[Volume 4 issue 06 June 2016]
PageNo.1514-1520 ISSN : 2320-7167

INTERNATIONAL JOURNAL OF MATHEMATICS AND COMPUTER RESEARCH

Power Series Solution of Non-linear Partial Differential Equations

Roopashree N S[1] and Achala L. Nargund[2]

[1] Assistant Professor and Head, Department of Mathematics, Basaveshwara College of Commerce, Arts & Science, 2nd Block, Rajajinagar, Bangalore-560010.
Email ID: ns.roopashree@gmail.com

[2] Professor and Head, P.G. Department of Mathematics and Research Centre in Applied Mathematics, M.E.S College of Arts, Commerce and Science,15th Cross, Malleshwaram,Bangalore-560003.
Email ID: anargund1960@gmail.com.

Abstract: The method of Power Series Solution is a strong method for Ordinary Differential Equation (ODE) and Partial Differential Equation (PDE). In this paper non-linear PDE is solved by using this method. This method ensures the theoretical exactness of the approximate solution and comparisons of the approximate solutions with the exact ones are demonstrated.

Keywords: *Non-linear Partial differential equation, Power series solution, Charpits method, Exact solution*, *Domb-Syke plot, Convergent series.*

Academic Classification: Mathematics.

Mathematical Classification: 35E15

1. INTRODUCTION:

There is an old American saying "he who lacks a sense of the past is condemned to live in the narrow darkness of his own generation". In the same way mathematics without history is mathematics stripped of its greatness. For like the other acts, mathematics is one of the supreme arts of civilization. Here we will try to analyze the importance, role and applications of Differential Equations (Particularly Partial differential equation) a part of mathematics and this branch is called as Applied Mathematics.

The field of partial differential equations (PDE for short) has a long history going back several hundred years, beginning with the development of calculus [11]. In this regard, the field is a traditional area of mathematics, although more recent than such classical fields as number theory, algebra, and geometry. As in many areas of mathematics, the theory of PDE has undergone a radical transformation in the past hundred years, fueled by the development of powerful analytical tools, notably, the theory of functional analysis and more specifically of function spaces [5]. The discipline has also been driven by rapid developments in science and engineering, which present new challenges of modeling and simulation and promote broader investigations of properties of PDE models and their solutions. As the theory and application of PDE have developed, profound unanswered questions and unresolved problems have been identified.

Historically, comparatively little was known about the extraordinary range of behavior exhibited by the solutions to nonlinear partial differential equations [8]. Many of the most fundamental phenomena that now drive modern-day research, including solutions, stability, blow-up and singularity formation, asymptotic properties, etc., remained undetected or at best dimly perceived in the pre-computer era [9]. The last sixty years has witnessed a remarkable blossoming in our understanding, due in large part to the insight offered by the availability of high performance computers coupled with great advances in the understanding and development of suitable numerical approximation schemes.

New analytical methods, new mathematical theories, coupled with new computational algorithms have precipitated this revolution in our understanding and study of nonlinear systems, an activity that continues to grow in intensity and breadth [4]. Each leap in computing power coupled with theoretical advances hassled to yet deeper understanding of nonlinear phenomena, while simultaneously demonstrating how far we have yet to go. There are some parts of Mathematics, perhaps number theory and abstract algebra in which high standards of rigorous proof may be appropriate at all levels [9]. But in elementary differential equations a narrow elementary differential equations a narrow insistence on doctrinaire exactitude tends to squeeze the juice out of the subject.

Meanwhile, mathematicians use analysis to probe new applications and to develop numerical simulation algorithms that are provably accurate and efficient. Such capability is of considerable importance, given the explosion of experimental and observational data and the spectacular acceleration of computing power [5]. Leaving aside quantum mechanics, which remains to date an inherently linear theory, most real-world physical systems, including gas dynamics, fluid mechanics, elasticity, relativity, ecology, neurology, thermodynamics, and many more, are modeled by nonlinear partial differential equations [9]. Attempts to survey, in such a small space, even a tiny fraction of such an all-encompassing range of phenomena, methods, results, and mathematical developments, are doomed to failure. So we will be content to introduce a handful of prototypical, seminal examples that arise in the study of nonlinear waves and that serve to highlight some of the most significant physical and mathematical phenomena not encountered in simpler linear system of equations [6]. The importance of differential equations lies in the abundance of their occurrence and their utility in understanding the Sciences. Differential equations are merely mathematical representation of physical phenomenon [10][12]. The primary purpose of differential equation is to serve as a tool for the study of change in the physical world.

Many ODE can be solved by Power series method. But very a few literature is available for PDE [1]. Hence we have tried to verify the power series solution by considering some non-linear partial differential equation which have exact solution.

2. METHODOLOGY: POWER SERIES SOLUTION

In mathematics the power series is an infinite series that can be written as a polynomial with an infinite number of terms. In general the power series [3] can be written as,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad ,$$

where x is a variable and a_n are constants which are called co-efficient of the series. The power series will converge [7] for all the values of x in the series within certain interval, particularly, whenever the absolute value of x is less than some positive number *r*, known as radius of convergence.

In this paper we concentrate on the power series of two variables which can be written in the form of Taylor series. These series also have a center where they always converge to the function value. If the center be (x_0, y_0), then the series of two variable in the basic polynomial form is given by

$$\sum a_{ij}(x - x_0)^i (y - y_0)^j \quad ,$$

where the indices $i \; and \; j$ run (independently) from 0 to 1. For instance, the terms up to order 2 can be written as,

$$a_{00} + a_{10}(x - x_0) + a_{01}(y - y_0) + a_{20}(x - x_0)^2 + a_{11}(x - x_0)(y - y_0) + a_{02}(y - y_0)^2$$

In Calculus one learns to understand the behavior of all such quadratic polynomials in two variables. In order to write down a Taylor series completely, we need to explain how to get the coefficients a_{ij} in terms of the function. As before, the zero order or constant term is just $a_{00} = f(x_0, y_0)$ and the other terms involve derivatives of the function f evaluated at (x_0, y_0) and factorials.

Let us explain by considering an example, $f(x, y) = tan^{-1}(x)$ at the center point $(x_0, y_0) = (1,1)$.

To get a_{10}, treat y as a constant, and differentiate with respect to x. Substitute the center (x_0, y_0) into the resulting expression. The expression that is partial derivative w.r.t. x obtained by differentiating is denoted by f_x at x and may also simply be called the “first x-partial,” or the “first homogeneous x-partial”.

Next by evaluating this expression at $(x_0, y_0) = (1,1)$ we obtain the coefficient of the first order linear term in x is $a_{10} = {}^{-1}/_{2}$. In the same manner we can find a_{ij} for different values of i and j.

Here our intension is to check the solution obtained by power series which is same as its exact solution. So now by applying this power series method, we try to verify the series solution of the following examples which have already exact solutions.

3. EXAMPLE-1:

Consider the partial differential equation,

$$px + qy = pq\,, \tag{1}$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ are the partial derivatives.

The exact solution of this equation by Charpits method [2] is given by,

$$2cz = (cx + y)^2 + 2a\,, \tag{2}$$

where a and c are arbitrary constants. This equation can also be written as

$$Z = \frac{a}{c} + xy + \frac{c}{2}x^2 + \frac{1}{2c}y^2 + \cdots\,, \tag{3}$$

Equation (3) is in the series form. So we try to obtain the series solution using power series method which matches with equation (3).

Now let us assume the solution of (1) as a power series in x and y as below,

$$z(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\, x^i y^j\,, \tag{4}$$

by differentiating both sides of equation (4) with respect to x and y, we get the series expansion of p and q as follows,

$$p = \frac{\partial z}{\partial x} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{(i+1)j}\,(i+1)x^i y^j\,, \tag{5}$$

$$q = \frac{\partial z}{\partial y} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i(j+1)}(j+1)x^i y^j\,, \tag{6}$$

$$pq = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\sum_{p=0}^{i}\sum_{q=0}^{j}(p+1)(j-q+1)a_{(p+1)q}\,a_{(i-p)(j-q+1)}x^i y^j\right]. \tag{7}$$

Substituting (5), (6), (7) into equation(1) , we obtain the following recurrence relation,

$$a_{ij} = \frac{1}{i+j}\left[\sum_{p=0}^{i}\sum_{q=0}^{j}(p+1)(j-q+1)a_{(p+1)q}a_{(i-p)(j-q+1)}\right], \tag{8}$$

where

$$a_{i,0} = \begin{Bmatrix} \frac{a}{c} & if \;\; i=0 \\ \frac{c}{2} & if \;\; i=2 \\ 0 & otherwise \end{Bmatrix}. \tag{9}$$

By applying the recurrence relation (8) for several values of $i \; and \; j$ the polynomial approximate series solution obtained is as follows,

$$Z = \frac{a}{c} + \frac{c}{2}x^2 + xy + \frac{1}{2c}y^2 + \cdots \tag{10}$$

Comparing equation (10) with equation (3), the power series solution exactly matches with exact solution obtained by Charpits method.

4. EXAMPLE-2:

Consider the partial differential equation

$$p(1+q) = zq\,, \tag{11}$$

The exact solution of this equation is given by

$$z = \frac{1}{a} + ce^{(x+ay)}\,, \tag{12}$$

where a and c are arbitrary constants. By taking the arbitrary constants as a=1 and c=1 this equation (12) can also be written as series form in the following way,

$$z = 2 + x + y + \frac{x^2}{2} + \frac{y^2}{2} + xy + \frac{x^3}{6} + \frac{y^3}{6} + \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \frac{1}{6}xy^3 + \cdots \tag{13}$$

This series solution tells us that the given equation can be solved by power series method. Hence we apply power series solution method for two variables to example 2 and show that the obtained series exactly matches with that of equation (13).

Let us assume the solution of equation (11) as a power series in $x \; and \; y$ as below

$$z(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\, x^i y^j \ , \tag{14}$$

by differentiating both sides of equation (14) with respect to x and y , we get the series expansion of p and q as follows,

$$p = \frac{\partial z}{\partial x} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{(i+1)j}\,(i+1)x^i y^j \ , \tag{15}$$

$$q = \frac{\partial z}{\partial y} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i(j+1)}(j+1)x^i y^j \ , \tag{16}$$

$$zq = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\sum_{p=0}^{i}\sum_{q=0}^{j}(j-q+1)a_{pq}a_{(i-p)(j-q+1)}x^i y^j\right] . \tag{17}$$

Substituting equations (15), (16), (17) into equation (11), we obtain the following recurrence relation,

$$\boldsymbol{a_{(i+1)j} = \frac{1}{i+1}\left[\sum_{p=0}^{i}\sum_{q=0}^{j}(j-q+1)a_{(i-p)(j-q+1)}\left[a_{pq}-(p+1)a_{(p+1)q}\right]\right]}, \tag{18}$$

where

$$a_{ij} = \begin{Bmatrix} a_{00} = 2 & if & i=j=0 \\ a_{0j} = \dfrac{1}{j!} & if & i=0 \ and \ j\neq 0 \end{Bmatrix} . \tag{19}$$

Applying the recurrence relation (18) for several values of i and j, the polynomial approximate series solution obtained is as follows,

$$\boldsymbol{z = 2 + x + y + \frac{x^2}{2} + \frac{y^2}{2} + xy + \frac{x^3}{6} + \frac{y^3}{6} + \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \frac{1}{6}xy^3 + \cdots} \tag{20}$$

Comparing equation (20) with equation (13), the series solution obtained by power series method exactly matches with exact solution obtained.

5. RESULT AND CONCLUSION:

In this paper we verify that, it is possible to solve non-linear partial differential equation with the power series method. This power series method is a semi analytic technique that could permit to obtain, in an easier and exact way, the solution of difficult non-linear differential equation with an approximated closed form expression. It is the general technique to solve any kind of non-linear differential equation.

Here the solution of equation (1) and equation (11) obtained by power series method exactly matches with the exact solution (3) and (13) respectively. This verification forces us to conclude that the method works for almost all problems. But one should carefully analyze obtained series solution's authenticity by testing its convergence and uniqueness, which can be done by ratio test and Domb-Sykes plot.

6. ACKNOWLEDGEMENT:

We heartily thank University Grant Commission (UGC), SWRO, Bangalore for giving us an opportunity and providing financial assistance to carry out the project under MRP(S)-0361/13-14/KABA009/ UGC-SWRO.

7. REFERENCES:

[1] Ameina S. Nuseir (Jordan University of Science and Technology), Abeer Al-Hasoon (Jordan University of Science and Technology), "Power Series Solutions for Nonlinear Systems of Partial Differential Equations", Applied Mathematical Sciences, Vol. 6, no. 104, 5147 – 5159, 2012.

[2] George F. Simmons, "Differential Equations with Applications and Historical notes", T.M.H Edition, 1974.

[3] Walter Rudin, "Real analysis", Macgraw Hills, 3rd Edition, 1976.

[4] P.L.Sachdev, "Nonlinear ordinary differential equations and their applications", IISc, Bangalore, India, Maacel Dekker, Inc, 1990.

[5] Emmanuelle Di Benedetto, "Partial Differential equation", 2nd edition, 2010.

[6] Richard.S.Palasis, "Differential Equations Mechanics and Computation", American Mathematical Society, Volume 51, First Indian Edition 2012.

[7] N. L. Carothers, "Real Analysis", Cambridge University Press, 2000.

[8] D. W. Jordan and P. Smith, "Non-linear Differential Equations", Clarendon, Oxford, 1977.

[9] Mark A. Pinsky, "Partial Differential equations and Boundary value problems", third Edition, American Mathematical Society, 2013.

[10] Y.L.Luke, "Mathematical Functions and Their Approximation", Academic, Newyork, 1975.

[11] Lawrence C. Evans, "Partial Differential Equations", 2nd Edition, Graduate Studies in Mathematics, Volume 19, 2014.

[12] Stuart P.Hastings and J. Bryce McLeod, "Classical Methods in Ordinary Differential Equations", American Mathematical Society, Volume129, First Indian Edition 2013.