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Dynamical Equations with Integral Boundary Conditions On Time Scales

G. V. Ramana, K. Swamy, A.V.S.N.Murty,

Aditya Engineering college, Surampalem, A.P, India.

Email: *ramanaginjala9@gmail.com*.

Sai Aditya Engineering college, Surampalem, A.P, India.

Email: *kswamymani@gmail.com*.

Aditya Engineering college, Surampalem, A.P, India.

Email: *avsnmurthy2005@gmail.com*.

Abstract: In this paper, we proved a new comparison result and the monotone iterative technique is developed to study the existence of solutions of dynamical equations with integral boundary conditions on time scales.

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1. INTRODUCTION

The study of dynamic equations on time scales, which has been created in order to unify the study of differential and difference equations, is an area of mathematics that has recently received a lot of attention; moreover, many results on this issue have been well documented in the monographs [2,3,15]. The concept is particularly useful in modeling stop-start processes where continuous and discrete time may be present at different stages. Let T be a time scale (any non empty closed subset of the real numbers). Without loss of generality we assume that 0, a $\in T$ and denote $J = [0, \sigma(a)] \subset T$ being a closed interval. In this paper, we consider the following first order dynamic equation on time scales

$$x^{\Delta}(t) = f(t, x(t)), t \in J, \qquad (1.1)$$

$$x(0) + \int_0^{\sigma(a)} x(s)\Delta s = x(\sigma(a)),$$

where $f \in C[J \times R, R]$. The monotone iterative technique is a powerful method used to approximate solutions of several problems [10,13,14]. The purpose of this paper is to show that it can be applied successfully to dynamical equations with integral boundary conditions on time scales . This technique combined with the method of lower and upper solutions play an important roles in constructing monotone sequences with converge to the solutions of our problem. In presence of a lower solution α and an upper solution β with reversed ordering condition $\beta \le \alpha$, we prove that under suitable conditions equation (1.1) has the maximal and minimal solutions between the lower solution and upper solution.

2. PRELIMINARY RESULTS

Definition 2.1: The mappings σ and $\rho : T \to R,$ where T is any closed subset of reals, are defined as $\sigma(t) = \inf\{s \in T : s > t\}$ and $\rho(t) = \sup\{s \in T : s < t\}$.

Definition 2.2: A non-maximal element t in T is called right dense if $\sigma(t) = t$; right scattered if $\sigma(t) > t;$ left dense if $\rho(t) = t$ and left scattered if $\rho(t) < t.$

Definition 2.3: If T has a left scattered maximum m, then $T^k = T - \{m\},$ otherwise, $T^k = T.$ T^k is called the degenerate set.

Definition 2.4: The function $\mu^* = T^k \to \quad R^+$ defined by $\mu^*(t) = \mu(\sigma(t), t)$ for $t \in T$ is called graininess. If t is right dense, then $\mu^* = 0$ and if t is right scattered, then $\mu^* = \sigma(t) - t.$

Definition 2.5: A mapping g : $T \to R$ is called rd-continuous if

a. It is continuous in each right dense or maximal $t \in T$;
b. Left side limit g(t^-) exists in each left dense t.

Remark 2.6: If (b) is replaced by g being continuous at each left dense point, then g is said to be a continuous function on T.

We define $C[J, R]$ = {u (t) is continuous on J }, and $C^1[J, R]$={ $u^\Delta(t)$ is continuous on J }.

Definition 2.7: Let f: $T^k \to R$ and if t is right scattered, then the Delta derivative $f^\Delta(t)$ of $f(t)$ is defined as

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

Definition 2.8: (Schauder Fixed Point Theorem) If E is a closed, bounded and convex subset of a Banach space B and T : E $\to$E is completely continuous, then T has a fixed point.

Definition 2.9: A function P : $T \to R$ is said to be regressive if $1 + \mu(t)P(t) \neq 0$ for all t in T^k. The set of all regressive and rd- continuous functions will be denoted in this paper by

R(T, R). For two functions p , $q \in$ R(T, R) define a plus $\oplus$ and a minus $\ominus$ by

$$(p(t) \oplus q(t)) = p(t) + q(t) + \mu(t)\, p(t) q(t),$$

$$(\ominus\, p\, (t))(t) = -\frac{p(t)}{1+\mu(t)p(t)}.$$

Definition 2.10: If $p \in R,$ then the exponential function is defined as

$$e_p(t,s) = \exp\left[\int_s^t \xi_{\mu(t)}(p(t))\Delta t\right] \text{ for } s,t \in T,$$

Where $\xi_{\mu(t)}(p(t)) = \dfrac{1}{\mu(t)} Log[1 + p(t)\mu(t)]$ where Log is principal logarithmic function.

When $T = R$, then $e_\alpha(t,t_0) = e^{\alpha(t-t_0)}$ and when $T = Z$ then $e_\alpha(t,t_0) = (1+\alpha)^{t-t_0}$.

Definition 2.11: If $p \in R$, then the first order linear dynamic equation

$$y^\Delta(t) = p(t)y(t) \qquad (2.1)$$

is called regressive.

Theorem 2.12: If $p \in R$, and for fixed $t_0 \in T$, $e_p(t,t_0)$ is a solution of the initial value problem (2.1) satisfying the initial condition $y(t_0) = 1$on T.

Remark 2.13: If $p(t) \geq 0$ for $t \geq t_0$, clearly $1 + \mu(t)p(t) \geq 1$. Therefore $\xi_{\mu(t)}(p(t)) \geq 0$ and $e_p(t,t_0) \geq 1$.

Lemma 2.14: If $p, q \in$R(T, R), then

i. $e_0(t,s) \equiv 1$ and $e_p(t,t) \equiv 1$;

ii. $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

iii. $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

iv. $e_p(t,r)e_p(r,s) = e_p(t,s)$;

v. $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$.

3. MAIN RESULTS

Lemma 3.1: Assume that there exists a positive function $m(t)$ continuous on J and

$$x^\Delta(t) \geq m(t)x(t), t \in J, \qquad (3.1)$$

$$x(0) \geq x(\sigma(a)).$$

Then $x(t) \leq 0$ for all $t \in J$ provided that $\frac{2M_0 e_m(\sigma(a),0)\sigma(a)}{1-K_0} \leq 1$, where $M_0 = \max_{t\in J}\{m(t)\}$ and $K_0 = \max_{t\in J}\{m(t)\mu(t)\} < 1$.

Proof: Let $v(t) = x(t)e_{\ominus(-m)}(t,0)$. Considering $\ominus(-m) = \frac{m(t)}{1-\mu(t)m(t)}$, we have

$$v^\Delta(t) \geq \frac{2m(t)v(t)e_{(-m)}(t,t)}{1-m(t)\mu(t)} \qquad (3.2)$$

$$v(0) \geq v(\sigma(a))e_{(-m)}(\sigma(a),0).$$

Obviously,
v and x have the same symbol, we need only to prove that $v(t) \leq 0, t \in$ J. If it is not true, then there exists two cases

(i) $v(t) \geq 0$ for all $t \in J$.

(ii) there exists $r_1, r_2 \in J$ such that $v(r_1) > 0$ and $v(r_2) < 0$.

In case (i), suppose that $v(t) \geq 0$ for all $t \in J$, then $v^{\Delta}(t) \geq 0, t \in J$ by (3.2). This shows that $v(t)$ is nondecreasing on J, and so, $v(\sigma(a)) \geq v(r_1) > 0, x(\sigma(a)) = v(\sigma(a))e_{(-m)}(\sigma(a),0) > 0, v(0) \leq v(\sigma(a))$. However, from (3.1), we have $v(0) = x(0) \geq x(\sigma(a)) > x(\sigma(a))\, e_{\ominus(-m)}(\sigma(a), 0) = v(\sigma(a))$, this is a contradiction. Hence, $v(t) < 0$ for some $t \in J$.

In case (ii), Let $v(r_2) = \min_{t \in J}\{v(t)\} = p$, then $p < 0$. From (3.2), we obtain

$$v^{\Delta}(t) \geq \frac{2pm(t)e_{(-m)}(t,t)}{1 - m(t)\mu(t)}$$

$$\geq \frac{2pM_0 e_{(-m)}(t,t)}{1 - K_0} \tag{3.3}$$

Next, we consider again two possible cases.

Case(a): $r_2 > r_1$. By (3.3), we have

$$v(r_2) - v(r_1) \geq \frac{2pM_0}{1 - K_0}\int_{r_1}^{r_2} e_{(-m)}(t,t)\Delta t$$

$$p = v(r_2) \geq v(r_1) + \frac{2pM_0}{1 - K_0}\int_{r_1}^{r_2} e_{(-m)}(t,t)\Delta t$$

$$> \frac{2pM_0}{1 - K_0}\int_{r_1}^{r_2} e_{(-m)}(t,t)\Delta t$$

Since $p < 0, e_m(\sigma(a), 0) > 1$, then

$$1 < \frac{2M_0}{1 - K_0}\int_{r_1}^{r_2} e_{(-m)}(t,t)\Delta t$$

$$< \frac{2M_0 e_m(\sigma(a),0)}{1 - K_0}\int_{0}^{\sigma(a)} e_{(-m)}(t,t)\Delta t$$

$$= \frac{2M_0 e_m(\sigma(a),0)}{1 - K_0}\int_{0}^{\sigma(a)} \Delta t$$

$$\leq \frac{2M_0 e_m(\sigma(a),0)\sigma(a)}{1 - K_0}.$$

This is a contradiction.

Case(b): $r_2 < r_1$. By (3.3), we have

$$v(r_2) - v(0) \geq \frac{2pM_0}{1 - K_0}\int_0^{r_2} e_{(-m)}(t,t)\Delta t$$

$$p = v(r_2) \geq v(0) + \frac{2pM_0}{1-K_0}\int_0^{r_2} e_{(-m)}(t,t)\Delta t \tag{3.4}$$

and

$$\int_{r_1}^{\sigma(a)} v^{\Delta}(t)\Delta t \geq \frac{2pM_0}{1-K_0}\int_{r_1}^{\sigma(a)} e_{(-m)}(t,t)\Delta t$$

$$v(\sigma(a)) \geq v(r_1) + \frac{2pM_0}{1-K_0}\int_{r_1}^{\sigma(a)} e_{(-m)}(t,t)\Delta t$$

$$> \frac{2pM_0}{1-K_0}\int_{r_1}^{\sigma(a)} e_{(-m)}(t,t)\Delta t. \tag{3.5}$$

Since $v(0) \geq v(\sigma(a))e_{(-m)}(\sigma(a),0)$, from (3.4)and (3.5), we get

$$p \geq v(\sigma(a))e_{(-m)}(\sigma(a),0) + \frac{2pM_0}{1-K_0}\int_0^{r_2} e_{(-m)}(t,t)\Delta t$$

$$p > \frac{2pM_0 e_{(-m)}(\sigma(a),0)}{1-K_0}\int_{r_1}^{\sigma(a)} e_{(-m)}(t,t)\Delta t + \frac{2pM_0}{1-K_0}\int_0^{r_2} e_{(-m)}(t,t)\Delta t.$$

Noting that $p < 0, e_m(\sigma(a),0) > 1, r_2 < r_1$, one can obtain

$$1 < \frac{M_0 e_m(\sigma(a),0)}{1-K_0}\int_{r_1}^{\sigma(a)} e_{(-m)}(t,t)\Delta t + \frac{2M_0}{1-K_0}\int_0^{r_2} e_{(-m)}(t,t)\Delta t$$

$$< \frac{2M_0 e_m(\sigma(a),0)}{1-K_0}\int_0^{\sigma(a)} e_{(-m)}(t,t)\Delta t$$

$$\leq \frac{2M_0 e_m(\sigma(a),0)\sigma(a)}{1-K_0}.$$

This is a contradiction. Combining case (i) and case (ii), we know that
$v(t) \leq 0$, for all $t \in J$, and thus $x(t) \leq 0$, for all $t \in J$.

Lemma 3.2:
Assume that there exists a positive function $m(t)$ and a non negative function $n(t)$ continuous on J.

If $\dfrac{\sigma(a)e_m(t,0)}{e_m(\sigma(a),0) - 1} < 1$, then the equation

$$x^{\Delta}(t) - m(t)x(t) = n(t), t \in J, \tag{3.6}$$

$$x(0) + \int_0^{\sigma(a)} x(s)\Delta s = x(\sigma(a)),$$

has a unique solution.

Proof: We shall prove the conclusion by Banach's contraction principle. First define a Banach space as follows.

$X = \{x(t) \in C_{rd}[J, R]\}$ with $\|x(t)\| = \max_{t \in J}\{x(t)\}$.

Now define an operator on X as

$$S: x(t) = e_m(t,0)\{\frac{1}{e_m(\sigma(a),0)-1}\int_0^{\sigma(a)} x(s)\Delta s - \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0)-1}\int_0^{\sigma(a)} e_m(0,\sigma(s))n(s)\Delta s + \int_0^t e_m(0,\sigma(s))n(s)\Delta s\}, t \in J.$$

For any two functions $\emptyset, \psi \in X$,

$$|(S\emptyset)(t) - (S\psi)(t)| = \left|\frac{e_m(t,0)}{e_m(\sigma(a),0)-1}\int_0^{\sigma(a)} [\emptyset(S) - \psi(S)]\Delta s\right|$$

$$\leq \frac{e_m(t,0)\sigma(a)}{e_m(\sigma(a),0)-1}|\emptyset(s) - \psi(s)|.$$

Thus, we have

$$\|(S\phi)(t) - (S\psi)(t)\| \leq \frac{e_m(t,0)\sigma(a)}{e_m(\sigma(a),0)-1}\|\emptyset - \psi\|$$

This implies by condition $\frac{\sigma(a)e_m(t,0)}{e_m(\sigma(a),0)-1} < 1$ that S is a contraction on X and

therefore by Banach's contraction principle there exists exactly one $x(t) \in X$ such that

$$x(t) = e_m(t,0)\{\frac{1}{e_m(\sigma(a),0)-1}\int_0^{\sigma(a)} x(s)\Delta s - \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0)-1}\int_0^{\sigma(a)} e_m(0,\sigma(s))n(s)\Delta s + \int_0^t e_m(0,\sigma(s))n(s)\Delta s\}, t \in J.$$

Next, we shall show that $x(t)$ is a solution of (3.6). We have

$$x^\Delta(t) = m(t)e_m(t,0)\{\frac{1}{e_m(\sigma(a),0)-1}\int_0^{\sigma(a)} x(s)\Delta s - \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0)-1}\int_0^{\sigma(a)} e_m(0,\sigma(s))n(s)\Delta s + \int_0^t e_m(0,\sigma(s))n(s)\Delta s\} + e_m(\sigma(t),0)e_m(0,\sigma(t))n(t)$$

$$= m(t)x(t) + n(t).$$

Moreover,

$$x(\sigma(a)) = e_m(\sigma(a),0)\{\frac{1}{e_m(\sigma(a),0)-1}\int_0^{\sigma(a)} x(s)\Delta s - \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0)-1}\int_0^{\sigma(a)} e_m(0,\sigma(s))n(s)\Delta s + \int_0^{\sigma(a)} e_m(0,\sigma(s))n(s)\Delta s\}$$

$$= e_m(\sigma(a),0)\{\frac{1}{e_m(\sigma(a),0)-1}\int_0^{\sigma(a)} x(s)\Delta s + [1 - \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0)-1}]\int_0^{\sigma(a)} e_m(0,\sigma(s))n(s)\Delta s$$

$$= \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0)-1}\int_0^{\sigma(a)}[x(s) - e_m(0,\sigma(s))n(s)]\Delta s$$

$$= x(0) + \int_0^{\sigma(a)} x(s)\Delta s.$$

Thus we complete the proof.

Definition 3.3: Functions $\alpha, \beta \in C^1(J,R)$ are said to be the lower and upper solutions

of (1.1)respectively if

$$\alpha^\Delta(t) \le f(t,\alpha(t)), t \in J, \quad \alpha(0) + \int_0^{\sigma(a)} \alpha(s)\Delta s \le \alpha(\sigma(a)).$$

$$\beta^\Delta(t) \ge f(t,\beta(t)), t \in J, \quad \beta(0) + \int_0^{\sigma(a)} \beta(s)\Delta s \ge \beta(\sigma(a)).$$

Let $\Omega_{(u,v)} = \{y: u(t) \le y(t) \le v(t), t \in J\}$ if $u(t) \le v(t)$ for $t \in J$.

We introduce the following assumptions.

$(H_1)\alpha_0, \beta_0 \in C^1(J,R)$ are lower and upper solutions of (1.1) respectively and $\beta_0(t) \le \alpha_0(t)$ for $t \in J$;

(H_2) $f \in C(J \times R, R)$;

(H_3) there exist positive, rd-continuous function $m(t)$ such that

$f(t,x_2) - f(t,x_1) \le m(x_2 - x_1)$ if $\beta_0 \le x_1 \le x_2 \le \alpha_{0,} t \in J$;

$(H_4)\frac{2M_0 e_m(\sigma(a),0)\sigma(a)}{1-K_0} \le 1$, where $M_0 = \max_{t\in J}\{m(t)\}$ and $K_0 = \max_{t\in J}\{m(t)\mu(t)\} < 1$;

$(H_5)\frac{\sigma(a)e_m(t,0)}{e_m(\sigma(a),0)-1} < 1.$

Lemma 3.4: Assume that $(H_1)-(H_5)$ hold. If

$$y^\Delta(t) - m(t)y(t) = f(t,\alpha_0(t)) - m(t)\alpha_0(t), t \in J,$$

$$y(0) + \int_0^{\sigma(a)} \alpha_0(s)\Delta s = y(\sigma(a)),$$

$$z^\Delta(t) - m(t)z(t) = f(t,\beta_0(t)) - m(t)\beta_0(t), t \in J,$$

$$z(0) + \int_0^{\sigma(a)} \beta_0(s)\Delta s = z(\sigma(a)),$$

then

$$\beta_0(t) \le z(t) \le y(t) \le \alpha_0(t), t \in J. \qquad (3.7)$$

and y, z are lower and upper solutions of (1.1) respectively.

Proof: From lemma 3.2, we know that there exists unique solutions for y and z.

$$put\ p = y - \alpha_0, q = \beta_0 - z, then$$

$$p(\sigma(a)) = y(\sigma(a)) - \alpha_0(\sigma(a))$$

$$\le y(0) + \int_0^{\sigma(a)} \alpha_0(s)\Delta s - \alpha_0(0) - \int_0^{\sigma(a)} \alpha_0(s)\Delta s$$

$$= y(0) - \alpha_0(0)$$

$$= p(0),$$

$$q(\sigma(a)) = \beta_0(\sigma(a)) - z(\sigma(a))$$

$$\le \beta_0(0) + \int_0^{\sigma(a)} \beta_0(s)\Delta s - z(0) - \int_0^{\sigma(a)} \beta_0(s)\Delta s$$

$$= \beta_0(0) - z(0)$$

$$= q(0),$$

and

$$p^\Delta(t) = y^\Delta(t) - \alpha_0{}^\Delta(t)$$

$$\ge f(t, \alpha_0(t)) + m(t)y(t) - m(t)\alpha_0(t) - f(t, \alpha_0(t))$$

$$= m(t)[y(t) - \alpha_0(t)]$$

$$m(t)p(t), t \in J,$$

$$q^\Delta(t) = \beta^\Delta(t) - z^\Delta(t)$$

$$\ge f(t, \beta_0(t)) - m(t)z(t) + m(t)\beta_0(t) - f(t, \beta_0(t))$$

$$= m(t)[\beta_0(t) - z(t)]$$

$$m(t)q(t), t \in J.$$

From lemma 3.1, we have $p(t) \le 0, q(t) \le 0, t \in J$ and so $y(t) \le \alpha_0(t), \beta_0(t) \le z(t), t \in J$.

Now let $p(t) = z(t) - y(t)$, then

$$p(\sigma(a)) = z(\sigma(a)) - y(\sigma(a))$$

$$= z(0) + \int_0^{\sigma(a)} \beta_0(s)\Delta s - y(0) - \int_0^{\sigma(a)} \alpha_0(s)\Delta s$$

$$= z(0) - y(0) + \int_0^{\sigma(a)} [\beta_0(s) - \alpha_0(s)]\Delta s$$

$$\le z(0) - y(0)$$

$$= p(0),$$

From assumption (H_3), we have

$$p^{\Delta}(t) = z^{\Delta}(t) - y^{\Delta}(t)$$
$$= f(t, \beta_0(t)) + m(t)z(t) - m(t)\beta_0(t) - f(t, \alpha_0(t)) - m(t)y(t) + m(t)\alpha_0(t)$$
$$\geq -m(t)[\alpha_0(t) - \beta_0(t)] + m(t)[z(t) - y(t)] + m(t)[\alpha_0(t) - \beta_0(t)]$$
$$= m(t)[z(t) - y(t)]$$
$$= m(t)p(t)$$

By lemma 3.1, one can get $p(t) \leq 0, t \in J$, then $z(t) \leq y(t), t \in J$. It proves that (3.7) holds. Now we need to show that y, z are lower and upper solutions of (1.1), respectively. Using again assumption (H_3), we have

$$y^{\Delta}(t) = f(t, \alpha_0(t)) + m(t)[y(t) - \alpha_0(t)] - f(t, y(t)) + f(t, y(t))$$
$$\leq m(t)[\alpha_0(t) - y(t)] + m(t)[y(t) - \alpha_0(t)] + f(t, y(t))$$
$$= f(t, y(t))$$
$$z^{\Delta}(t) = f(t, \beta_0(t)) + m(t)[z(t) - \beta_0(t)] - f(t, z(t)) + f(t, z(t))$$
$$\geq -m(t)[z(t) - \beta_0(t)] + m(t)[z(t) - \beta_0(t)] + f(t, z(t))$$
$$= f(t, z(t))$$
$$y(0) + \int_0^{\sigma(a)} y(s)\Delta s \leq y(0) + \int_0^{\sigma(a)} \alpha_0(s)\Delta s$$
$$= y(\sigma(a)),$$
$$z(0) + \int_0^{\sigma(a)} z(s)\Delta s \geq z(0) + \int_0^{\sigma(a)} \beta_0(s)\Delta s$$
$$= z(\sigma(a)).$$

It shows that y, z are lower and upper solutions of (1.1), respectively.

Theorem 3.5: Suppose that $(H_1) - (H_5)$ hold. Then there exist monotone sequences $\{\alpha_n, \beta_n\}$ such that $\alpha_n \to \alpha, \beta_n \to \beta, t \in J$ as $n \to \infty$ and this convergence is uniformly and monotonically on J. Moreover, α, β are maximal and minimal solutions of (1.1) in $[\beta_0, \alpha_0] = \{u \in C^1(J, R) : \beta_0 \leq u \leq \alpha_0\}$.

Proof: Consider

$$\alpha_{n+1}^{\Delta}(t) - m(t)\alpha_{n+1}(t) = f(t, \alpha_n(t)) - m(t)\alpha_n(t), t \in J,$$
$$\alpha_{n+1}(0) + \int_0^{\sigma(a)} \alpha_n(s)\Delta s = \alpha_{n+1}(\sigma(a)),$$
$$\beta_{n+1}^{\Delta}(t) - m(t)\beta_{n+1}(t) = f(t, \beta_n(t)) - m(t)\beta_n(t), t \in J,$$
$$\beta_{n+1}(0) + \int_0^{\sigma(a)} \beta_n(s)\Delta s = \beta_{n+1}(\sigma(a)), \tag{3.8}$$

for n = 0,1,2,…. Lemma 3.4, shows $\beta_0(t) \leq \beta_1(t) \leq \alpha_1(t) \leq \alpha_0(t), t \in J$ and α_1, β_1 are lower and upper solutions of (1.1), respectively. Assume that

$\beta_0(t) \le \beta_1(t) \le ... \le \beta_k(t) \le \alpha_k(t) \le ... \le \alpha_1(t) \le \alpha_0(t), t \in J$ for some $k \ge 1$and let α_k, β_k be lower and upper solutions of (1.1), respectively. Then using again lemma 3.4, we get

$\beta_k(t) \le \beta_{k+1}(t) \le \alpha_{k+1}(t) \le \alpha_k(t), t \in J$, and $\alpha_{k+1}(t), \beta_{k+1}(t)$ are lower and upper solutions of (1.1), respectively. By induction, we have

$\beta_0(t) \le \beta_1(t) \le ... \le \beta_n(t) \le \alpha_n(t) \le ... \le \alpha_1(t) \le \alpha_0(t), t \in J$ for all n.

Hence $\beta_n(t) \to \beta(t), \alpha_n(t) \to \alpha(t), t \in J$ if $n \to \infty$. Indeed, taking the limit $n \to \infty$ on both sides of (3.8), we know that α and β are solutions of (1.1). Next, we are going to show that α, β are maximal and minimal solutions of (1.1) in $[\beta_0, \alpha_0]$. To do it, we need to show that if $w(t)$ is any solution of (1.1) such that $\beta_0(t) \le w(t) \le \alpha_0(t), t \in J$, then $\beta_0(t) \le \beta(t) \le w(t) \le \alpha(t) \le \alpha_0(t), t \in J$. Assume that for some k, $\beta_k(t) \le w(t) \le \alpha_k(t), t \in J$. Let $p(t) = \beta_{k+1}(t) - w(t), q(t) = w(t) - \alpha_{k+1}(t)$. Then

$$
\begin{aligned}
p(\sigma(a)) &= \beta_{k+1}(\sigma(a)) - w(\sigma(a)) \\
&= \beta_{k+1}(0) + \int_0^{\sigma(a)} \beta_k(s)\Delta s - w(0) - \int_0^{\sigma(a)} w(s)\Delta s \\
&= \beta_{k+1}(0) - w(0) + \int_0^{\sigma(a)} [\beta_k(s) - w(s)]\Delta s \\
&\le \beta_{k+1}(0) - w(0) \\
&= p(0),
\end{aligned}
$$

$$
\begin{aligned}
q(\sigma(a)) &= w(\sigma(a)) - \alpha_{k+1}(\sigma(a)) \\
&= w(0) + \int_0^{\sigma(a)} w(s)\Delta s - \alpha_{k+1}(0) - \int_0^{\sigma(a)} \alpha_k(s)\Delta s \\
&= w(0) - \alpha_{k+1}(0) + \int_0^{\sigma(a)} [w(s) - \alpha_k(a)]\Delta s \\
&\le w(0) - \alpha_{k+1}(0) \\
&= q(0).
\end{aligned}
$$

From assumption (H_3), we have

$$
\begin{aligned}
p^{\Delta}(t) &= \beta_{k+1}^{\Delta}(t) - w^{\Delta}(t) \\
&= f(t, \beta_k(t)) - f(t, w(t)) + m(t)[\beta_{k+1}(t) - \beta_k(t)] \\
&\ge -m(t)[w(t) - \beta_k(t)] + m(t)[\beta_{k+1}(t) - \beta_k(t)] \\
&= m(t)[\beta_{k+1}(t) - w(t)] \\
&= m(t)p(t)
\end{aligned}
$$

$$q^{\Delta}(t) = w^{\Delta}(t) - \alpha_{k+1}^{\Delta}(t)$$

$$= f(t, w(t)) - f(t, \alpha_k(t)) - m(t)[\alpha_{k+1}(t) - \alpha_k(t)]$$

$$\geq -m(t)[\alpha_k(t) - w(t)] - m(t)[\ \alpha_{k+1}(t) - \alpha_k(t)]$$

$$= m(t)[w(t) - \alpha_{k+1}(t)]$$

$$= m(t)q(t), t \in J.$$

By lemma 3.1, we can obtain $p(t) \leq 0, q(t) \leq 0, t \in J$, this shows $\beta_{k+1}(t) \leq w(t) \leq \alpha_{k+1}(t), t \in J$. It proves, by induction, that $\beta_n(t) \leq w(t) \leq \alpha_n(t), t \in J$ for all n. Taking the limit $n \to \infty$, we have $\beta_0(t) \leq \beta(t) \leq w(t) \leq \alpha(t) \leq \alpha_0(t), t \in J$, so the assertion of Theorem 3.5 is true.

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