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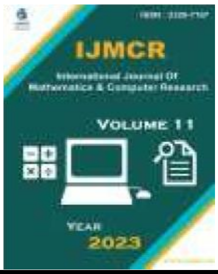

A Family of 82q Exponential Hybrid Methods for Solving IVPs in Odes

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ARTICLE INFO	ABSTRACT
Published Online: 19 December 2023 Corresponding Author: **Alex M.**	In a recent paper, a new class of exponential general linear methods of second stage seventh order for the numerical solution of first order initial value problems in ordinary deferential equations was derived. In this paper, we derive the new class of higher order extended exponential general linear methods for the numerical solution of first order initial value problems in ordinary deferential equations. The numerical results obtained by the new method for some problems show its superiority in efficiency and accuracy for solving problems for which the proposed general linear method is appropriate.

KEYWORDS: General Linear Methods Extended Exponential General Linear Methods, Order Conditions

1 INTRODUCTION

In this paper, a family of two stage exponential general linear methods for solving IVPs in ordinary deferential equation (ODE) of the form

$$u'(t) - Wu(t) = N(u(t)),\ 0 \le t \le T, \text{given} \quad u(0).$$

(1)

is discussed .

General linear methods are multivalve-multistage methods in which the input to a step $u^{[n-1]}$ and the output from the step $u^{[n]}$ are related to the stage values U and the stage derivatives $F = f(U)$ by the equations

$$\begin{bmatrix} U \\ u^{[n]} \end{bmatrix} = \begin{bmatrix} A & M \\ B & D \end{bmatrix} \begin{bmatrix} hF \\ u^{[n-1]} \end{bmatrix}$$

We denote the internal values of step n by $u_1^{[n]}, u_2^{[n]}, u_3^{[n]}, \ldots, u_s^{[n]}$ .where we define s as the number of internal stages and n as the step number and the derivatives evaluated at the steps by $f(u_1^{[n]}), f(u_2^{[n]}), f(u_3^{[n]}), \ldots, f(u_s^{[n]})$ as the start of step number $n; r$ quantities denoted by $u_1^{[n-1]}, u_2^{[n-1]}, u_3^{[n-1]}, \ldots, u_r^{[n]-1}$

Different types of General linear methods for the numerical solution of (1) have been proposed in the past. See the following references [1, 2, 3, 4, 5]. Many of these methods require starting values.

The development of numerical integrators for this class of problems (1) has attracted considerable interest and it is this interest that inspires this research.

2 MATHEMATICAL FORMULATION

The aim of this paper is to develop a new approach to the development and numerical investigations of the Exponential General Linear Methods for solving problems (1). The practical General Linear Methods was constructed by [2] with a considerable advantages over [4]. However, the extension of the internal stages to the second level was carried out by [5]. This extension enables for the derivation of methods of higher order. However, this present study is concerned with the construction of a step two order eight via a new extension. This extension has not been seen anywhere in literature.

The theoretical approximation u_{n+1} at time t_{n+1} , $n \le q-1,$ is given by the recurrence relation or formula

$$u_{n+1} = e^{hW} y_n + h\sum_{i=1}^{s} B_i(hW)N(U_{ni}) + h\sum_{k=1}^{q-1} D_k(hW)N(u_{n-k}) \qquad (2)$$

The internal stages U_{ni} ,$1 \leq i \leq s$, are defined through

$$U_{ni} = e^{c_i hW} u_n + h\sum_{j=1}^{i-1} A_{ij}(hW)N(U_{nj}) + h\sum_{k=1}^{q-1} M_{ik}(hW)N'(u_{n-k}) \qquad (3)$$

Our interest is to extend (2) and (3) by a higher exponential and its related matrix functions. The extended methods becomes

$$u_{n+1} = e^{hW} u_n + h\sum_{i=1}^{s} B_i(hW)N(U_{ni}) + h\sum_{k=1}^{q-1} D^{(1)}{}_k(hW)N(u_{n-k}) +$$

$$h^2\sum_{k=1}^{q-1} D^{(2)}{}_k(hW)N'(u_{n-k}) \qquad (4)$$

And the extended internal stages U_{ni} ,$1 \leq i \leq s$ are defined through

$$U_{ni} = e^{c_i hW} u_n + h\sum_{j=1}^{i-1} A_{ij}(hW)N(U_{nj}) + h\sum_{k=1}^{q-1} M_{ik}^{(1)}(hW)N'(u_{n-k})$$

$$+ h^2\sum_{k=1}^{q-1} M_{ik}^{(2)}(hW)N'(u_{n-k}) \qquad (5)$$

We assume in this paper that these conditions $M_{ik}(hW) = 0$ which implies $c_1 = 0$ and thus $u_{n1} = u_n$ are satisfied. The coefficients can be represented in a tablau as seen above

Before constructing methods arising from this method class, we derive the order conditions.

The order conditions are very crucial in the construction of different order in general linear Methods. In the section that follows, we introduce the order conditions of the propose scheme and thereafter show the construction of our two -stage eight order Exponential general linear Methods.

3 THE ORDER CONDITIONS FOR THE PROPOSED SCHEME.

To achieve the derivation of the order conditions for the method (2), we require that the nonlinearity evaluated at the exact solution $f(t) = N(u(t))$ is sufficiently often differentiable with respect to t , for $0 < t < T$

$$u_{n+1} = e^{hW} u_n + h\sum_{i=1}^{s} B_i(hW)f(t_{ni}) + h\sum_{k=1}^{q-1} D^{(1)}{}_k(hW)f(t_n - kh) +$$

$$h^2\sum_{k=1}^{q-1} D^{(2)}{}_k(hW)f'(t_n - kh) \qquad (6)$$

with

$$U_{ni} = e^{c_i hW} u_n + h\sum_{j=1}^{i-1} A_{ij}(hW)f(t_{nj}) + h\sum_{k=1}^{q-1} M_{ik}^{(1)}(hW)f(t_n - kh)$$

$$+ h^2\sum_{k=1}^{q-1} M_{ik}^{(2)}(hW)f'(t_n - kh) \qquad (7)$$

[5, 6] derived the order conditions of this class of method by Expanding the functions in (6) and (7) and obtained the order conditions as

$$c_i^{\lambda}\psi_{\lambda}(hW) = \sum_{j=1}^{i-1}\frac{c_j^{\lambda-1}}{(\lambda-1)!}A_{ij}(hW) + \sum_{k=1}^{q-1}\frac{(-k)^{\lambda-1}}{(\lambda-1)!}M^{(1)}{}_{ik}(hW)$$

$$+ \sum_{k=1}^{q-1}\frac{(-k)^{\lambda-2}}{(\lambda-2)!}M^{(2)}{}_{ik}(hW) \qquad (8)$$

$$\psi_{\lambda}(hW)=\sum_{i=1}^{s}\frac{c_i^{\lambda-1}}{(\lambda-1)!}B_i(hW)+\sum_{k=1}^{q-1}\frac{(-k)^{\lambda-1}}{(\lambda-1)!}D^{(1)}{}_k(hW)$$

$$+\sum_{k=1}^{q-1}\frac{(-k)^{\lambda-2}}{(\lambda-2)!}D^{(2)}{}_k(hW) \qquad (9)$$

4 Construction of Families of 2-stage method of order Eight

The new scheme (823) is given as

$$u_{n+1}=e^{hW}u_n+hB_1(hW)N(U_{n1})+hB_2(hW)N(U_{n2})+hD^{(1)}{}_1(hW)N(u_{n-1})$$

$$+hD^{(1)}{}_2(hW)N(u_{n-2})+h^2D^{(2)}{}_1(hW)N'(u_{n-1})+h^2D^{(2)}{}_2(hW)N'(u_{n-2}) \qquad (10)$$

$$U_{n2}=e^{c_2hW}u_n+hA_{21}(hW)N(u_n)+hM^{(1)}{}_{21}(hW)N(u_{n-1})+hM^{(2)}{}_{22}(hW)N(u_{n-2})$$

$$+h^2M^{(2)}{}_{21}(hW)N'(u_{n-1})+h^2M^{(2)}{}_{22}(hW)N'(u_{n-2}) \qquad (11)$$

With the order conditions (8) and (9) above, the coefficient matrix of the order eight step two stage order three scheme (known as method 823) is given as

$$c_1^1A_{21}=(-1)^1\mathrm{M}_{21}^{(1)}+(-2)^1M_{22}^{(1)}+(-1)^0\mathrm{M}_{21}^{(3)}+(-2)^0M_{22}^{(2)}=\theta_2$$

$$=-\mathrm{M}_{21}^{(1)}-2M_{22}^{(1)}+\mathrm{M}_{21}^{(3)}+M_{22}^{(2)}=\theta_2 \qquad (12)$$

$$\frac{c_1^2A_{21}}{2!}=\frac{(-1)^2M_{21}^{(1)}}{2!}+\frac{(-2)^2M_{22}^{(1)}}{2!}+(-1)^1M_{21}^{(2)}+(-2)^1M_{22}^{(2)}=\theta_3$$

$$=\frac{M_{21}^{\{1\}}}{2}+\frac{4M_{22}^{\{1\}}}{2}-M_{21}^{(2)}-2M_{22}^{(2)}=\theta_3 \qquad (13)$$

$$\frac{c_1^3A_{21}}{3!}=\frac{(-1)^3M_{21}^{(1)}}{3!}+\frac{(-2)^3M_{22}^{(1)}}{3!}+\frac{(-1)^2M_{21}^{(2)}}{2!}+\frac{(-2)^2M_{22}^{(2)}}{2!}=\theta_4$$

$$=-\frac{M_{21}^{\{1\}}}{6}-\frac{8M_{22}^{\{1\}}}{6}+\frac{M_{21}^{(2)}}{2}+\frac{4M_{22}^{(2)}}{2}=\theta_4 \qquad (14)$$

$$\frac{c_1^4A_{21}}{4!}=\frac{(-1)^4M_{21}^{(1)}}{4!}+\frac{(-2)^4M_{22}^{(1)}}{4!}+\frac{(-1)^3M_{21}^{(2)}}{3!}+\frac{(-2)^3M_{22}^{(2)}}{3!}=\theta_5$$

$$=\frac{M_{21}^{\{1\}}}{24}+\frac{16M_{22}^{\{1\}}}{24}-\frac{M_{21}^{(2)}}{6}-\frac{8M_{22}^{(2)}}{6}=\theta_5 \qquad (15)$$

Solving equations (12) to (15) gives

$$M_{21}^{(1)} = \frac{8}{31}(6\theta_2 - 2\theta_3 - 27\theta_4 - 33\theta_5)$$

$$M_{22}^{(1)} = \frac{1}{31}(6\theta_2 - 29\theta_3 - 66\theta_4 - 60\theta_5)$$

$$M_{21}^{(2)} = \frac{1}{31}(28\theta_2 - 32\theta_3 + 126\theta_4 + 216\theta_5)$$

$$M_{22}^{(2)} = \frac{1}{31}(\theta_2 - 10\theta_3 - 42\theta_4 - 72\theta_5)$$

Similarly,

$$c_1^1 B_1 + c_2^1 B_2 - D_1^{(1)} - 2D_2^{(1)} + D_1^{(2)} + D_2^{(2)} = \theta_2$$

$$B_2 - D_1^{(1)} - 2D_2^{(1)} + D_1^{(2)} + D_2^{(2)} = \theta_2 \qquad (16)$$

$$\frac{c_1^2 B_1}{2!} + \frac{c_2^2 B_2}{2!} + \frac{D_1^{(1)}}{2!} + \frac{4D_2^{(1)}}{2!} - D_1^{(2)} - 2D_2^{(2)} = \theta_3$$

$$\frac{B_2}{2} + \frac{D_1^{(1)}}{2} + \frac{4D_2^{(1)}}{2} - D_1^{(2)} - 2D_2^{(2)} = \theta_3 \qquad (17)$$

$$\frac{c_1^3 B_1}{3!} + \frac{c_2^3 B_2}{3!} - \frac{D_1^{(1)}}{3!} - \frac{8D_2^{(1)}}{3!} + \frac{D_1^{(2)}}{2!} + \frac{4D_2^{(2)}}{2!} = \theta_4$$

$$\frac{B_2}{6} - \frac{D_1^{(1)}}{6} - \frac{8D_2^{(1)}}{6} + \frac{D_1^{(2)}}{2} + \frac{4D_2^{(2)}}{2} = \theta_4 \qquad (18)$$

$$\frac{c_1^4 B_1}{4!} + \frac{c_2^4 B_2}{4!} + \frac{D_1^{(1)}}{4!} + \frac{16D_2^{(1)}}{4!} - \frac{D_1^{(2)}}{3!} - \frac{8D_2^{(2)}}{3!} = \theta_5$$

$$\frac{B_2}{24} + \frac{D_1^{(1)}}{24} + \frac{16D_2^{(1)}}{24} - \frac{D_1^{(2)}}{6} - \frac{8D_2^{(2)}}{6} = \theta_5 \qquad (19)$$

$$\frac{c_1^5 B_1}{5!} + \frac{c_2^5 B_2}{5!} + \frac{D_1^{(1)}}{5!} + \frac{32D_2^{(1)}}{5!} - \frac{D_1^{(2)}}{4!} - \frac{16D_2^{(2)}}{4!} = \theta_6$$

$$\frac{B_2}{120} + \frac{D_1^{(1)}}{120} + \frac{32D_2^{(1)}}{120} - \frac{D_1^{(2)}}{24} - \frac{8D_2^{(2)}}{24} = \theta_6 \qquad (20)$$

Solving the systems of equations (16) to (20) we have

$$B_2 = 3(\frac{1}{27}\theta_2 + 4\theta_3 + 13\theta_4 + 24\theta_5 + 20\theta_6)$$

$$D_1^{(1)} = \frac{1}{2}(-2\theta_2 + 4\theta_3 + 9\theta_4 - 24\theta_5 - 60\theta_6)$$

$$D_2^{(1)} = \frac{-10\theta_2}{9}\frac{-19\theta_3}{6} + \frac{23\theta_4}{6} + 38\theta_5 + \frac{170\theta_6}{3})$$

$$D_1^{(2)} = 2(-\theta_2 - 2\theta_3 + \frac{9}{2}\theta_4 + 24\theta_5 + 30\theta_6)$$

$$D_2^{(2)} = 3(\frac{\theta_2}{6} - \frac{\theta_3}{3} + \frac{4}{3}\theta_4 + 4\theta_5 + \frac{40\theta_6}{6})$$

The above method is represented in Extended Butcher Tableau.
The coefficients can be represented in a tablau as

Table 1: Extended Butcher Tableau

$A_{21}^{(1)}$	$M_{21}^{(1)} \cdots U_{2,q-1}^{(1)}$	$M_{21}^{(2)} \cdots M_{2,q-1}^{(2)}$
$\therefore \ddots$	$\vdots \quad \vdots$	$\vdots$
$A_{s1}^{(1)} \cdots A_{s,s-1}^{(1)}$	$M_{s1}^{(1)} \cdots M_{s,q-1}^{(1)}$	$M_{s1}^{(2)} \cdots M_{s,q-1}^{(2)}$
$B_1 \cdots B_{s-1} \quad B_s$	$D_1^{(1)} \cdots D^{(1)}{}_{q-1}$	$D_1^{(2)} \cdots D^{(2)}{}_{q-1}$

The coefficients can be represented in a tablau as

Table 2: Coefficients Tableau of EEGLM with z=1

$\frac{0000}{0000}$		$\frac{-555153}{1000000}$	$\frac{276269}{50000}$	$\frac{-118997}{200000}$	$\frac{-9331}{50000}$
$\frac{0000}{0000}$	$\frac{276269}{50000}$	$\frac{-217287}{100000}$	$\frac{-78899}{500000}$	$\frac{-3971}{31250}$	$\frac{124327}{500000}$

Table 3: Coefficients Tableau of EEGLM with z=2

$\frac{0000}{0000}$		$\frac{978522}{1000000}$	$\frac{-231692}{1000000}$	$\frac{10382}{100000}$	$\frac{-1794}{10000}$
$\frac{0000}{0000}$	$\frac{730537}{1000000}$	$\frac{-52001}{125000}$	$\frac{-131781}{100000}$	$\frac{-103449}{50000}$	$\frac{96961}{250000}$

5. DISCUSSIONS

In this section, we discuss by comparing the accuracies of step two order eight exponential general linear methods with other related studies in literatures .

Problem1. Consider the initial value problem $u' = -2xu^2$ with $u(0) = 1$

The theoretical solution is given as

$$u(x) = \frac{1}{1+x^2}$$

The accumulated errors of the proposed scheme (823), [1,2,3,5] for the above problem with their corresponding mesh sizes are shown in the figure 1 below

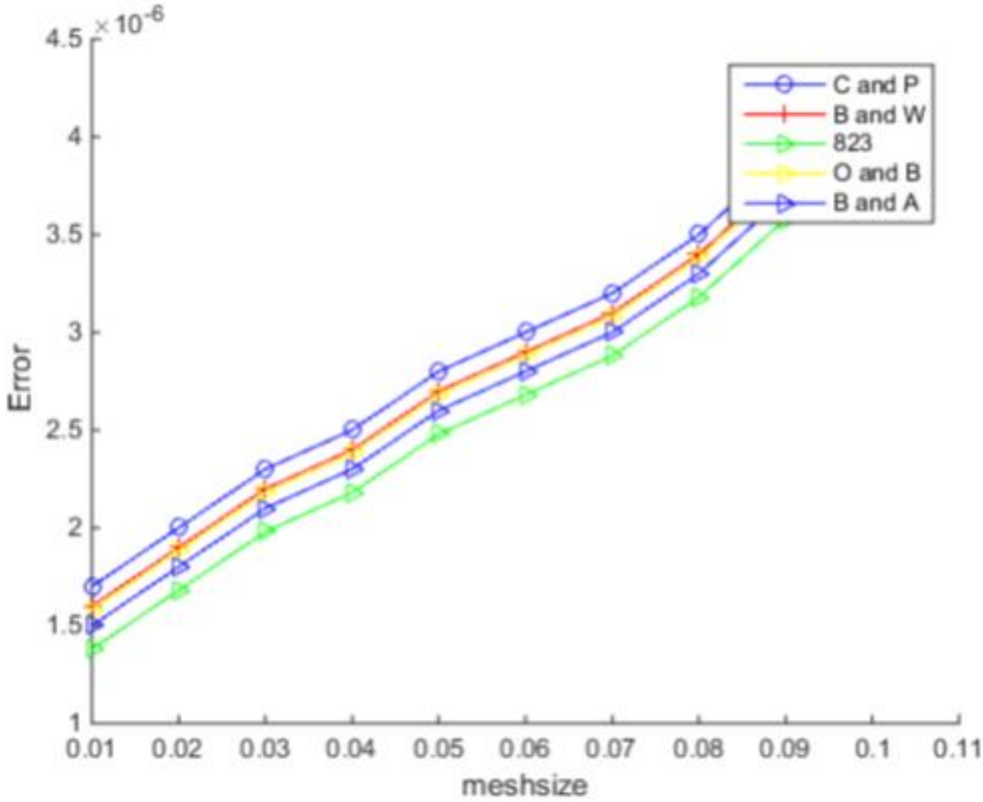

Figure 1:The relationship between the proposed method and other Methods

From the graph, step two order eight exhibits remarkable improvement in terms of accuracies over [1, 3, 4, 5].

Problem 2.Consider the initial value problem

$$y_1' = -20y_1 - 0.25y_2 - 19.75y_3,$$

$$y_2' = 20y_1 - 20.25y_2 + 0.25y_3,$$

$$y_3' = 20y_1 - 19.75y_2 - 0.25y_3$$

$$y_1(0) = 1, y_2(0) = 0, y_3(0) = -1$$

The theoretical solution is given by

$$y_1 = \frac{[\exp(0.5t) + \exp(20t)(\cos(20t) + \sin(20t))]}{2},$$

$$y_2 = \frac{[\exp(0.5t) + \exp(20t)(\cos(20t) - \sin(20t))]}{2},$$

$$y_3 = \frac{[\exp(0.5t) + \exp(20t)(\cos(20t) + \sin(20t))]}{2}$$

This is illustrated in table 4 below.

Table 4: Numerical Results of the different schemes

T	H	RKM_4	SHOKRI	PROPOSED SCHEME
50	0.005	$7.1e-26$	$1.38e-20$	$4.42e-28$
		$7.1e-26$	$1.38e-20$	$4.42e-28$
		$7.1e-26$	$1.38e-20$	$4.42e-28$
100	0.01	4.3e - 33	3.57e - 31	6.3e - 42
		4.3e - 33	3.57e - 31	6.3e - 42
		4.3e - 33	3.57e - 31	6.3e - 42

From Table 4 above, ,it can be seen that step two order eight extended exponential general linear methods exhibits remarkable improvement in terms of accuracies over the other methods.

6. CONCLUSION

We have introduced a new method to the derivation of exponential general linear methods. The numerical results obtained through step two order eight scheme as indicated in figure 1 and Table 4,exhibit a considerable improvement over [6] .Numerical results presented also show that our scheme is accurate and efficient in handling the given IVP.

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