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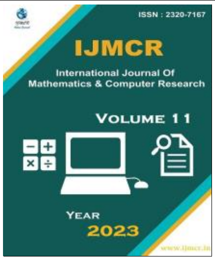

A Note on Visible Sets

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ABSTRACT
In this paper, we investigate some properties of visible sets and its characterization. Also we obtained relation between visible set and union of convex sets with nonempty intersection.

KEYWORDS: visible set, convex set, balanced set, absorbing set

1. INTRODUCTION

In this paper, we study the concept of visible set which can be considered as the generalizations of convex sets [1,7]. The aim of this work is to look at which of the properties of convex sets are extendable to that of properties of visible sets and what additional properties does this set possess. We investigated some characteristics of the mentioned set. Accordingly, we have seen that some of algebraic properties of convex sets are not extendable to those visible sets. For example, the intersection of visible sets is not visible set and union of arbitrary convex sets with nonempty common is always visible set which might not be convex set. The most remarkable result is that every visible set can be expressed as the union of convex sets. In addition, we tried to develop the conditions that enable us to determine whether the given visible set can be expressed as the union of finite number of convex sets or not.

2. PRELIMINARIES

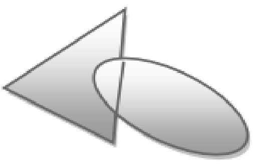
Definition 2.1 (visible points in sets) [6] **:** Two points in a set V are said to be visible to each other with respect to V if the line segment determined by them lies in the set V.

Definition 2.2 (Visible set) [6]**:** A set V is said to be visible set if there exists a point x in V such that each other point in V is visible to it. If such an x exists then it is called visible center of the set V and it may not be unique.

From the definition of visible set, we can easily verify that every convex set is a visible set. Therefore, convex set can be redefined as a visible set. Every point in convex set V is visible centre y of the set V.

Example 1: If $X = R^2$ and $V = \{(x, y): 0 \le x \le 1$ and $y = 0 \; or \; 0 \le y \le 1 \; and \; x = 0\}$,
then V is a visible set with visible center $x_o = (0,0)$.

Example 2:. a)



b)

c)

The sets in (a) and (b) represents a visible set while (c) does not. Note that (a), (b) and (c) are convex sets.

Example 3: Let X be a normed linear space and let $V_1 = \{x \in X: \|x\| \le 1\}$,
$V_2 = \{y \in X: \|y - y_0\| \le 1, y_0 \text{ not in } V_1\}$. If $V_1 \cap V_2 \neq \emptyset$, then $V = V_1 \cup V_2$ is a visible set.

Proof: Let $V_1 \cap V_2 \neq \emptyset, x_o \in V_1 \cap V_2$, $\alpha \in [0,1]$ and let $y \in V$ be arbitrary element in V. We need to show that $\alpha x_o + (1-\alpha)y \in V$. But $y \in V$ implies either $y \in V_1$ or $y \in V_2$ or $y \in V_1 \cap V_2$.

If $y \in V_1$ then $\|y\| \le 1$ and

$$\|\alpha x_o + (1-\alpha)y\| \le \alpha\|x_o\| + (1-\alpha)\|y\| \le \alpha + (1-\alpha) = 1.$$

Thus $\alpha x_o + (1-\alpha)y \in V_1$ and hence in V.

If $y \in V_2$ then $\|y_o - (\alpha x_o + (1-\alpha)y)\| = \|y_o - (\alpha x_o + (1-\alpha)y) + \alpha y_o - \alpha y_o\|$

$\leq (1-\alpha)\|y_o - y\| + \alpha\|y_o - x_o\| = 1$.
Thus $\alpha x_o + (1-\alpha)y \in V_2$ and hence $\alpha x_o + (1-\alpha)y$ lies in V.
Similarly, we can verify the other case.
Example (2(a)) shows that if two convex sets have nonempty intersection, then their union is visible set.

3. SOME PROPERTIES OF VISIBLE SETS AND ITS CHARACTERIZATION

Definition 3.1: Let X be a linear space over a field F and let V be a visible set in X. For $y \in X$, $\alpha \in F$ and $\emptyset \neq A \subseteq X$, we define

(i) $V + y = \{x + y : x \in V\}$
(ii) $\alpha V = \{\alpha x : x \in V\}$
(iii) $V + A = \{x + y : x \in V, y \in A\}$.

Theorem 3.2: Let X be a linear space over a field F and let V and M are visible sets in X. If $y \in X$, $\alpha \in F$ then (i) $V + y$ (ii) αV (iii) $V + M$ are visible sets.

Proof: (i) Since V is visible set, there is at least one point x_o in V such that, $\alpha x_o + (1-\alpha)z \in V$ for all z in V and for all $\in [0,1]$.
Claim: $x_o + y$ is a visible center for $V + y$.
If $z \in V + y$, then $z = k + y$ *for some* $k \in V$. So, for $\alpha \in [0,1]$, we have
$\alpha(x_o + y) + (1-\alpha)z = \alpha(x_o + y) + (1-\alpha)(k + y) = \alpha x_o + (1-\alpha)k + y \in V + y$. Since z is arbitrary, $x_o + y$ is a visible center for $V + y$. Thus $V + y$ is a visible set. Similarly αx_o is a visible center for αV so that (ii) holds true.
(iii). Let x_o *and* y_o be visible centers for V and M respectively.

Claim : $x_o + y_o$ is a visible center for $V + M$.
Let $z \in V + M$ then $z = x + y$ for some $x \in V$ *and* $y \in M$.
For $\alpha \in [0,1]$, we have
$\alpha(x_o + y_o) + (1-\alpha)(x + y\ ) = \alpha x_0 + (1-\alpha)x + \alpha y_0 + (1-\alpha)y \in V + M$. Since z is arbitrary, $x_o + y_o$ is visible center for $M + V$, showing V + M is a visible set.

Theorem 3.3 : Let X be a linear space and let $\emptyset \neq V \subseteq X$. Then V is visible set if and only if V is union of convex sets with nonempty intersection.

Proof : Without loss of generality we assume the index set countable.
Suppose $\{V_i : i \in I, I$ an index set $\}$ is the collection of convex sets such that $\bigcap_{i\in I} V_i \neq \emptyset$. Let $V = \bigcup_i V_i$. We need to show V is visible set. *Let* $\alpha \in [0,1]$, $z \in V$ be arbitrary. Then by definition of V, there exists $j \in I$ such that $z \in V_j$. Since $\bigcap_{i\in I} V_i \neq \emptyset$ there exists $x_o \in V_i\ \forall\ i \in I$ and $\alpha x_0 + (1-\alpha)z \in V_j$.
Therefore $\alpha x_0 + (1-\alpha)z \in V$. Since z is arbitrary we have x_o is a visible center for V, consequently V is a visible set.

Conversely, let V be a visible set. Here now construct the collection of convex sets such that $V = \bigcup_{i\in I} V_i$ and $\bigcap_{i\in I} V_i \neq \emptyset$.
Notation: x $\sim$y, we mean x and y are visible to each other with respect to a given set.
Now we have different cases to consider.
Case I: (Every member of V is a visible center for V). In this case by definition V is convex set and we have done.
Case II: Suppose V has a unique visible center, say, x_0 **:** If $V = \{x_0\}$, V is convex set and hence the Theorem. If $V \neq \{x_0\}$, then for $y_1 \in V$, $y_1 \neq x_0$, define $V_1 = \{y \in V : y \sim y_1$ with $x \sim y_1$ and $y \sim y_1 \Longrightarrow y \sim x\}$. $V_1 \neq \emptyset$, because $y_1 , x_o \in V_1$ and $x_o \sim y_1$. Given x *, and* $y \in V_1$, we have $x \sim y_1$ *and* $y \sim y_1 \Longrightarrow x \sim y$ for all $x, y \in V_1$ (by definition of V_1) .Thus every member of V_1 is a visible center forV_1. Therefore V_1 is convex set.
If $V = V_1$, we have done; otherwise choose $y_2 \in V_1^C$ (complement is taken relative to V) and define $V_k = \{y \in V : y \sim y_k$ with $x \sim y_k$ and $y \sim y_k \Longrightarrow x \sim y\}$. Similar to the case I, V_2 is convex and $V_2 \neq \emptyset$
If $V = V_1 \cup V_2$, we have done; otherwise we continue by the same manner as
$V_k = \{y \in V : y \sim y_k\ with\ x \sim y_k\ and\ y \sim y_k \Longrightarrow x \sim y\}$,
$y_k \in \left(\bigcup_{i=1}^{k-1} V_i\right)^C$, $k \geq 2$.
Put $B = \bigcup_{i\in I} V_i$, where I is the index set composed of the preceding procedure. We want to show that $V = B$. The definition of V_i's , $i \in I$shows that $B \subseteq V$.So , we need to show that $V \subseteq B$.
If $z \notin B$, then z is not member of V_i , $i \in I$. Since $x_o \in V_i$ *for all* $i \in I$, we have z is not visible for x_o . That is $z \notin V$ .Thus, $V \subseteq B$. therefore, $V = B$.

Case III: Suppose V contains more than one visible centers:
Let $A = \{x \in V :\ x$ is a visible center of $V\}$. Similar to **case II**, we can construct collection of convex sets containing set A such that $V = \bigcup_{i\in I} V_i$ and $\bigcap_{i\in I} V_i = A$. Hence the theorem is proved.
It worth to note that, each V_k contains at least y_k and x_o. Consequently V_k contains infinitely many vectors, provide that $V_k \neq x_0$.
Note 3.4 : Closure of any convex set is convex.

Corollary 3.5 : If V is visible set, then its closure is visible set.
Proof : By Theorem 3.3, there exists convex sets V_i such that $V = \bigcup_{i\in I} V_i$. Since $cl(V_i)$is convex for each i, $\bigcap_{i\in I} cl(V_i) \neq \emptyset$, and $cl(V) = \bigcup cl(V)_i$, closure of V is visible set.
It is well-known fact that the intersection of convex sets is a convex set. But the neither union nor intersection of visible sets is visible set in general.
Remark 3.6: In general, (i) Intersection of visible sets may not be visible set.
(ii) Union of visible sets is not visible set.

According to Theorem 3.3, Every Visible set can be expressed as union of convex sets with nonempty intersection.

Now the question is "is it possible to express a given visible set as a union of finite number of convex sets ? " As the following result reveals, the answer to the above question depends strongly on the behavior of the boundary set of a given visible set.

Definition 3.7: Let X be a linear space and let $\emptyset \neq V \subset X$ bounded closed visible set. The boundary of V, denoted by ∂V is defined as ∂V :={x∈V : every neighborhood U_x of x contains a point $y \notin V, and\ z \in V$}.

Definition 3.8: (Interior of a visible set): Let X a linear space and $\emptyset \neq V \subset X$ be visible set. Interior of V, denoted by int(V) is defined as $intV :=\{x \in V$: there exists a neighborhood $U_x \subset V$ of x such that $U_x \cap \partial V = \emptyset\}$.

If V is open and bounded, then the boundary of V coincides with the boundary of closure of V. Hence we can define the boundary of an open visible set with respect to its closure.

We call a collection $\{A_i : i \in I\}$ partition of boundary set if $\partial V = \cup A_i$, and $A_i \cap A_j , i \neq j$ have at most finite common points.

Theorem 3.9 : Let X be a linear space and let $\emptyset \neq V \subset X$ be a bounded visible set. Then V is expressed as the union of finite number of convex sets if and only if there exists a finite collection $\{A_i : i \in I\}$ (partition of boundary set) such that for $x, y \in A_i$, $\alpha x + (1-\alpha)y \in V$, $\alpha \in [0,1]$.

Proof: If $V = \cup_{i=1}^{n} V_i$, then define $A_i = \partial V_i \Longrightarrow \forall x, y \in A_i , \alpha x + (1-\alpha)y \in V_i \Longrightarrow \alpha x + (1-\alpha)y \in V$. Since, there are finite V_i's, we have finite A_i's with $\partial V = \cup_{i=1}^{n} A_i$.

Conversely, suppose $\partial V = \cup_{i=1}^{n} A_i$ and $\forall x, y \in A_i , \alpha x + (1-\alpha)y \in A_i$, we need to construct a collection of convex sets $\{V_i : i = 1,2,..,n\}$ such that $V = \cup_{i=1}^{n} V_i$.

Let V_i ={$x \in V$:x is visible to some vector in A_i such that if x and y are visible to some vector in A_i⟹x is visible to y}

Since, $A_i \subset V_i$, $V_i \neq \emptyset$ and V_i is convex set, moreover, $\cap_{i=1}^{n} V_i \neq \emptyset$ (since $A_i \subset V_i$ and V_i contains every visible center of V).

Claim: $\boldsymbol{V = \cup_{i=1}^{n} V_i}$:Clearly, $\cup_{i=1}^{n} V_i \subset V$. So, we need to show that $\cup_{i=1}^{n} V_i \supset V$. Now suppose $x \notin \cup_{i=1}^{n} V_i \Longrightarrow x \notin A_i \Longrightarrow x$ is not visible in any vector $A_i , \forall\, i, i = 1,2, \dots, n \Longrightarrow x \notin V$ (each member of V is visible to a vector in boundary set).Thus, $\cup_{i=1}^{n} V_i \supset V$, and hence the theorem is proved.

The preceding two theorems show that, every visible set is the union of convex sets. Moreover, if the boundary set of a visible set satisfies some conditions, then it can be expressed as the union of finite number of convex sets.

Example 3.10: Let $X = R^2$

a. $V_1 = \{(x, y) \in X: 2y \le 5x, 3y \ge 4x, (x-2)^2 + (y-3)^2 \le 4\} \cup$
$\left\{(x, y) \in X: 2y \le 5x, 3y \ge 4x, 0 \le y \le \frac{45}{29}, 0 \le x \le 2\right\}$
and $V_2 = \{(x, y) \in X: 4x \le 3y, 3x \ge y, y \ge 0\}$. If $V = V_1 + V_2$. Since V_i , i=1,2 is convex and $V_1 \cap V_2 = \{(0,0)\}$, V is visible set . If $A_i = \partial V_i, i = 1,2$, then for all, $x, y \in A_I \Longrightarrow \alpha x + (1-\alpha)y \in V_i$ ($\because V_i'$s are convex) $\Longrightarrow \alpha x + (1-\alpha)y \in V$. Hence A_i's satisfied the conditions of Theorem 3.9 and boundary set has shape as

b. $V_1 = \{(x, y) \in X: 2y \le 5x, 3y \ge 4x, (x-6)^2 + (y-8)^2 \ge 4 , x \le 6\}$ and
$V_2 = \{(x, y) \in X: 4x \le 3y, 3x \ge y, y \ge 0\}$. If $V = V_1 + V_2$ Since V_i's, i=1, 2 are visible sets and $V_1 \cap V_2 = \{(0,0)\}$, V is visible set. Consider a portion of boundary set ∂V, $A_1 = \{(x, y) \in X: 2y \le 5x, 3y \ge 4x, (x-6)^2 + (y-8)^2 = 4 , x \le 6\}$, then $x = (6,6)$, $y = (6,10) \in A_I$ implies that $\frac{1}{2}x + \frac{1}{2}y = (6,8) \notin V_1 \Longrightarrow \frac{1}{2}x + \frac{1}{2}y = (6,,8) \notin V$.

Hence A_i's do not satisfy the conditions of Theorem 3.9. Here V can't be expressed as union of finite number of convex sets. Indeed, $V = V_2 \cup \left(\cup_{x \in A_1}\{\alpha x: \alpha \in [0,1]\}\right)$.

Theorem 3.11 : Let$X$ be a linear space,Vbe nonempty visible subset of X, and $y_o \in V$ be a visible center. Given $y \in V$, there are vectors $y_1 , y_2 , y_3 , \dots, y_n$ and $\beta_0 , \beta_1 , \beta_2 , \dots, \beta_n \in [0,1]$ such that $y = \sum_{j=0}^{n} \beta_j y_j$ and $1 = \sum_{j=0}^{n} \beta_j$

Proof : Given $y \in V$ and $y_o \in V$ a visible center for V, there is a vector $z \in V$ such that $y = \beta y_0 + (1-\beta)z$. Since V is a visible set there is a collection of convex set $\{V_i : i \in I\}$ such that $V = \cup_{i \in I} V_i$ and $\cap_{i \in I} V_i \neq \emptyset$. Therefore there exists,$i \in I$ such that $z \in V_i$. Consequently, there exist vectors $y_1 , y_2 , y_3 , \dots, y_n$ $in\ V_i$ and $\alpha_1 , \alpha_2 , \dots, \alpha_n \in [0,1]$ such that$z = \sum_{j=1}^{n} \alpha_j y_j$,$\sum_{j=1}^{n} \alpha_j = 1$ (because V_i is a convex set)

But $y = \beta y_o + (1-\beta)z = \beta y_o + (1-\beta)\sum_{j=1}^{n} \alpha_j y_j = \beta y_o + \sum_{j=1}^{n} (1-\beta)\alpha_j y_j = \sum_{j=0}^{n} \beta_j y_j$

where $\beta_0 = \beta$, $\beta_j = (1-\beta)\alpha_j$, $j \ge 1$.

Clearly, $\beta_j \in [0,1]$ and $\sum_{j=0}^{n} \beta_j = 1$.

Hence the result holds true.

Definition 3.12: A nonempty subset V of linear space X is said to be

a. Balanced set if $x \in V$, and $\alpha \in F$ $with$ $|\alpha| \le 1$, $then\ \alpha x \in V$.
b. Absorbing set if for every $x \in X$ there exists $r > 0$ such that $x \in rV$.

Theorem 3.13 : Let X be a linear space and let V be nonempty visible and balanced subset of X. Then

a) If $x_o \in V$ is a visible center for V, then $(-x)$ is also visible center for V.
b) 0 is a visible center for V.

Proof : a) Since V is balanced set, we have $-V = V$. Let $x_0 \in V$ be a visible center for V. Since αx_o is a visible center for αV , $\forall \alpha \in F$, we have $-x_0$ is a visible center for V.

b) Since V is nonempty balanced set, $0 \in V$ (taking $\alpha = 0$). Consequently for all $\alpha \in [0,1]$ and for arbitrary x in V $\alpha 0 + (1-\alpha)x \in V$.

Theorem 3.14 : Let Xbe a linear space and let V, V_1, V_2 be nonempty visible and balanced sets in X.Then

a) For any $\alpha \in F$, αV a visible and balanced set.
b) V_1+V_2 is visible and balanced set.
c) 0 is a visible center for V_1+V_2

Proof : We apply Theorem 1 and definition of balanced set.

Theorem 3.15 : Let X be a linear space and let V_1, V_2 be nonempty visible and balanced sets in X. If $r_1, r_2 > 0$, then $r_1V_1 + r_2V_2 \subseteq (r_1+r_2)(V_1+V_2)$.

Proof : Let $y \in r_1V_1+r_2V_2$, where $r_i > 0, i = 1,2$. Then $y = r_1x_1+r_2x_2$, $x_i \in V_i$, $i = 1,2$. If x_o, y_o are visible centers for V_1 and V_2 respectively, then $r_1x_o + r_2y_o$ is a visible center for $r_1V_1 + r_2V_2$. Thus, $z = \alpha(r_1x_o+r_2y_o) + (1-\alpha)y \in r_1V_1+r_2V_2$ for any $\alpha \in [0,1]$ (because $r_1V_1+r_2V_2$ is visible set with visible center $r_1x_o+r_2y_o$). But $z = \alpha(r_1x_o+r_2y_o) + (1-\alpha)(r_1x_1+r_2x_2)$

$$= r_1(\alpha x_o+(1-\alpha)x_1) + r_2(\alpha yo + (1-\alpha)x_2).$$

Since V_i, $i = 1,2$ are balanced sets, we have $\frac{z}{r_1+r_2} = \frac{r_1}{r_1+r_2}(\alpha x_o+(1-\alpha)x_1) + \frac{r_2}{r_1+r_2}(\alpha yo + (1-\alpha)x_2) \in V_1+V_2$.

This implies that $z \in (r_1+r_2)(V_1+V_2)$, hence the theorem is proved.

Theorem 3.16 : Let X be a linear space and $\emptyset \neq V \subseteq X$. Suppose V is visible, balanced and absorbing set such that every member of V is a visible center for V. For each x in X, define $\|x\| = \inf\left\{r > 0 : \frac{x}{r} \in V\right\}$.

If V does not contain a nonzero subspace of X, then $\|.\|$ is a norm on X and

$$\{x \in X: \|x\| < 1\} \subseteq V \subseteq \{x \in X: \|x\| \leq 1\}.$$

Proof : We need to show that

i) $\|x\| \geq 0 \quad \forall x \in X$
ii) $\|x\| = 0 \Leftrightarrow x = 0$
iii) $\|\alpha x\| = |\alpha|\|x\| \ \forall x \in X \ and \ \forall \ \alpha \epsilon F$
iv) $\|x+y\| \leq \|x\| + \|y\| \ for \ \forall x, y \in X$.

Let $S_x = \left\{r > 0: \frac{x}{r} \in V\right\}$.$\|x\| = inf S_x$, by definition of $\|x\|$.

Since $S_x \subseteq (0,\infty) \Rightarrow \|x\| \geq 0, \forall x \in X$. Proving (i).

If $x \in V$, then $-x \in V$ (because V is balanced set). Since every member of V is visible center for, $0 = \frac{x}{2} + \frac{-x}{2} \in V \Rightarrow \frac{0}{r} \in V, \forall r > 0$.

Thus, $\|0\| = inf S_0 = inf(0,\infty) = 0$

Now let $0 \neq x \in X$. Then $Y = \{\alpha x : \alpha \in F\}$ is nonzero subspace of X. Since V contains only the zero subspace, Y is not contained in V. Thus there exists an α_1 in F such that $\alpha_1 x \notin V$. Since $0 \in V$, $\alpha_1 x \neq 0 \Rightarrow \alpha_1 \neq 0$. Suppose That $0 < r < \frac{1}{|\alpha_1|}$ and that $r \in S_x$ then $\frac{x}{r} \in V$ (by definition of S_x) , and hence $\alpha_1 x = (\alpha_1 r)\frac{x}{r} \epsilon V$ (because V is balanced set and $|\alpha_1 r| < 1$). This contradicts to $\alpha_1 x \notin V$. This contradiction shows that $r \in S_x \Rightarrow r > \frac{1}{|\alpha_1|} > 0$. Hence $\|x\| = 0 \Leftrightarrow x = 0$. (ii) is proved. Let $r \neq 0$, and $r \in S_{\alpha x}$. Then $\frac{\alpha x}{r} \in V$ (by definition ofS_x) $\Rightarrow \frac{|\alpha|}{r}x = \frac{|\alpha|}{\alpha}\left(\frac{\alpha x}{r}\right) \in V$

$\Rightarrow \frac{r}{|\alpha|} \epsilon S_x \Rightarrow \|x\| \leq \frac{r}{|\alpha|} \Rightarrow |\alpha|\|x\| \leq r, \forall r \in S_{\alpha x} \Rightarrow |\alpha|\|x\| \leq \|\alpha x\|$ (*)

Now by changing α to $\frac{1}{\alpha}$ and x to αx in (*) we obtain,

$\frac{1}{|\alpha|}\|\alpha x\| \leq \left\|\frac{1}{\alpha}(\alpha x)\right\| \Rightarrow \frac{\|\alpha x\|}{|\alpha|} \leq \|x\| \Rightarrow \|\alpha x\| \leq |\alpha|\|x\|$ (**)

By combining (*) and (**) we get $\|\alpha x\| = |\alpha|\|x\|$. Since$\|0\| = 0$, $0x = 0$, we have $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in F$ for all $x \ in \ X$. proved (iii).

Let $x, y \in X$. Given, $\varepsilon > 0$, we can find $r_1 \in S_x$ and $r_2 \in S_y$ such that $r_1 < \|x\| + \varepsilon$ and $r_2 < \|y\| + \varepsilon$. Since $r_1 \in S_x$ and $r_2 \in S_y$, we have $\frac{x}{r_1} \in V$ and

$$\frac{y}{r_2} \in V \Rightarrow \frac{x+y}{r_1+r_2} = \frac{r_1}{r_1+r_2}\left(\frac{x}{r_1}\right) + \frac{r_2}{r_1+r_2}\left(\frac{y}{r_2}\right) \in V$$

(because V is visible set and every member of V is a visible center for V)

$$\Rightarrow \|x+y\| \leq r_1 + r_2 < \|x\| + \|y\| + 2\varepsilon$$

By letting $\varepsilon \to 0$, we obtain that $\|x+y\| \leq \|x\| + \|y\|$ for all x, y in X. Thus $\|.\|$ is a norm on X.

Now the question is what will happen is if every point of V is not visible center in the preceding Theorem? As the following Theorem shows that we can find similar results even if the condition "every member of V is a visible center is omitted " on the set $+V$.

Theorem 3.17 : let X be a linear space over the field F and $\emptyset \neq V \subseteq X$. Suppose V is visible, balanced and absorbing set. For each x in X, define

$$\|x\| = inf\left\{r > 0 : \frac{x}{r} \in V + V\right\}.$$

If V does not contain a nonzero subspace of X, then $\|.\|$ is a norm on X and

$$\{x \in X: \|x\| < 1\} \subseteq V + V \subseteq \{x \in X: \|x\| \leq 1\}.$$

Proof : Similar to Theorem 7, we need to show that

i) $\|x\| \geq 0 \quad \forall x \in X$
ii) $\|x\| = 0 \Leftrightarrow x = 0$
iii) $\|\alpha x\| = |\alpha|\|x\| \ \forall x \in Xand \ \forall \ \alpha \epsilon F$
iv) $\|x+y\| \leq \|x\| + \|y\| \ for \ \forall x, y \in X$.

Let $S_x = \left\{r > 0: \frac{x}{r} \in V + V\right\}$.$\|x\| = inf S_x$, by definition of $\|x\|$.

Since $S_x \subseteq (0,\infty) \Rightarrow \|x\| \geq 0, \forall x \in X$. Thus (i) is proved. Since V is visible and balanced, we have $V + V$ is visible and balanced set and $0 \in V + V$(by Theorem 3.14. Thus $0 \in V + V \Rightarrow \frac{0}{r} \in V + V, \forall r > 0$. Thus, $\|0\| = inf S_0 = inf(0,\infty) = 0$

Now let $0 \neq x \in X$. Then $M = \{\alpha x : \alpha \in F\}$ is non zero subspace of X. Since V does not contain any nonzero subspace of X, $V + V$ does not contain any nonzero subspace of X. Therefore, M is not subset of $V + V$. So, there exists $\alpha_1 \in F$ such that $\alpha_1 x \notin V + V$.

Since, $0 \in V + V \implies \alpha_1 x \neq 0 \implies \alpha_1 \neq 0$.

Suppose That $0 < r < \frac{1}{|\alpha_1|}$ and that $r \in S_x$, then $\frac{x}{r} \in V + V$ (by definition of S_x), and hence $\alpha_1 x = (\alpha_1 r)\frac{x}{r} \epsilon V + V$ (because $V + V$ is balanced set and $|\alpha_1 r| < 1$).

This contradicts to $\alpha_1 x \notin V + V$. This contradiction shows that $\in S_x \implies r > \frac{1}{|\alpha_1|} > 0$. Hence $\|x\| = 0 \Leftrightarrow x = 0$.

Hence (ii) is proved.

Let $\alpha \neq 0$ and $\alpha \in S_{\alpha x}$. Then $\frac{\alpha x}{r} \in V + V$ (by definition ofS_x).

$\implies \frac{|\alpha|}{r} x = \frac{|\alpha|}{\alpha}\left(\frac{\alpha x}{r}\right) \in V + V$ (Since $V + V$ is balanced and $\left|\frac{|\alpha|}{\alpha}\right| = 1$)

$\implies \frac{r}{|\alpha|} \epsilon S_x \implies \|x\| \leq \frac{r}{|\alpha|} \implies |\alpha|\|x\| \leq r$, $\forall r \in S_{\alpha x} \implies |\alpha|\|x\| \leq \|\alpha x\|$. (*)

Now by changing α to $\frac{1}{\alpha}$ and x to αx in (*) , we obtain,

$$\frac{1}{|\alpha|}\|\alpha x\| \leq \left\|\frac{1}{\alpha}(\alpha x)\right\| \implies \frac{\|\alpha x\|}{|\alpha|} \leq \|x\|$$

$$\implies \|\alpha x\| \leq |\alpha|\|x\| \quad (**)$$

By combining (*) and (**) we get $\|\alpha x\| = |\alpha|\|x\|$. Since$\|0\| = 0$, $0x = 0$,

we have $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in F$ for all x in X. Thus we proved (iii).

Let $x, y \in X$ be arbitrary points .Since V is absorbing set, there are positive numbers m_1, m_2 such that $\frac{x}{m_1}, \frac{y}{m_2} \in V$.

Moreover, $V \subseteq V + V$ (because $0 \in V$ as V is absorbing).

Thus $\frac{x}{m_1}, \frac{y}{m_2} \in V \implies \frac{x}{m_1}, \frac{y}{m_2} \in V + V$. So given $\varepsilon > 0$, we can find $r_1, r_2 > 0$ such that

$r_1 \in S_x$, and $r_2 \in S_y$ such that $\frac{x}{r_1}, \frac{y}{r_2} \in V$ and $r_1 < \|x\| + \frac{\varepsilon}{2}$ and $r_2 < \|y\| + \frac{\varepsilon}{2}$.

But $\frac{x}{r_1}, \frac{y}{r_2} \in V \implies \frac{x}{r_1}, \frac{y}{r_2} \in V + V \implies r_1 \in S_x$, and $r_2 \in S_y$ (by definition of S_x and S_y).

Since $0 \in V$ is a visible center for V (as V is balanced set), there exist x_1,and x_2 such that

$$\frac{x}{r_1} = \lambda o + (1 - \lambda)x_1 \text{ , and } \frac{y}{r_2} = \mu o +$$

(V is visible set and 0 is visible center for V)

$$\implies x = r_1(1 - \lambda)x_1 \text{ , and } y = r_2(1 - \mu)x_2$$

$$\implies \frac{x+y}{r_1+r_2} = \frac{r_1}{r_1+r_2}\left((1 - \lambda)x_1\right) + \frac{r_2}{r_1+r_2}(r_2(1 - \mu)x_2) \in V + V$$

(because V is balanced set).

$$\implies r_1 + r_2 \in S_{x+y} \implies \|x + y\| \leq r_1 + r_2 < \|x\| + \|y\| + \varepsilon$$

$$\implies \|x + y\| \leq \|x\| + \|y\| \text{ , by letting } \varepsilon \to 0 \ (\varepsilon \text{ was arbitrary}).$$

Hence the Theorem is proved.

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