1+x)\right] e^{-\theta x}} \right)$$

3.7 Mean Residual function

Mean Residual function of length biased Loai distribution is given by

$$M(x) = \frac{1}{S(x;\alpha,\theta)} \int_x^\infty t\,f(t;\alpha,\theta)dt - x$$

$$= \frac{1}{1 - \left(\frac{(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta x) + \frac{1}{\theta+1}\big(\theta\,\gamma(2,\theta x) + \gamma(3,\theta x)\big)\right] \right)} \times \int_x^\infty \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\,t^2\,\left[\frac{1}{2}\,\alpha\theta\,t^2 + \frac{1}{\theta+1}(1+t)\right] e^{-\theta t}\,dt - x$$

Put $\theta x = t, \quad x = \frac{t}{\theta}, \quad dx = \frac{1}{\theta}\,dt$

When $x \to 0, t \to 0$ and $x \to x, t \to \theta x$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\int_{\theta x}^\infty \frac{1}{2}\,\alpha\theta\,\left(\frac{t}{\theta}\right)^4\, e^{-t}\frac{1}{\theta}\,dt + \frac{1}{\theta+1}\int_{\theta x}^\infty \left(\frac{t^2}{\theta^2} + \left(\frac{t}{\theta}\right)^3\right) e^{-t}\frac{1}{\theta}\,dt \right] - x$$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta \int_{\theta x}^\infty \frac{t^4}{\theta^4}\, e^{-t}\frac{1}{\theta}\,dt + \frac{1}{\theta+1}\int_{\theta x}^\infty \left(\frac{t^2}{\theta^2} + \frac{t^3}{\theta^3}\right) e^{-t}\frac{1}{\theta}\,dt \right] - x$$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta\int_{\theta x}^{\infty}\frac{t^4}{\theta^4}\,e^{-t}\frac{1}{\theta}\,dt+\frac{1}{\theta+1}\int_{\theta x}^{\infty}\left(\frac{\theta t^2+t^3}{\theta^3}\right)e^{-t}\frac{1}{\theta}\,dt\right]-x$$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\,\frac{1}{\theta^3}\int_{\theta x}^{\infty}t^4\,e^{-t}\,dt+\frac{1}{\theta+1}\int_{\theta x}^{\infty}\frac{1}{\theta^3}(\theta t^2+t^3)e^{-t}\,dt\right]-x$$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\times\frac{1}{\theta^4}\left[\frac{\alpha}{2}\int_{\theta x}^{\infty}t^3\,e^{-t}\,dt+\frac{1}{\theta+1}\int_{\theta x}^{\infty}(\theta t+t^2)e^{-t}\,dt\right]-x$$

$$= \frac{(\theta+1)}{\theta(3\,\alpha\,(\theta+1)+\theta+2)}\left[\frac{\alpha}{2}\int_{\theta x}^{\infty}t^3\,e^{-t}\,dt+\frac{1}{\theta+1}\left(\int_{\theta x}^{\infty}\theta t\,e^{-t}\,dt+\int_{\theta x}^{\infty}t^2\,e^{-t}\,dt+\right)\right]-x$$

After solving the integral, we get

$$M_l(x;\alpha,\theta)=\frac{(\theta+1)}{\theta\left[3\,\alpha\,(\theta+1)+\theta+2-\theta+1\left(\left[\frac{\alpha}{2}\gamma(4,\theta x)+\frac{1}{\theta+1}\left(\theta\,\gamma(2,\theta x)+\gamma(3,\theta x)\right)\right]\right)\right]}$$
$$\times\left[\frac{\alpha}{2}\Gamma(5,\theta x)+\frac{1}{\theta+1}\left(\theta\Gamma(3,\theta x)+\Gamma(4,\theta x)\right)\right]-x$$

4 STATISTICAL PROPERTIES

In this section, we derived the structural properties, the moment generating function, Characteristic function and r^{th} moment for the length biased Loai distribution random variable are derived. Also, the mean, variance, coefficient of variance, standard deviation, dispersion, hormonic mean is investigated.

4.1 Moments

Let X_l denoted the random variable following length biased Loai distribution then r^{th} order moments $E(X^r)$ is obtained as

$$E(X^r)=\mu_r'=\int_0^{\infty}x^r\,f_l(x;\alpha,\theta)\,dx$$

$$=\int_0^{\infty}x^r\,\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\,x\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}\,dx$$

$$=\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\int_0^{\infty}x^{r+1}\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}\,dx$$

$$=\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{1}{2}\,\alpha\theta\int_0^{\infty}x^{r+3}\,e^{-\theta x}\,dx+\frac{1}{\theta+1}\left(\int_0^{\infty}x^{r+1}\,e^{-\theta x}\,dx+\int_0^{\infty}x^{r+2}\,e^{-\theta x}\,dx\right)\right]$$

$$=\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{1}{2}\,\alpha\theta\,\frac{\Gamma(r+4)}{\theta^{r+4}}+\frac{1}{\theta+1}\left(\frac{\Gamma(r+2)}{\theta^{r+2}}+\frac{\Gamma(r+3)}{\theta^{r+3}}\right)\right]$$

$$=\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\times\frac{1}{\theta^{r+3}}\left[\frac{\alpha}{2}\,\Gamma(r+4)+\frac{1}{\theta+1}\left(\theta\Gamma(r+2)+\Gamma(r+3)\right)\right]$$

$$\mu_r'=\frac{(\theta+1)}{\theta^r\,[3\,\alpha\,(\theta+1)+\theta+2]}\left[\frac{\alpha}{2}\,\Gamma(r+4)+\frac{1}{\theta+1}\left(\theta\Gamma(r+2)+\Gamma(r+3)\right)\right]$$

$$\mu_r'=\frac{(\theta+1)}{\theta^r\,[3\,\alpha\,(\theta+1)+\theta+2]}\left[\frac{\alpha\,\Gamma(r+4)}{2}+\frac{\left(\theta\Gamma(r+2)+\Gamma(r+3)\right)}{\theta+1}\right] \qquad (8)$$

Putting $r=1,2,3,4$ in equation (8), the mean of length biased Loai distribution is obtained as

$$\mu_1'=\frac{12\alpha\,(\theta+1)+2\theta+6}{\theta\,[3\,\alpha\,(\theta+1)+\theta+2]}$$

$$\mu_2'=\frac{60\alpha\,(\theta+1)+6\theta+24}{\theta^3\,[3\,\alpha\,(\theta+1)+\theta+2]}$$

$$\mu_3'=\frac{360\alpha\,(\theta+1)+24\theta+120}{\theta^3\,[3\,\alpha\,(\theta+1)+\theta+2]}$$

$$\mu_4'=\frac{2520\alpha\,(\theta+1)+120\theta+720}{\theta^4\,[3\,\alpha\,(\theta+1)+\theta+2]}$$

Variance $=\mu_2'-(\mu_1')^2$

$$\sigma^2=\left[\frac{60\alpha\,(\theta+1)+6\theta+24}{\theta^3\,[3\,\alpha\,(\theta+1)+\theta+2]}\right]-\left[\frac{12\alpha\,(\theta+1)+2\theta+6}{\theta\,[3\,\alpha\,(\theta+1)+\theta+2]}\right]^2$$

$$\sigma^2 = \frac{60\alpha\,(\theta+1)+6\theta+24}{\theta^3\,[3\,\alpha\,(\theta+1)+\theta+2]} - \frac{[12\alpha\,(\theta+1)+2\theta+6]^2}{\theta^2[3\,\alpha\,(\theta+1)+\theta+2]^2}$$

After simplification, we get

$$var(X) = \sigma^2 = \frac{36\alpha^2\theta^2+72\alpha^2\theta+36\alpha^2+15\alpha\theta^2-3\alpha\theta-18\alpha-20\theta-3\theta^2-32}{\theta^2[3\,\alpha\,(\theta+1)+\theta+2]^2}$$

Standard Deviation

$$S.D(\sigma) = \frac{\sqrt{36\alpha^2\theta^2+72\alpha^2\theta+36\alpha^2+15\alpha\theta^2-3\alpha\theta-18\alpha-20\theta-3\theta^2-32}}{\theta\,[3\,\alpha\,(\theta+1)+\theta+2]}$$

Coefficient of Variation

$$C.V\left(\frac{\sigma}{\mu}\right) = \frac{\sqrt{36\alpha^2\theta^2+72\alpha^2\theta+36\alpha^2+15\alpha\theta^2-3\alpha\theta-18\alpha-20\theta-3\theta^2-32}}{[12\alpha\,(\theta+1)+2\theta+6]}$$

Dispersion

Dispersion= $\frac{\sigma^2}{\mu}$

$$= \frac{36\alpha^2\theta^2+72\alpha^2\theta+36\alpha^2+15\alpha\theta^2-3\alpha\theta-18\alpha-20\theta-3\theta^2-32}{\theta\,[3\,\alpha\,(\theta+1)+\theta+2][12\alpha\,(\theta+1)+2\theta+6]}$$

4.2 Hormonic Mean

The Hormonic mean of the length biased Loai distribution is defined as

$$H.M = E\left[\frac{1}{x}\right]$$

$$H.M = \int_0^\infty \frac{1}{x} f_l(x;\alpha,\theta)\,dx$$

$$H.M = \int_0^\infty \frac{1}{x}\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\,x\left[\frac{1}{2}\,\alpha\theta\,x^2 + \frac{1}{\theta+1}(1+x)\right]e^{-\theta x}dx$$

$$H.M = \int_0^\infty \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{1}{2}\,\alpha\theta\,x^2 + \frac{1}{\theta+1}(1+x)\right]e^{-\theta x}dx$$

$$H.M = \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{1}{2}\,\alpha\theta\int_0^\infty x^2\,e^{-\theta x}dx + \frac{1}{\theta+1}\left(\int_0^\infty e^{-\theta x}dx + \int_0^\infty x\,e^{-\theta x}dx\right)\right]$$

$$H.M = \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{1}{2}\,\alpha\theta\,\frac{2!}{\theta^3} + \frac{1}{\theta+1}\left(\frac{1}{\theta}+\frac{1!}{\theta^2}\right)\right]$$

$$H.M = \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{\alpha}{\theta^2} + \frac{1}{\theta+1}\left(\frac{\theta}{\theta^2}+\frac{1}{\theta^2}\right)\right]$$

$$H.M = \frac{\theta\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\alpha + \frac{(\theta+1)}{\theta+1}\right]$$

$$H.M = \frac{\theta\,(\theta+1)\alpha+1}{3\,\alpha\,(\theta+1)+\theta+2}$$

4.3 Moment Generating function and Characteristic function

Let X_l follows length biased Loai distribution then the moment generating function (mgf) of X is obtained as

$$M_{X_l}(t) = E(e^{tx}) = \int_0^\infty e^{tx}\;f_l(x;\alpha,\theta)\,dx$$

Using Taylor's series

$$M_{X_l}(t) = \int_0^\infty \left[1+tx+\frac{(tx)^2}{2!}+\cdots\right]f_l(x;\alpha,\theta)\,dx$$

$$M_{X_l}(t) = \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!}x^j\;f_l(x;\alpha,\theta)\,dx$$

$$M_{X_l}(t) = \sum_{j=0}^\infty \frac{t^j}{j!}\int_0^\infty x^j\;f_l(x;\alpha,\theta)\,dx$$

$$M_{X_l}(t) = \sum_{j=0}^\infty \frac{t^j}{j!}\;E\left(X_l^j\right)$$

$$M_{X_l}(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{(\theta+1)}{\theta^j\,[3\,\alpha\,(\theta+1)+\theta+2]}\left[\frac{\alpha}{2}\,\Gamma(j+4)+\frac{1}{\theta+1}\big(\theta\Gamma(j+2)+\Gamma(j+3)\big)\right]$$

$$M_{X_l}(t) = \frac{(\theta+1)}{[3\,\alpha\,(\theta+1)+\theta+2]}\sum_{j=0}^{\infty}\frac{t^j}{j!\,\theta^j}\left[\frac{\alpha}{2}\,\Gamma(j+4)+\frac{\big(\theta\Gamma(j+2)+\Gamma(j+3)\big)}{\theta+1}\right]$$

Similarly, we can get the characteristic function of length biased Loai distribution can be obtained as

$$\phi_{X_l}(t) = M_{X_l}(it)$$

$$\phi_{X_l}(t) = \sum_{j=0}^{\infty}\frac{(it)^j}{j!}\;\mu_j'$$

$$\phi_{X_l}(t) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \frac{(\theta+1)}{\theta^j\,[3\,\alpha\,(\theta+1)+\theta+2]}\left[\frac{\alpha}{2}\,\Gamma(j+4)+\frac{1}{\theta+1}\big(\theta\Gamma(j+2)+\Gamma(j+3)\big)\right]$$

$$\phi_{X_l}(t) = \frac{(\theta+1)}{[3\,\alpha\,(\theta+1)+\theta+2]}\sum_{j=0}^{\infty}\frac{(it)^j}{j!\,\theta^j}\left[\frac{\alpha}{2}\,\Gamma(j+4)+\frac{\big(\theta\Gamma(j+2)+\Gamma(j+3)\big)}{\theta+1}\right]$$

5. MEAN DEVIACTION

Let X be a random variable from length biased Loai distribution with mean μ. Then the deviation from mean is defined as

$$D(\mu) = E(|X-\mu|)$$

$$D(\mu) = \int_0^{\infty} |X-\mu|\; f(x)dx$$

$$D(\mu) = \int_0^{\mu} (\mu-x)f(x)dx + \int_{\mu}^{\infty}(x-\mu)f(x)dx$$

$$D(\mu) = \mu\int_0^{\mu} f(x)dx - \int_0^{\mu} x\,f(x)dx + \int_{\mu}^{\infty} x\,f(x)dx - \int_{\mu}^{\infty}\mu\,f(x)dx$$

$$D(\mu) = \mu F(\mu) - \int_0^{\mu} x\,f(x)dx - \mu[1-F(\mu)] + \int_{\mu}^{\infty} x\,f(x)dx$$

$$D(\mu) = 2\mu F(\mu) - 2\int_0^{\mu} x\,f(x)dx$$

Now,

$$\int_0^{\mu} x\,f(x)dx = \int_0^{\mu} x\,\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\,x\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}\,dx$$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\int_0^{\mu}\left[\frac{1}{2}\,\alpha\theta\,x^4+\frac{1}{\theta+1}(x^2+x^3)\right]e^{-\theta x}\,dx$$

Put $\theta x = t,\quad x = \frac{t}{\theta},\quad dx = \frac{1}{\theta}dt$

When $x \to 0, t \to 0$ *and* $x \to x, t \to \theta\mu$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\int_0^{\theta\mu}\left[\frac{1}{2}\alpha\theta\left(\frac{t}{\theta}\right)^4+\frac{1}{\theta+1}\left(\left(\frac{t}{\theta}\right)^2+\left(\frac{t}{\theta}\right)^3\right)\right]e^{-t}\frac{1}{\theta}dt$$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\int_0^{\theta\mu}\left[\frac{\alpha}{2}\frac{t^4}{\theta^3}+\frac{1}{\theta+1}\left(\frac{\theta t^2}{\theta^3}+\frac{t^3}{\theta^3}\right)\right]e^{-t}\frac{1}{\theta}dt$$

$$= \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\int_0^{\theta\mu}\frac{1}{\theta^4}\left[\frac{\alpha}{2}t^4+\frac{1}{\theta+1}(\theta t^2+t^3)\right]e^{-t}dt$$

$$= \frac{(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\int_0^{\theta\mu}\left[\frac{\alpha}{2}t^4+\frac{1}{\theta+1}(\theta t^2+t^3)\right]e^{-t}dt$$

$$= \frac{(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{\alpha}{2}\int_0^{\theta\mu} t^4\,e^{-t}dt+\frac{1}{\theta+1}\left(\theta\int_0^{\theta\mu} t^2\,e^{-t}dt+\int_0^{\theta\mu} t^3\;e^{-t}dt\right)\right]$$

After solving the integral, we get

$$D(\mu) = \frac{2\mu\ (\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta\mu)+\frac{1}{\theta+1}\big(\theta\ \gamma(2,\theta\mu)+\gamma(3,\theta\mu)\big)\right] - \frac{2\ (\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(5,\theta\mu)+\frac{1}{\theta+1}\big(\theta\ \gamma(3,\theta\mu)+\gamma(4,\theta\mu)\big)\right]$$

6. MEAN DIVIATION FROM MEDIAN

Let X be a random variable from length biased Loai distribution with median M. Then the mean deviation from median is defined as

$$D(M) = E(|X-M|)$$

$$D(M) = \int_0^\infty |X-M|\ f(x)dx$$

$$D(M) = \int_0^M (M-x)f(x)dx + \int_M^\infty (x-M)f(x)dx$$

$$D(M) = MF(M) - \int_0^M x\ f(x)dx - M[1-F(M)] + \int_M^\infty x\ f(x)dx$$

$$D(M) = \mu - 2\int_0^M x\ f(x)dx$$

Now,

$$\int_0^M x\ f(x)dx = \int_0^M x\ \frac{\theta^3\ (\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\ x\ \left[\frac{1}{2}\ \alpha\theta\ x^2 + \frac{1}{\theta+1}(1+x)\right]e^{-\theta x}\ dx$$

$$= \frac{\theta^3\ (\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\int_0^M \left[\frac{1}{2}\ \alpha\theta\ x^4 + \frac{1}{\theta+1}(x^2+x^3)\right]e^{-\theta x}\ dx$$

After solving the integral, we get

$$D(M) = \mu - \frac{2\ (\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(5,\theta M)+\frac{1}{\theta+1}\big(\theta\ \gamma(3,\theta M)+\gamma(4,\theta M)\big)\right]$$

7. ORDER STATISTICS

In this section, we derived the distributions of order statistics from the length biased Loai distribution.

Let $X_{(1)}, X_{(2)}, X_{(3)}, \dots, X_{(n)}$ be the order statistics of the random sample $X_1, X_2, X_3, \dots, X_n$ selected from length biased Loai distribution. Then the probability density function of the k^{th} order statistics $X_{(k)}$ is defined as.

$$f_{X_{w(k)}}(x) = \frac{n!}{(r-1)!\ (n-k)!}\ f_X(x)\ [F_X(x)]^{k-1}\ [1-F_X(x)]^{n-k} \tag{9}$$

Inserting equation (5) and (7) in equation (9), the probability density function of k^{th} order statistics $X_{(k)}$ of the length biased Loai distribution is given by

$$f_{X_{(k)}}(x) = \frac{n!}{(n-1)!\ (n-k)!}\left[\frac{(\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta x)+\frac{1}{\theta+1}\big(\theta\ \gamma(2,\theta x)+\gamma(3,\theta x)\big)\right]\right]^{k-1} \times \left[1-\frac{(\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta x)+\frac{1}{\theta+1}\big(\theta\ \gamma(2,\theta x)+\gamma(3,\theta x)\big)\right]\right]^{n-k} \times \frac{\theta^3\ (\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\ x\ \left[\frac{1}{2}\ \alpha\theta\ x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}$$

The distribution of the minimum first order statistics $X_{(1)} = \min(X_1, X_2, X_3, \dots, X_n)$ and the largest order statistics $X_{(n)} = max(X_1, X_2, X_3, \dots, X_n)$ can be computed by replacing k in the previous equation by 1and n, so we get.

$$f_{X_{(1)}}(x) = \frac{n\ \theta^3\ (\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\ x\ \left[\frac{1}{2}\ \alpha\theta\ x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x} \times \left[1-\frac{(\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta x)+\frac{1}{\theta+1}\big(\theta\ \gamma(2,\theta x)+\gamma(3,\theta x)\big)\right]\right]^{n-1}$$

$$f_{X_{(n)}}(x) = \frac{n\ \theta^3\ (\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\ x\ \left[\frac{1}{2}\ \alpha\theta\ x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x} \times \left[\frac{(\theta+1)}{3\ \alpha\ (\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta x)+\frac{1}{\theta+1}\big(\theta\ \gamma(2,\theta x)+\gamma(3,\theta x)\big)\right]\right]^{n-1}$$

Quantile function

The quantile function of a distribution with cumulative distribution function $F_l(x;\alpha,\theta)$ is defined by $q = F_l(x_q;\alpha,\theta)$, where $0 < q < 1$. Thus, the quantile function of length biased Loai distribution is the real solution of the equation.

$$1 - q = 1 - \frac{(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\left[\frac{\alpha}{2}\gamma(4,\theta x_q) + \frac{1}{\theta+1}\left(\theta\,\gamma(2,\theta x_q) + \gamma(3,\theta x_q)\right)\right]$$

8. LIKELIHOOD RATIO TEST

In this section, we derive the likelihood ratio test from the length biased Loai distribution.

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample from the length biased Loai distribution.

To testing the hypothesis, we have the null and alternative hypothesis.

$H_0: f(x) = f(x,\theta) \quad against \quad H_1: f(x) = f_l(x;\alpha,\theta)$

In test whether the random sample of size n comes from the Loai distribution or length biased Loai distribution, the following test statistics is used.

$$\Delta = \frac{L_1}{L_2} = \prod_{i=1}^{n} \frac{f_l(x_i;\alpha,\theta)}{f(x_i,\theta)}$$

$$\Delta = \prod_{i=1}^{n}\left(\frac{\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\,x_i\left[\frac{1}{2}\,\alpha\theta\,x_i^2 + \frac{1}{\theta+1}(1+x_i)\right]e^{-\theta x_i}}{\frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\,\alpha\theta\,x_i^2 + \frac{1}{\theta+1}(1+x_i)\right]e^{-\theta x_i}}\right)$$

$$\Delta = \prod_{i=1}^{n}\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\,x_i\left[\frac{1}{2}\,\alpha\theta\,x_i^2 + \frac{1}{\theta+1}(1+x_i)\right]e^{-\theta x_i} \times \frac{\alpha+1}{\theta^2\left[\frac{1}{2}\alpha\theta\,x_i^2 + \frac{1}{\theta+1}(1+x_i)\right]e^{-\theta x_i}}$$

$$\Delta = \prod_{i=1}^{n}\left(\frac{\theta(\theta+1)\alpha+1}{3\,\alpha\,(\theta+1)+\theta+2}x_i\right)$$

$$\Delta = \frac{L_1}{L_2}\left(\frac{\theta(\theta+1)\alpha+1}{3\,\alpha\,(\theta+1)+\theta+2}\right)^n \prod_{i=1}^{n} x_i$$

We have rejected the null hypothesis if

$$\Delta = \left(\frac{\theta(\theta+1)\alpha+1}{3\,\alpha\,(\theta+1)+\theta+2}\right)^n \prod_{i=1}^{n} x_i > k$$

Equivalently, we also reject null hypothesis, were

$$\Delta^* = \prod_{i=1}^{n} x_i > k\left(\frac{3\,\alpha\,(\theta+1)+\theta+2}{\theta(\theta+1)\alpha+1}\right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i > k^* \; where \; k^* = k\left(\frac{3\,\alpha\,(\theta+1)+\theta+2}{\theta(\theta+1)\alpha+1}\right)^n$$

for large sample size n, 2 $\log\Delta$ is distribution as chi-square variates with one degree of freedom. Thus, we rejected the null hypothesis, when the probability value is given by $p(\Delta^* > \alpha^*)$,

where

$\alpha^* = \prod_{i=1}^{n} x_i$ is less than level of significance and $\prod_{i=1}^{n} x_i$ is the observed value of the statistics Δ^*.

9. BONFERRONI AND LORENZ CURVES AND GINI INDEX

In this section, we have derived the Bonferroni and Lorenz curves and Gini index from the length biased Loai distribution. The Bonferroni and Lorenz curve is a powerful tool in the analysis of distributions and has applications in many fields, such as economies, insurance, income, reliability, and medicine. The Bonferroni and Lorenz cures for a X be the random variable of a unit and $f(x)$ be the probability density function of x. $f(x)dx$ will be represented by the probability that a unit selected at random is defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x\,f(x)dx$$

$$B(p) = \frac{1}{p\mu}\int_0^q x\,f_l(x;\alpha,\theta)dx$$

And

$$L(p) = \frac{1}{\mu}\int_0^q x\,f_l(x;\alpha,\theta)dx$$

Where, $q = F^{-1}(p); \quad q\epsilon[0,1]$

and $\mu = E(X)$

Hence the Bonferroni and Lorenz corves of our distribution are, given by

$$\mu = \frac{12\alpha\,(\theta+1) + 2\theta + 6}{\theta\,[3\,\alpha\,(\theta+1) + \theta + 2]}$$

$$B(p) = \frac{\theta\,[3\,\alpha\,(\theta+1)+\theta+2]}{p[12\alpha\,(\theta+1)+2\theta+6]} \times \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2} \int_0^q x^2 \left[\frac{1}{2}\,\alpha\theta\,x^2 + \frac{1}{\theta+1}(1+x)\right] e^{-\theta x}\,dx$$

$$B(p) = \frac{\theta^4\,(\theta+1)}{p[12\alpha\,(\theta+1)+2\theta+6]} \int_0^q \left[\frac{1}{2}\,\alpha\theta\,x^4 + \frac{1}{\theta+1}(x^2+x^3)\right] e^{-\theta x}\,dx$$

Put $\theta x = t, \quad x = \frac{t}{\theta}, \quad dx = \frac{1}{\theta}dt$

When $x \to 0, t \to 0$ and $x \to x, t \to \theta q$

$$B(p) = \frac{\theta^4\,(\theta+1)}{p[12\alpha\,(\theta+1)+2\theta+6]} \int_0^{\theta q} \left[\frac{1}{2}\alpha\theta\left(\frac{t}{\theta}\right)^4 + \frac{1}{\theta+1}\left(\left(\frac{t}{\theta}\right)^2 + \left(\frac{t}{\theta}\right)^3\right)\right] e^{-t}\frac{1}{\theta}dt$$

$$B(p) = \frac{\theta^4\,(\theta+1)}{p[12\alpha\,(\theta+1)+2\theta+6]} \int_0^{\theta q} \left[\frac{\alpha}{2}\frac{t^4}{\theta^3} + \frac{1}{\theta+1}\left(\frac{\theta t^2}{\theta^3} + \frac{t^3}{\theta^3}\right)\right] e^{-t}\frac{1}{\theta}dt$$

$$B(p) = \frac{\theta^4\,(\theta+1)}{p[12\alpha\,(\theta+1)+2\theta+6]} \int_0^{\theta q} \frac{1}{\theta^4}\left[\frac{\alpha}{2}t^4 + \frac{1}{\theta+1}(\theta t^2 + t^3)\right] e^{-t}dt$$

$$B(p) = \frac{(\theta+1)}{p[12\alpha\,(\theta+1)+2\theta+6]} \int_0^{\theta q} \left[\frac{\alpha}{2}t^4 + \frac{1}{\theta+1}(\theta t^2 + t^3)\right] e^{-t}dt$$

$$B(p) = \frac{(\theta+1)}{p[12\alpha\,(\theta+1)+2\theta+6]} \left[\frac{\alpha}{2}\int_0^{\theta q} t^4\,e^{-t}dt + \frac{1}{\theta+1}\left(\theta\int_0^{\theta q} t^2\,e^{-t}dt + \int_0^{\theta q} t^3\,e^{-t}dt\right)\right]$$

After the simplification, we get

$$B(p) = \frac{(\theta+1)}{p[12\alpha\,(\theta+1)+2\theta+6]} \left[\frac{\alpha}{2}\gamma(5,\theta q) + \frac{1}{\theta+1}\big(\theta\gamma(3,\theta q) + \gamma(4,\theta q)\big)\right]$$

Where,

$$L(p) = pB(p)$$

$$L(p) = \frac{p\,(\theta+1)}{p[12\alpha\,(\theta+1)+2\theta+6]} \left[\frac{\alpha}{2}\gamma(5,\theta q) + \frac{1}{\theta+1}\big(\theta\gamma(3,\theta q) + \gamma(4,\theta q)\big)\right]$$

$$L(p) = \frac{(\theta+1)}{[12\alpha\,(\theta+1)+2\theta+6]} \left[\frac{\alpha}{2}\gamma(5,\theta q) + \frac{1}{\theta+1}\big(\theta\gamma(3,\theta q) + \gamma(4,\theta q)\big)\right]$$

Gini Index

The information in the Lorenz Curve is often summarized in a single measure called the Gini index (proposed in a 1912 paper by the Italian statistician Corrado Gini. It is often used as a gauge of economic inequality, measuring income distribution. The Gini index is defined as

Therefore, the Gini index is for length biased Loai distribution

$$G = 1 - 2\int_0^1 L(p)dp$$

$$= 1 - 2\int_0^1 \frac{(\theta+1)}{[12\alpha\,(\theta+1)+2\theta+6]} \left[\frac{\alpha}{2}\gamma(5,\theta q) + \frac{1}{\theta+1}\big(\theta\gamma(3,\theta q) + \gamma(4,\theta q)\big)\right] dp$$

$$= 1 - 2\frac{(\theta+1)}{[12\alpha\,(\theta+1)+2\theta+6]} \left[\frac{\alpha}{2}\gamma(5,\theta q) + \frac{1}{\theta+1}\big(\theta\gamma(3,\theta q) + \gamma(4,\theta q)\big)\right]$$

10. STOCHASTIC ORDERING

Stochastic ordering is an important tool in finance and reliability to assess the comparative performance of the models. Let X and Y be two random variables with pdf, cdf, and reliability functions $f(x), f(y), F(x), F(y). S(x) = 1 - F(x) and\ F(y)$.

1- Likelihood ratio order $(X \leq_{LR} Y)$ if $\frac{f_{X_l}(x)}{f_{Y_l}(x)}$ decreases in x
2- Stochastic order $(X \leq_{ST} Y)$ if $F_{X_l}(x) \geq F_{Y_l}(x)$ for all x
3- Hazard rate order $(X \leq_{HR} Y)$ if $h_{X_l}(x) \geq h_{Y_l}(x)$ for all x
4- Mean residual life order $(X \leq_{MRL} Y)$ if $MRL_{X_l}(X) \leq MRL_{Y_w}(X)$ for all x

Show that length biased Loai distribution satisfies the strongest ordering (likelihood ratio ordering)

Assume that X and Y are two independent Random variables with probability distribution function $f_{l_x}(x;\alpha,\theta)$ and $f_{l_y}(x;\beta,\lambda)$. If $\alpha<\beta$ and $\theta<\lambda$, then

$$\Lambda=\frac{f_{l_x}(x;\alpha,\theta)}{f_{l_y}(x;\beta,\lambda)}$$

$$=\frac{\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\;x\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}}{\frac{\lambda^3\,(\lambda+1)}{3\,\beta\,(\lambda+1)+\lambda+2}\;x\left[\frac{1}{2}\,\beta\lambda\,x^2+\frac{1}{\lambda+1}(1+x)\right]e^{-\lambda x}}$$

$$=\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\;x\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}\times\frac{3\,\beta\,(\lambda+1)+\lambda+2}{\lambda^3\,(\lambda+1)x\left[\frac{1}{2}\,\beta\lambda\,x^2+\frac{1}{\lambda+1}(1+x)\right]}\,e^{\lambda x}$$

$$=\left[\frac{\theta^3\,(\theta+1)3\,\beta\,(\lambda+1)+\lambda+2}{\lambda^3\,(\lambda+1)3\,\alpha\,(\theta+1)+\theta+2}\right]\times\frac{\left[\frac{1}{2}\,\alpha\theta\,x^3+\frac{(x+x^2)}{\theta+1}\right]}{\left[\frac{1}{2}\,\beta\lambda\,x^3+\frac{(x+x^2)}{\lambda+1}\right]}e^{-(\theta-\lambda)x}$$

Therefore,

$$\log[\Lambda]=\log\left[\frac{\theta^3\,(\theta+1)3\,\beta\,(\lambda+1)+\lambda+2}{\lambda^3\,(\lambda+1)3\,\alpha\,(\theta+1)+\theta+2}\right]+\log\left[\frac{1}{2}\,\alpha\theta\,x^3+\frac{(x+x^2)}{\theta+1}\right]-\log\left[\frac{1}{2}\,\beta\lambda\,x^3+\frac{(x+x^2)}{\lambda+1}\right]-(\theta-\lambda)x$$

Differentiating with respect to x, we get.

$$\frac{\partial\log[\Lambda]}{\partial x}=\left[\frac{6}{\alpha\theta x}+\frac{(1+2x)(\theta+1)}{x(1+x)}\right]-\left[\frac{6}{\beta\lambda x}+\frac{(1+2x)(\lambda+1)}{x(1+x)}\right]+(\lambda-\theta)$$

Hence, $\frac{\partial\log[\Lambda]}{\partial x}<0$ if $\alpha<\beta,\theta<\lambda$.

11. ENTROPIES

In this section, we derived the Shannon entropy, Rényi entropy, and Tsallis entropy from the length biased Loai distribution.

It is well known that entropy and information can be considered measures of uncertainty or the randomness of a probability distribution. It is applied in many fields, such as engineering, finance, information theory, and biomedicine. The entropy functionals for probability distribution were derived on the basis of a variational definition of uncertainty measure.

9.1 Shannon Entropy

Shannon entropy of the random variable X such that length biased Loai distribution is defined as

$$S_\lambda=-\int_0^\infty f(x)\log\big(f(x)\big)dx \qquad ;\lambda>0,\lambda\neq 1$$

$$S_\lambda=-\int_0^\infty f_l(x;\alpha,\theta)\log\big(f_l(x;\alpha,\theta)\big)dx$$

$$S_\lambda=-\int_0^\infty \frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\;x\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}\times\log\left(\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\;x\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}\right)dx$$

9.2 Rényi Entropy

Entropy is defined as a random variable X is a measure of the variation of the uncertainty. It is used in many fields, such as engineering, statistical mechanics, finance, information theory, biomedicine, and economics. The entropy measure is the Rényi of order which is defined as

$$R_\lambda=\frac{1}{1-\lambda}\log\int_0^\infty [f(x)]^\lambda\;dx \qquad ;\lambda>0,\lambda\neq 1$$

$$R_\lambda=\frac{1}{1-\lambda}\log\int_0^\infty [f_l(x;\alpha,\theta)]^\lambda\,dx$$

$$R_\lambda=\frac{1}{1-\lambda}\log\int_0^\infty\left[\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\;x\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}\right]^\lambda dx$$

$$R_\lambda=\frac{1}{1-\lambda}\log\left[\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\right]^\lambda\int_0^\infty x^\lambda\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]^\lambda e^{-\lambda\theta x}\;dx \qquad (10)$$

Using Binomial expansion, we get

$$\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]^\lambda=\left[\frac{\alpha\theta}{2}\;x^2+\frac{1}{\theta+1}(1+x)\right]^\lambda$$

$$\left[\frac{\alpha\theta}{2}\ x^2+\frac{1+x}{\theta+1}\right]^\lambda=\sum_{k=0}^{\lambda}\binom{\lambda}{k}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}x^{2(\lambda-k)}\left(\frac{1}{\theta+1}\right)^k(1+x)^k$$

$$\left[\frac{\alpha\theta}{2}\ x^2+\frac{1+x}{\theta+1}\right]^\lambda=\sum_{k=0}^{\lambda}\binom{\lambda}{k}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k x^{2(\lambda-k)}(1+x)^k$$

$$(1+x)^k=\sum_{l=0}^{k}\binom{k}{l}1^{k-l}\ x^l$$

$$(1+x)^k=\sum_{k=0}^{\lambda}\binom{\lambda}{k}\sum_{l=0}^{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k x^{2(\lambda-k)}\,x^l$$

$$(1+x)^k=\sum_{k=0}^{\lambda}\sum_{l=0}^{k}\binom{\lambda}{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k x^{2(\lambda-k)}\,x^l \qquad (11)$$

Substituting equation (11) in (10) we get,

$$R_\lambda=\frac{1}{1-\lambda}\log\left[\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\right]^\lambda\int_0^\infty\sum_{k=0}^{\lambda}\sum_{l=0}^{k}\binom{\lambda}{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k x^\lambda\,x^{2(\lambda-k)}\,x^l\,e^{-\lambda\theta x}\,dx$$

$$R_\lambda=\frac{1}{1-\lambda}\log\left[\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\right]^\lambda\sum_{k=0}^{\lambda}\sum_{l=0}^{k}\binom{\lambda}{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k\int_0^\infty x^{\lambda+2(\lambda-k)+l}\,e^{-\lambda\theta x}\,dx$$

$$R_\lambda=\frac{1}{1-\lambda}\log\left[\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\right]^\lambda\sum_{k=0}^{\lambda}\sum_{l=0}^{k}\binom{\lambda}{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k\int_0^\infty x^{(\lambda+2(\lambda-k)+l+1)-1}\,e^{-\lambda\theta x}\,dx$$

$$R_\lambda=\frac{1}{1-\lambda}\log\left[\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\right]^\lambda\sum_{k=0}^{\lambda}\sum_{l=0}^{k}\binom{\lambda}{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k\frac{\Gamma(\lambda+2(\lambda-k)+l+1)}{(\lambda\theta)^{\lambda+2(\lambda-k)+k+1}}$$

$$R_\lambda=\frac{1}{1-\lambda}\log\left[\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\right]^\lambda\sum_{k=0}^{\lambda}\sum_{l=0}^{k}\binom{\lambda}{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k\left(\frac{1}{\lambda\theta}\right)^{\lambda+2(\lambda-k)+l+1}\times\Gamma(\lambda+2(\lambda-k)+l+1)$$

9.3 Tsallis Entropy

The Boltzmann-Gibbs (B-G) statistical properties initiated by Tsallis have received a great deal of attention. This generalization of (B-G) statistics was first proposed by introducing the mathematical expression of Tsallis entropy (Tsallis, (1988) for continuous random variables, which is defined as

$$T_\lambda=\frac{1}{\lambda-1}\left[1-\int_0^\infty[f(x)]^\lambda\,dx\right]\qquad ;\lambda>0,\lambda\neq1$$

$$T_\lambda=\frac{1}{\lambda-1}\left[1-\int_0^\infty[f_l(x;\alpha,\theta)]^\lambda\,dx\right]$$

$$T_\lambda=\frac{1}{\lambda-1}\left[1-\int_0^\infty\left(\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\,x\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}\right)^\lambda dx\right]$$

$$T_\lambda=\frac{1}{\lambda-1}\left[1-\left(\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\right)^\lambda\int_0^\infty x^\lambda\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\lambda\theta x}\,dx\right]\qquad (12)$$

Using Binomial expansion, we get

$$\left[\frac{1}{2}\,\alpha\theta\,x^2+\frac{1}{\theta+1}(1+x)\right]^\lambda=\left[\frac{\alpha\theta}{2}\ x^2+\frac{1+x}{\theta+1}\right]^\lambda$$

$$=\sum_{k=0}^{\lambda}\sum_{l=0}^{k}\binom{\lambda}{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k x^{2(\lambda-k)}\,x^l \qquad (13)$$

Substituting equation (13) in (12) we get,

$$T_\lambda=\frac{1}{\lambda-1}\left[1-\left(\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\right)^\lambda\int_0^\infty\sum_{k=0}^{\lambda}\sum_{l=0}^{k}\binom{\lambda}{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k x^\lambda\,x^{2(\lambda-k)}\,x^l\,e^{-\lambda\theta x}\,dx\right]$$

$$T_\lambda=\frac{1}{\lambda-1}\left[1-\left(\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\right)^\lambda\sum_{k=0}^{\lambda}\sum_{l=0}^{k}\binom{\lambda}{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k\int_0^\infty x^{(\lambda+2(\lambda-k)+l+1)-1}\,e^{-\lambda\theta x}\,dx\right]$$

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\left(\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\right)^\lambda \sum_{k=0}^{\lambda}\sum_{l=0}^{k}\binom{\lambda}{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k \frac{\Gamma(\lambda+2(\lambda-k)+l+1)}{(\lambda\theta)^{\lambda+2(\lambda-k)+k+1}}\right]$$

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\left(\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\right)^\lambda \sum_{k=0}^{\lambda}\sum_{l=0}^{k}\binom{\lambda}{k}\binom{k}{l}\left(\frac{\alpha\theta}{2}\right)^{\lambda-k}\left(\frac{1}{\theta+1}\right)^k \left(\frac{1}{\lambda\theta}\right)^{\lambda+2(\lambda-k)+l+1} \times\Gamma(\lambda+2(\lambda-k)+l+1)\right]$$

12. ESTIMATIONS OF PARAMETER

In this section, the maximum likelihood estimates and Fisher's information matrix of the length biased Loai distribution parameter is given.

12.1 Maximum Likelihood estimation (MLE)and Fisher's Information Matrix

Consider $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n from the length biased Loai distribution with parameter α, θ the likelihood function, which is defined as

$$L = (x;\alpha,\theta) = \prod_{i=1}^{n} f_l(x_i;\alpha,\theta)$$

$$L = \prod_{i=1}^{n}\left[\frac{\theta^3\,(\theta+1)}{3\,\alpha\,(\theta+1)+\theta+2}\, x_i\left[\frac{1}{2}\,\alpha\theta\, x_i^2 + \frac{1}{\theta+1}(1+x_i)\right]e^{-\theta x_i}\right]$$

Then, the log-likelihood function is

$$\ell = \log L = n\log[\theta^3\,(\theta+1)] - n\log[3\,\alpha\,(\theta+1)+\theta+2] + \sum_{i=1}^{n}\log\left[x_i\left[\frac{1}{2}\,\alpha\theta\, x_i^2+\frac{1}{\theta+1}(1+x_i)\right]e^{-\theta x_i}\right]$$

$$\ell = n\log[\theta^3\,(\theta+1)] - n\log[3\,\alpha\,(\theta+1)+\theta+2] + \sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\left[\frac{1}{2}\,\alpha\theta\, x_i^2+\frac{1}{\theta+1}(1+x_i)\right] + \log e^{-\theta\sum_{i=1}^{n}x_i} \qquad (14)$$

Deriving (14) partially with respect to α and θ we have.

$$\frac{\partial \log L}{\partial\alpha} = -n\left[\frac{3\theta+3}{3\,\alpha\,(\theta+1)+\theta+2}\right] + \sum_{i=1}^{n}\left[\frac{\frac{\theta x_i^2}{2}}{\frac{1}{2}\,\alpha\theta\, x_i^2}\right]$$

$$\frac{\partial \log L}{\partial\alpha} = -n\left[\frac{3\theta+3}{3\,\alpha\,(\theta+1)+\theta+2}\right] + \left(\frac{1}{\alpha}\right) = 0 \qquad (15)$$

$$\frac{\partial \log L}{\partial\theta} = n\left[\frac{4\theta^3+3\theta^2}{\theta^3\,(\theta+1)}\right] - n\left[\frac{3\alpha+3}{3\,\alpha\,(\theta+1)+\theta+2}\right] + \sum_{i=1}^{n}\left[\frac{\frac{\alpha x_i^2}{2}}{\frac{1}{2}\alpha\theta\, x_i^2}\right] + \sum_{i=1}^{n}\left[\frac{-\frac{1+x_i}{(\theta+1)^2}}{\frac{1}{\theta+1}(1+x_i)}\right] - \sum_{i=1}^{n}x_i$$

$$\frac{\partial \log L}{\partial\theta} = n\left[\frac{4\theta^3+3\theta^2}{\theta^3\,(\theta+1)}\right] - n\left[\frac{3\alpha+3}{3\,\alpha\,(\theta+1)+\theta+2}\right] + \left(\frac{1}{\theta}\right) - \left(\frac{1}{\theta+1}\right) - \sum_{i=1}^{n}x_i = 0 \qquad (16)$$

The equation (15) and (16) gives the maximum likelihood estimation of the parameters for the length biased Loai distribution. However, the equation cannot be solved analytically, thus we solved numerically using R programming with data set.

To obtain confidence interval we use the asymptotic normality results. We have that if $\hat{\lambda} = (\hat{\theta}, \hat{\alpha})$ denotes the MLE of $\lambda = (\theta, \alpha)$ we can state the results as follows:

$$\sqrt{n}(\hat{\lambda}-\lambda) \to N_2(0, I^{-1}(\lambda))$$

Where $I(\lambda)$ is Fisher's Information Matrix? i.e.,

$$I(\lambda) = -\frac{1}{n}\begin{bmatrix} E\left[\frac{\partial^2 \log L}{\partial\theta^2}\right] & E\left[\frac{\partial^2 \log L}{\partial\theta\partial\alpha}\right] \\ E\left[\frac{\partial^2 \log L}{\partial\alpha\partial\theta}\right] & E\left[\frac{\partial^2 \log L}{\partial\alpha^2}\right]\end{bmatrix}$$

Where,

$$\left[\frac{\partial^2 \log L}{\partial\theta^2}\right] = E\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \log L}{\partial\theta}\right)\right]$$

$$\left[\frac{\partial^2 \log L}{\partial\theta^2}\right] = -n\left(\frac{4\theta^2+6\theta+3}{\theta^2(\theta+1)^2}\right) + n\left(\frac{9\alpha^2+12\alpha+3}{(3\alpha\theta+3\alpha+\theta+2)^2}\right) - \frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}$$

$$\left[\frac{\partial^2 \log L}{\partial \alpha^2}\right] = E\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial \log L}{\partial \alpha}\right)\right]$$

$$\left[\frac{\partial^2 \log L}{\partial \alpha^2}\right] = n\left(\frac{9\theta^2 + 12\theta + 3}{(3\alpha\theta + 3\alpha + \theta + 2)^2}\right) - \frac{1}{\alpha^2}$$

$$E\left[\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right] = -n\left(\frac{3}{(3\alpha\theta + 3\alpha + \theta + 2)^2}\right)$$

$$E\left[\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right] = -n\left(\frac{3}{(3\alpha\theta + 3\alpha + \theta + 2)^2}\right)$$

Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence interval for θ, α.

13. APPLICATIONS

Data set 1: This data consists of the life time (in years) of 40-blood cancer (leukemia) patients from one of ministry of health hospitals in Sdudhi Arabia reported in [01]. This actual data is

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162
2.211	2.370	2.532	2.693	2.805	2.910	2.912	3.192	3.263	3.348
3.427	3.499	3.534	3.767	3.751	3.858	3.986	4.049	4.244	4.323
4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074	5.381	

Data set 2: The data under consideration are the life times of 20 leukemia patients who were treated by a certain drug [15]. The data are

1.013	1.034	1.109	1.226	1.509	1.533	1.563	1.716	1.929	1.965	2.061	2.344	2.546
2.626	2.778	2.951	3.413	4.118	5.136							

Data set 3: [20] Consider a simulated data represents the survival times (in days) of 73 patients who diagnosed with acute bone cancer, as follows

0.09	0.76	1.81	1.10	0.72	2.49	1.00	0.53	31.61	0.60
0.20	1.61	1.88	0.70	1.36	0.43	3.16	1.57	4.93	11.07
1.63	1.39	4.54	3.12	76.01	1.92	0.92	4.04	1.16	2.26
0.20	0.94	1.82	3.99	1.46	2.75	1.38	2.76	1.86	2.68
1.76	0.67	1.29	1.56	2.83	0.71	1.48	2.41	0.66	0.65
2.36	1.29	13.75	0.67	3.70	0.76	3.63	0.68	2.65	0.95
2.30	2.57	0.61	1.56	1.29	9.94	1.67	1.42	4.18	1.37

Data set 4: The data set is reported by [05] and, which corresponds to the survival times (in years) of a group of patients group of patients given by chemotherapy treatment alone.

0.047	0.115	0.121	0.132	0.164	0.197	0.203	0.260	0.282	0.296
0.334	0.395	0.458	0.466	0.501	0.507	0.529	0.534	0.534	0.540
0.570	0.641	0.644	0.696	0.841	0.863	1.099	1.219	1.271	1.326
1.447	1.485	1.553	1.581	1.581	1.589	2.178	2.343	2.461	2.444
2.825	2.830	3.578	3.658	3.743	3.978	4.003	4.033		

To compare to the goodness of fit of the fitted distribution, the following criteria: Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Akaike Information Criteria Corrected (AICC) and $-2\log L$.

AIC, BIC, AICC and $-2\log L$ can be evaluated by using the formula as follows.

$$AIC = 2k - 2\log L, \qquad BIC = k\log n - 2\log L \ and \ AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$$

Where, $k =$ number of parameters, n sample size and $-2\log L$ is the maximized value of loglikelihood function.

Table 1: MLEs AIC, BIC, AICC, and -2log L of the fitted distribution for the given data set 1

Distribution	ML Estimates	-2log L	AIC	BIC	AICC
Length biased Loai distribution	$\hat{\alpha} = \mathbf{0.0010000}$ (NaN) $\hat{\theta} = \mathbf{0.93825814}(\mathbf{0.0874753})$	**74.37574**	**78.37574**	**81.70287**	**78.7000**
Loai distribution	$\hat{\alpha} = 0.0010000$ (NaN) $\hat{\theta} = 0.5320351$ (0.0612795)	90.90448	94.90448	98.2316	95.2288
Aradhana	$\hat{\theta} = 0.75060122$ (0.07108124)	149.4283	151.4283	153.0918	151.5335
Ishita	$\hat{\theta} = 0.80668240$ (0.06521656)	147.9967	149.9967	151.6603	150.1019
Akshaya	$\hat{\theta} = 0.98152890$ (0.07981806)	144.7945	146.7945	148.458	146.8997
Shanker	$\hat{\theta} = 0.54972161$ (0.05806214)	144.7945	155.9545	157.6181	156.0597
Rama	$\hat{\theta} = 1.10146523$ (0.08055189)	143.3158	145.3158	147.1023	145.4210
Exponential	$\hat{\theta} = 0.31893857$ (0.05107054)	167.1353	169.1353	170.7988	169.0405
Lindley	$\hat{\theta} = 0.2577071$ (0.06161721)	156.5028	158.5028	160.1664	158.6080
Akash	$\hat{\theta} = 0.80168363$ (0.07120997)	149.0561	151.0561	152.7196	151.1613

Table 2: MLEs AIC, BIC, AICC, and -2log L of the fitted distribution for the given data set 2

Distribution	ML Estimates	-2log L	AIC	BIC	AICC
Length biased Loai distribution	$\hat{\alpha} = \mathbf{0.0010000}$ (NaN) $\hat{\theta} = \mathbf{1.3262294}$ $(\mathbf{0.1775588})$	**16.20981**	**20.20981**	**22.09868**	**20.9598**
Loai distribution	$\hat{\alpha} = 0.0010000$ (NaN) $\hat{\theta} = 0.7574830$ (0.1254249)	28.89377	32.89377	34.78265	33.6437
Aradhana	$\hat{\theta} = 0.985545$ (0.135948)	60.60053	62.60053	63.54497	62.8227
Ishita	$\hat{\theta} = 0.9975990$ (0.1134076)	62.74297	64.74297	65.68741	64.9651
Akshaya	$\hat{\theta} = 1.2738546$ (0.1506672)	58.05546	60.05546	60.9999	60.2776
Shanker	$\hat{\theta} = 0.7124395$ (0.10777871)	63.08856	65.08856	66.033	65.3107
Rama	$\hat{\theta} = 1.3784229$ (0.1415338)	62.41991	64.41991	65.36435	64.6421
Exponential	$\hat{\theta} = 0.4463246$ (0.1023934)	68.65501	70.65501	71.59945	70.8772
Lindley	$\hat{\theta} = 0.7076860$ (0.1200725)	64.02158	66.02158	66.96602	66.2438
Akash	$\hat{\theta} = 0.0297001$ (0.1317933)	62.69158	64.69158	65.63602	64.9138

Table 3: MLEs AIC, BIC, AICC, and -2log L of the fitted distribution for the given data set 3

Distribution	ML Estimates	-2log L	AIC	BIC	AICC
Length biased Loai distribution	$\hat{\alpha} = \mathbf{0.001000(0.03789741)}$ $\hat{\theta} = \mathbf{0.78486207\ (0.057891)}$	**293.1362**	**297.1362**	**301.7172**	**297.3076**
Loai distribution	$\hat{\alpha} = 0.0010000$ (NaN) $\hat{\theta} = 0.18180524\ (0.01575308)$	300.1188	304.1188	308.4366	304.2902
Aradhana	$\hat{\theta} = 0.66657919\ (0.04589789)$	405.5844	407.5844	409.8748	407.6407
Ishita	$\hat{\theta} = 0.7626730\ \ (0.0449009)$	425.5164	427.5164	429.8069	427.5727
Akshaya	$\hat{\theta} = 0.87506525\ (0.05179087)$	450.401	452.401	454.6915	452.4573
Shanker	$\hat{\theta} = 0.51244299\ (0.03919024)$	373.2109	375.2109	377.5014	375.2672
Rama	$\hat{\theta} = 0.99007435\ (0.05367476)$	483.9594	485.9594	488.2499	486.0157
Exponential	$\hat{\theta} = 0.27638027\ (0.03234744)$	333.7534	335.7534	338.0438	335.8097
Lindley	$\hat{\theta} = 0.46503064\ (0.03949368)$	365.8631	367.8631	370.1536	367.9294
Akash	$\hat{\theta} = 0.71628684\ (0.04656348)$	419.7666	421.7666	424.051	421.8230

Table 4: MLEs AIC, BIC, AICC, and -2log L of the fitted distribution for the given data set 4

Distribution	ML Estimates	-2log L	AIC	BIC	AICC
Length biased Loai distribution	$\hat{\alpha} = \mathbf{0.0010000}$ (**NaN**) $\hat{\theta} = \mathbf{2.2639683\ (0.189936)}$	**8.550961**	**12.55096**	**16.29336**	**12.8300**
Loai distribution	$\hat{\alpha} = 0.0010000$ (NaN) $\hat{\theta} = 1.3077943\ \ (0.1291839)$	12.04766	16.04766	19.79006	16.3267
Aradhana	$\hat{\theta} = 1.4963630\ \ (0.1351993)$	124.5748	126.5748	128.446	126.6657
Ishita	$\hat{\theta} = 1.4066951\ \ (0.1027368)$	122.1315	124.1315	126.0027	124.2224
Akshaya	$\hat{\theta} = 1.8857589\ \ (0.1463088)$	127.6693	129.6693	131.5405	129.7602
Shanker	$\hat{\theta} = 1.1087323\ \ (0.1066882)$	122.5482	124.5482	126.4194	124.6392
Rama	$\hat{\theta} = 1.8610360\ \ (0.1239909$	123.0723	125.0723	126.9435	125.1632
Exponential	$\hat{\theta} = 0.7607586\ \ (0.1098058)$	122.2503	124.2505	126.1215	124.3415
Lindley	$\hat{\theta} = 1.119663\ \ \ (0.123191)$	122.6538	124.6538	126.525	124.7457
Akash	$\hat{\theta} = 1.484596\ \ \ (0.124779)$	122.7795	124.7795	126.6507	124.8705

From table 1, 2, 3, and 4 it can be clearly observed and seen from the results that the length biased Loai distribution have the lesser AIC, BIC, AICC, -2log L, and values as compared to the Loai, Aradhana, Ishita, Akshaya, Shanker, Rama, Exponential, Lindley, Akash distributions, which indicates that the length biased Loai distribution better fits than the Loai, Aradhana, Ishita, Akshaya, Shanker, Rama, Exponential, Lindley, Akash distributions. Hence, it can be concluded that the length biased Loai distribution leads to a better fit over the other distributions.

14. CONCLUSIONS

Research has focused a great deal of attention on selecting an appropriate model for fitting survival data. In this paper, the Loai distribution is extended to provide a new distribution called the length biased Loai distribution for the model's lifetime data. It has various special cases that have been presented in the paper. The statistical properties of the distribution have been studied, including survival and hazard functions, moments, mean and, median deviations, moment generating functions, Entropies, Bonferroni and Lorenz curve, and order statistics. The Inference of parameters for a length biased Loai distribution was obtained using the method of maximum likelihood estimates. When the parameters have been estimated using the maximum likelihood method, a good performance is seen. The application of statistical distributions is critical for medical research and can significantly affect public health, especially for cancer patients. Thus, the usefulness of this distribution is illustrated through its applications to the survival of some cancer patients, including both complete and censored cases. In using various goodness-of-fit criteria, including AIC, BIC, AICC, and -2logL, the results demonstrate the superior performance of the length biased Loai distribution. Overall, it is intended that the length-biased distribution that is given in tables 1, 2, 3, and 4 will offer a better fit than other existing distributions for simulating real data in survival analysis, specifically for cancer data.

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