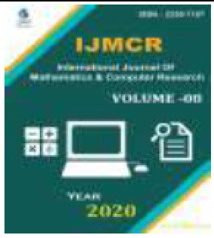
International Journal of Mathematics and Computer Research
ISSN: 2320-7167
Volume 11 Issue 06 June 2023, Page no. – 3479-3493
Index Copernicus ICV: 57.55, Impact Factor: 7.362
DOI: 10.47191/ijmcr/v11i6.03



Strongly Hopfian Objets and Strongly Cohopfian Objets in the Categories of $AGr(A-Mod)$ and $COMP\ (AGr(A-Mod))$

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Abstract

1. Let M be a graded left$A$-module and M_* the associate complex of M. Then :
   (a) If M_* is noetherian (resp. artinian) then M_* is strongly hopfian (resp. strongly cohopfian) ;
   (b) If M_* is strongly hopfian (resp. cohopfian), then M_* is hopfian (resp. cohopfian) ;
2. Let M be a graded left$A$-module, M_* the associate complex of M, N a submodule of M, and N_* fully invariant subcomplex of M_*. Then :
   If N_* and M_*/N_* strongly hopfian, then M_* is strongly hopfian.
3. Let M a graded left$A$-module, N a submodule of M and M_* the associate complex of M. Then :
   (a) if all subcomplex of M_* is cohopfian, then M_* is cohopfian.
   (b) if M_*/N_* is strongly hopfian, then M_* is strongly hopfian

Keywords :
Graded rings, graded module, category, sequence complex, chains complex, strongly cohopfian objects, and strongly hopfian object.

1 Introduction

In this paper, the ring A is supposed to be associative, unitary and not necessarily commutative, every left A-module is unifere.
The aim of this article is to study the strongly hopfian and strongly cohopfian objects in the category of graded left $A-$modules denoted $AGr(A-Mod)$ and in the category of associated complex of graded left $A-$modules denoted $COMP(AGr(A-Mod))$. In particular we give conditions over M_* and N_* such that M_*/N_* be strongly cohopfian (respectively strongly hopfian) and conditions over M_*/N_* and N_* such that M_* be strongly hopfian.
We define $AGr(A-Mod)$ and $COMP(AGr(A-Mod))$:

1. The category of graded of left A-modules denoted $AGr(A-Mod)$ where :
   (a) The objects are the graded left $A-$modules ;
   (b) The morphisms are the graded morphisms..
2. the category of complexes associate of graded left A-modules denoted $COMP(AGr(A-Mod))$ where
   (a) the objects are the complex sequences associated to a left A-modules ;
   (b) the morphisms are the complex chains associated to a graded morphisms.

The principal results of this article are given in the third and forth section, here are the results :

1. Let M be a graded left$A$-module and M_* the associate complex of M. Then :
   (a) If M_* is noetherian (resp. artinian) then M_* is strongly hopfian (resp. strongly cohopfian) ;
   (b) If M_* is strongly hopfian (resp. cohopfian), then M_* is hopfian (resp. cohopfian) ;
2. Let M be a graded left$A$-module, M_* the associate complex of M, N a submodule of M, and N_* fully invariant subcomplex of M_*. Then :
   If N_* and M_*/N_* strongly hopfian, then M_* is strongly hopfian.
3. Let M a graded left$A$-module, N a submodule of M and M_* the associate complex of M. Then :
   (a) if all subcomplex of M_* is cohopfian, then M_* is cohopfian.
   (b) if M_*/N_* is strongly hopfian, then M_* is strongly hopfian

2 Reminder and preliminary results

Définition 2.1
Let A be a ring, we say that A is a graded ring if there exists a family $\{A_n\}_{n\in\mathbb{Z}}$ of additive subgroup of A such that

1. $A=\bigoplus_{n\in\mathbb{Z}} A_n$;
2. $A_n\cdot A_m\subset A_{n+m},\ \forall\ n,\ m\in\mathbb{Z}$.

In all that follows, A and M are supposed unitary.

Remarque 2.1
Let A be a graded ring.
We say that A is positively graded if $A_n=0,\forall n<0$.

Définition 2.2
Let $A=\bigoplus_{n\in\mathbb{Z}}A_n$ be a graded ring and M be a left $A-$module, we say that M is a graded left $A-$module if there exists a suite $(M_n)_{n\in\mathbb{Z}}$ of sub-group of M such that :

1. $M=\bigoplus_{n\in\mathbb{Z}}M_n$;
2. $A_n\cdot M_d\subset M_{n+d},\ \forall\ n,\ d\in\mathbb{Z}$.

Définition 2.3
Lets $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *be a graded left* $A-$*module and* N *is a sub-module of* M*, then we say that* N *is a graded sub-module of* M*, if* $\forall x = \sum_{n\in\mathbb{Z}} x_n \in N$*, with* $x_n \in M_n$*, then* $x_n \in N$*,* $\forall n \in \mathbb{Z}$*.*

Proposition 2.1
Lets $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring and* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *is graded left* $A-$*module , then for all* $n \in \mathbb{Z}$ *fixed, we have*

$$M(n) = \bigoplus_{k\geq n} M_k = \bigoplus_{k\in\mathbb{N}} M_{n+k}$$

is a graded sub-module of M *and we have the descendant sequence :*

$$\cdots M(n+2) \subset M(n+1) \subset M(n) \subset \cdots.$$

Proof
For all $n \in \mathbb{Z}$ fixed, $M(n) = \bigoplus_{k\geq n} M_k$ is a sub-group of M and

$$A_s \cdot M(n)_k = A_s \cdot M_{n+k} \subset M_{n+k+s} = M_{n+(k+s)} = M(n)_{k+s}.$$

In the other hand, it suffices to remark that

$$M(n) = \bigoplus_{k\geq n} M_k = M_n \bigoplus M(n+1) = \bigoplus_{k\in\mathbb{N}} M_k.$$

Hence $M(n+1) \subset M(n)$. Thus

$$\cdots M(n+2) \subset M(n+1) \subset M(n) \subset \cdots.$$

Définition 2.4
Lets $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *and* $N = \bigoplus_{n\in\mathbb{Z}} N_n$ *two graded left* $A-$*modules and* $f : M \longrightarrow N$ *is a morphism of left* $A-$*modules, then we say that* f *is a graded morphism if for any* $m \in M_s$ *then* $f(m) \in N_{s+k}$*.*

Théorème et Définition 2.1
Let A *be a graded ring, then the following information :*

1. *The objects are the graded left* $A-$*modules ;*
2. *The morphisms are the graded morphisms..*

constitute a category called the category of graded left $A-$*module and which is denoted by* $Gr(A - Mod)$*-Mod.*

Proof
Lets M, N be two objects of $G_r(A-Mod)$, then M and N are two graded left $A-$modules and :

1. $Hom_{G_r(A-Mod)}(M, N) = \{$ Set of graded morphisms of M to N $\}$;
2. The morphisms are the graded morphisms then we have :
   (a) $\forall\ f \in Hom_{G_r(A-Mod)}(M, N)\,;\forall\ g \in Hom_{G_r(A-Mod)}(N, P)\,;\forall\ h \in Hom_{G_r(A-Mod)}(P, Q)$ we have : $(h \circ g) \circ f\ =\ h \circ (g \circ f)$, because f, g, h are a morphisms of left $A-$modules and if $m \in M$, then $f(m) \in N$ then $g(f(m)) \in P$, Hence $g \circ f$ is a graded morphism.

(b) Let M be an object of $G_r(A-Mod)$ then we have :

$$1_M : M \longrightarrow M$$

$$m \longmapsto m$$

1_M verifies $f \circ 1_M(m) = f(m), \quad \forall \ \ m \in M$
$\Rightarrow f \circ 1_M = f, \quad \forall \ \ f \in Hom_{G_r(A-Mod)}(M,N).$
Moreover $1_M \circ g(n) = 1_M(g(n)) = g(n), \quad \forall \ \ n \in N,$
thus $1_M \circ g = g \quad \forall \ \ g \in Hom_{G_r(A-Mod)}(N,M).$

Thus $Gr(A-Mod)$ is a category.

Définition 2.5
A complex sequence (C,d) : $\ldots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \overset{d_{n-1}}{\to} \ldots$ *is a sequence of morphisms of* $A-$ *modules satisfying* $d_n \circ d_{n+1} = 0$*, for all* $n \in \mathbb{Z}$.

Définition 2.6
A complex chain $f : (C,d) \to (C',d')$ *is a sequence of homomorphisms* $(f_n : C_n \longrightarrow C'_n)_{n\in\mathbb{Z}}$ *of* $A-$ *modules making the following diagram commute :*

$$\begin{array}{ccccccccc}
(C,d): \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\
\downarrow f & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\
(C',d'): \cdots & \longrightarrow & C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n & \xrightarrow{d'_n} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

i.e $d'_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}$*, for all* $n \in \mathbb{Z}$.

Proposition et Définition 2.1
We called the category of complexes of $A-$*modules and we denote* $COMP(A-Mod)$*, the category whose :*

1. *The objects are the sequences complex;*
2. *The morphisms are the complex chains.*

Proof
See [4]

Proposition 2.2
Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring and* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *be a graded left* $A-$*module, then we have the following associate complex sequence* M_* *of a grade* $A-$*module* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *:*

$$M_* : \cdots \to \ M(n+1) \overset{d_{n+1}}{\to} M(n) \overset{d_n}{\to} M(n-1) \to \cdots$$

with $M(n) = \bigoplus_{k\in\mathbb{Z}} M_{n+k}$ *and*

$$d_n : M(n) \longrightarrow M(n-1)$$

$$x = y + z \longmapsto y$$

with $(y,z) \in M_n \times M(n+1)$.

Proof
We have $M(n) = \bigoplus_{k\in\mathbb{Z}} M_{n+k} = \bigoplus_{k\geq n} M_k = M_n \bigoplus M_{n+1}$ and

$$M(n-1) = M_{n-1} \bigoplus M(n) = M_{n-1} \bigoplus M_n \bigoplus M(n+1).$$

Let $x \in M(n)$, then it exists a unique $(y,z) \in M_n \times M(n+1)$ such that $x = y+z$.
Put

$$d_n : M(n) \longrightarrow M(n-1)$$
$$x = y+z \longmapsto y$$

so $Im(d_n) = M_n$; on the other hand

$$d_{n-1} : M(n-1) \longrightarrow M(n-2)$$
$$w = u+v \longmapsto u$$

with $(u,v) \in M_{n-1} \times M(n)$ so $ker(d_{n-1}) = M(n)$ so $Im(d_n) \subset \ker(d_{n-1})$ so

$$d_{n-1} \circ d_n = 0$$

thus

$$M_* : \cdots \to \ M(n+1) \overset{d_{n+1}}{\to} M(n) \overset{d_n}{\to} M(n-1) \to \cdots$$

is a complex sequence.

Proposition 2.3
Let M be a graded leftA-module, N a graded submodule of M, M_* the complex associate to M and for all $n \in \mathbb{Z}$, $N(n)$ is a submodule of $M(n)$. Then

$$N_* : \cdots \to \ N(n+1) \overset{\delta_{n+1}}{\to} N(n) \overset{\delta_n}{\to} N(n-1) \to \cdots \ \textit{with } d_n(x) = \delta_n(n)(x)$$

is a subcomplex of M_*

Proof
We have $\delta_n : N(n+1) \longrightarrow N(n)$
let $x, y \in N(n+1) : x = y$
then $d_n(x) = d_n(y)$
$\Longrightarrow \delta_n(x) = \delta_n(y)$
$\Longrightarrow$ is well defined.
Let's calculate $\delta_n \circ \delta_{n+1}$
Let $x \in N(n+1)$, we have :
$\delta_n \circ \delta_{n+1}(x) = \delta_n(\delta_{n+1}(x))$
$= \delta_n(d_{n+1}(x)) = d_n(d_{n+1}(x)) = d_n \circ d_{n+1}(x) = 0$
Thus $\delta_n \circ \delta_{n+1} = 0$
hence N_* is a subcomplex of M_*.

Proposition 2.4
Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ are two graded left $A-$module $f : M \longrightarrow N$ is a graded morphism of a graded left $A-$modules, then for all $n \in \mathbb{Z}$

$$f(n) : M(n) \longrightarrow N(n)$$
$$m \quad \longmapsto \quad f(n)(m) = f(m)$$

is graded morphism of graded left $A-$modules.

Proof
We have $f : M \longrightarrow N$ is graded morphism of graded left $A-$modules, and $M(n)$ is a sub-module of graded left $A-$module M then let $m \in M(n)$, so

$$m = \sum_{i\in\mathbb{Z}} m_{i+n} \Longrightarrow f(n)(m) = f(m) = f(\sum_{i\in\mathbb{Z}} m_{i+n}) = \sum_{i\in\mathbb{Z}} f(m_{i+n})$$

or $f(m_{i+n}) \in N_{i+n+k} = (N(n))_{i+k}$ thus f is graded morphism of a graded left $A-$modules.

Corollaire 2.1
Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ are two graded left $A-$module $f : M \longrightarrow N$ is a graded morphism of a graded left $A-$modules, then $f : M \longrightarrow N(k)$ is graded morphism of graded left $A-$modules.

Proof
We have $f : M \longrightarrow N$ is graded morphism of graded left $A-$modules, and $N(k)$ is a sub-module of graded left $A-$module N then let $m \in M$, so

$$m = \sum_{i\in\mathbb{Z}} m_i \Longrightarrow f(m) = f(\sum_{i\in\mathbb{Z}} m_i) = \sum_{i\in\mathbb{Z}} f(m_i)$$

or $f(m_i) \in N_{i+k} = (N(k))_i$ thus f is graded morphism of a graded left $A-$modules.

Proposition 2.5
Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$, $N = \bigoplus_{n\in\mathbb{Z}} N_n$ are two graded left $A-$modules and $f : M = \bigoplus_{n\in\mathbb{Z}} M_n \longrightarrow N = \bigoplus_{n\in\mathbb{Z}} N_n$ is a graded morphism of a graded $A-$modules, then we have the following associated chain complex f_* of graded morphism $f : M = \bigoplus_{n\in\mathbb{Z}} M_n \longrightarrow N = \bigoplus_{n\in\mathbb{Z}} N_n$ of a graded $A-$modules :

$$\begin{array}{ccccccccc}
M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \\
\downarrow f_* & & \downarrow f(n+1) & & \downarrow f(n) & & \downarrow f(n-1) & & \\
N_* : \cdots & \longrightarrow & N(n+1) & \xrightarrow{d'_{n+1}} & N(n) & \xrightarrow{d'_n} & N(n-1) & \longrightarrow & \cdots
\end{array}$$

Proof
Prove that for all $n \in \mathbb{Z}$,

$$f(n) \circ d_{n+1} = d'_{n+1} \circ f(n+1).$$

Let $x \in M(n+1)$, then it exists the unique couple $(y, z) \in M_{n+1} \times M(n+2)$ such that $x = y + z$ so

$$(f(n) \circ d_{n+1})(x) = f(n)[d_{n+1}(x)] = f[d_{n+1}(x)] = f[y] = f(y)$$

and
$(d'_{n+1} \circ f(n+1))(x) = d'_{n+1}[f(n+1)(x)] = d'_{n+1}[f(x)] = d'_{n+1}[f(y+z)] = d'_{n+1}[f(y)+f(z)] = f(y)$
because $f(y) \in N_{n+1}$ and $f(z) \in N(n+2)$

$$\Longrightarrow (f(n) \circ d_{n+1})(x) = (d'_{n+1} \circ f(n+1))(x), \quad \forall\ x \ \in M(n+1)$$

so

$$f(n) \circ d_{n+1} = d^{'}_{n+1} \circ f(n+1)$$

thus f_* is a complex chain.

Théorème et Définition 2.2

Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a graded ring, then the following information :

1. *The objets are the associated complex sequences of a graded left $A-$modules ;*
2. *The morphisms are the associate complex chains of a graded morphism of a graded left $A-$modules.*

formed a category called the category of associate complex of a graded left $A-$modules and denoted by $COMP(AGr(A - Mod))$.

Proof

1. Lets M_* and N_* two complex sequences associated with graded left A-module M and N respectively.
Put $Hom_{COMP(AGr(A-Mod))}(M_*, N_*) =$ the class of complex chains associate to graded morphism of $M \longrightarrow N$. Then $Hom_{COMP(Gr(A-Mod))}(M_*, N_*)$ is a set, because the class of complex chain of $M_* \longrightarrow N_*$ of the category $COMP(A-Mod)(M_*, N_*)$ is a set (it suffices to remark also the class of graded of $M \longrightarrow N$ is a set).
2. $\forall \;\; f_* \in Hom_{COMP(Gr(A-Mod))}(M_*, N_*)$; $g_* \in Hom_{COMP(Gr(A-Mod))}(N_*, P_*)$ and $h_* \in Hom_{COMP(Gr(A-Mod))}(P_*, Q_*)$ we have :

$$\begin{array}{ccccccc}
M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & \ldots \\
\downarrow f_* & & \downarrow f(n+1) & & \downarrow f(n) & & \\
N_* : \cdots & \longrightarrow & N(n+1) & \xrightarrow{d^{'}_{n+1}} & N(n) & \xrightarrow{d^{'}_{n+}} & \ldots \\
\downarrow g_* & & \downarrow g(n+1) & & \downarrow g(n) & & \\
P_* : \cdots & \longrightarrow & P(n+1) & \xrightarrow{d^{''}_{n+1+k+r}} & P(n) & \xrightarrow{d^{''}_{n+k+r}} & \ldots \\
\downarrow h_* & & \downarrow h(n+1) & & \downarrow h(n) & & \\
Q_* : \cdots & \longrightarrow & Q(n+1) & \xrightarrow{d^{'''}_{n+1}} & Q(n) & \xrightarrow{d^{'''}_{n}} & \ldots
\end{array}$$

So $(h_* \circ g_*) \circ f_* = h_* \circ (g_* \circ f_*)$;
3. Let M_* the object of $COMP(Gr(A - Mod))$ we have :

$$1_{M_*} : M_* \longrightarrow M_*$$

$$\begin{array}{ccccccc}
M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & \ldots \\
\downarrow 1_{M_*} & & \downarrow 1(n+1) & & \downarrow 1(n) & & \\
M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & \ldots
\end{array}$$

1_{M_*} verified $f_* \circ 1_{M_*} = f_* \quad \forall \;\; f_* \in Hom_{COMP(Gr(A-Mod))}(M_*, N_*)$.
Furthermore $1_{M_*} \circ g_* = g_* \quad \forall \;\; g_* \in Hom_{COMP(Gr(A-Mod))}(N_*, M_*)$.
4. $\forall (M_*, N_*) \neq (M'_*, N'_*) \Longrightarrow Hom_{COMP(Gr(A-Mod))}(M_*, N_*) \neq Hom_{COMP(Gr(A-Mod))}(M'_*, N'_*)$

Thus $COMP(Gr(A - Mod))$ is a category.

Remarque 2.2
$COMP(AGr(A - Mod))$ is a subcategory of $COMP(A - Mod)$

Proposition et Définition 2.2
Let $(C, d) : \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$ be a complex sequence of left A-modules and (E_n) be a family of left A-modules such that for all n, E_n is a submodule of C_n. If $d_n(E_n) \subset E_{n-1}$, then the complex sequence of induced morphisms $(d_n : E_n \longrightarrow E_{n-1})$ is complex sequence of left A-modules called subcomplex of C

Proof
Suppose that $\delta_n : E_n \longrightarrow E_{n-1}$ the induced morphism of d_n, we show that δ_n is well defined.
Suppose that $x \in E_n$, hence $d_n(x) \in E_n$, then δ_n is well defined.
δ_n is a morphism, since it's composed of two morphisms.
Let's verify that $\delta_n \circ \delta_{n+1} = 0$
Let be $x \in E_{n+1}$, hence $\delta_{n+1}(x) = d_{n+1}(x) \in E_n$ and $\delta((d_{n+1})(x) = d_n \circ d_{n+1}(x) = 0$, then $\delta_n \circ \delta_{n+1} = 0$, for all $n \in \mathbb{Z}$.

Proposition et Définition 2.3
Let C be a complex and E be a subcomplex of C. Suppose that $K = (K_n)_{n\in\mathbb{Z}}$, where $K_n = C_n/E_n$. Then K is a complex sequence called quotient complex of C by E and denoted C/E

Proof
See [13]

Théorème 2.1
let $(C, d), (C', d')$ two objects of $COMP(A - Mod)$ and $f : C \longrightarrow C'$ be a complex chain. Then
f is a monomorphism of $COMP(A - Mod)$ if, and only, if $ker f = 0$.

Proof
Suppose that f is a monomorphism of C into C', hence $f \circ u = f \circ v \Longrightarrow u = v$, then for all $n \in \mathbb{Z}$, $f_n \circ u_n = f_n \circ_n \Longrightarrow u_n = v_n$, so f_n is a monomorphism, thus $ker f_n = 0$, hence $ker f = 0$.
Reciprocally, suppose $ker f = 0$, hence $ker f$ is a zero complex, so each term is zero, thus f_n is a monomorphism of left A-modules then for all $n \in \mathbb{Z}$ it gives if $f_n \circ u_n = f_n \circ v_n$ hence $u_n = v_n$, so $(f \circ u)_n = (f \circ v)_n \Longrightarrow (u)_n = (v)_n$, finally, f is a monomorphism of complex chains

Théorème 2.2
let $(C, d), (C', d')$ two objects of $COMP(A-Mod)$ and $f : (C, d) \longrightarrow (C', d')$ be a complex chain. Then
f is an epimorphism of $COMP(A - Mod)$ if, and only, if $Im f = C'$.

Proof
Suppose that f is an epimorphism of $COMP(A - Mod)$, then $u \circ f = v \circ f \Longrightarrow u = v$, hence for all $n \in \mathbb{Z}$, $(u \circ f)_n = (v \circ f)_n \Longrightarrow u_n = v_n$, f_n is an epimorphism of left A-modules, then $Im f_n = C'_n$ for all $n \in \mathbb{Z}$, so $Im f = C'$.
Reciprocally, suppose that $Im f = C'$, hence for all $n \in \mathbb{Z}$, $Im f_n = C'_n$, then f_n is an

epimorphism of left A-modules so for all $n \in \mathbb{Z}$, $u_n \circ f_n = v_n \circ f_n$, then $(u \circ f)_n = (v \circ f_n) \Longrightarrow u_n = v_n$ then f is an epimorphism of complex chains.

3 Strongly hopfian objects in the catégories $AGr(A - Mod)$ and $COMP(AGr(A - Mod))$

Définitions 3.1

Let M_* be an object of $COMP(AGr(A-Mod))$ and f_* a complex endomorphism of M_* :

$$\begin{array}{ccccccccc} M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \\ f_* \downarrow & & f(n+1) \downarrow & & f(n) \downarrow & & f(n-1) \downarrow & & \\ M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \end{array}$$

We call $f_* \circ f_*$ the chain complex compounded of f_* by itself denoted f_*^2.
We also define $(f^2)_n = f_n \circ f_n$, for all $n \in \mathbb{Z}$.
We also define f_*^k such that $f^k(n) = f(n) \circ f(n) \circ \ldots f(n)$, with $k factors$, for all $n \in \mathbb{Z}$.

Proposition 3.1

Let M_* be an object of $COMP(AGr(A-Mod))$ and f_* a complex endomorphism of M_* :

$$\begin{array}{ccccccccc} M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \\ f_* \downarrow & & f(n+1) \downarrow & & f(n) \downarrow & & f(n-1) \downarrow & & \\ M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \end{array}$$

Given $\Delta^k(n+1) : ker f^k(n+1) \longrightarrow ker f^k(n)$
$x \longmapsto d_{n+1}(x)$
where $f^k(n+1) = f(n+1) \circ f(n+1) \circ \ldots \circ f(n+1)$, with k factors, then $\Delta^k(n+1)$ is the induced morphism by $\Delta^{k+1}(n+1)$.

Proof

Considering $\Delta^k(n+1) : ker f^k(n+1) \longrightarrow ker f^k(n)$
$x \longmapsto d_{n+1}(x)$
and $\Delta^{k+1}(n+1) : ker f^{k+1}(n+1) \longrightarrow ker f^{k+1}(n)$
$x \longmapsto d_{n+1}(x)$
We obtain $ker f^k(n+1) \subset ker f^{k+1}(n+1)$ and $ker f^k(n) \subset ker f^{k+1}(n)$, therefore $\Delta^k(n+1)$ is the induced by Δ^{k+1}_{k+1}.

Définitions 3.2

Let M_* be an object of $COMP(AGr(A-Mod))$ and f_* a complex endomorphism of M_* :

$$\begin{array}{ccccccccc} M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \\ f_* \downarrow & & f(n+1) \downarrow & & f(n) \downarrow & & f(n-1) \downarrow & & \\ M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \end{array}$$

Then the chain complex

$$(ker f_*^k) : \ldots \to \; ker f^k(n+1) \overset{\Delta^k_{n+1}}{\to} ker f^k(n) \overset{\Delta^k_n}{\to} ker f^k(n-1) \overset{\Delta^k_{n-1}}{\to} \ldots$$

is stationary if, and only if, it exists $k_0 \in \mathbb{N}^*$ such that $(kerf_*^{k_0}) = (kerf_*^{k_0+s})$ for all $s \in \mathbb{N}$.

Proposition 3.2

Let M_* be an object of $COMP(AGr(A-Mod))$ and f_* a complex endomorphism of M_* :

$$\begin{array}{ccccccccc} M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \\ \downarrow f_* & & \downarrow f(n+1) & & \downarrow f(n) & & \downarrow f(n-1) & & \\ M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \end{array}$$

Then $(kerf_*^k)$ is stationary if, and only if, $(kerf^k(n))$ stabilizes for all $n \in \mathbb{Z}$

Proof

If $(kerf_*^k)$ is stationary, then it exists $k_0 \in \mathbb{N}^*$ such that $(kerf_*^k) = (kerf_*^{k+s})$, for all $s \in \mathbb{N}$, hence $kerf^{k_0}(n) = kerf^{k_0+s}(n)$, for all $n \in \mathbb{Z}$. This implies that $(kerf^k(n))$ stabilizes for all $n \in \mathbb{Z}$.
Reciprocally, assume that $(kerf^k(n))$ for all $n \in \mathbb{Z}$, then it exists $k_0 \in \mathbb{N}^*$ such that for all $s \in \mathbb{N}$ then $kerf^{k_0}(n) = kerf^{k_0+s}(n)$, for all $n \in \mathbb{Z}$.
So $(kerf_*^k)$ is stationary for all $k \in \mathbb{N}$

Définitions 3.3

Let M_* be an object of $COMP(AGr(A-Mod))$ and f_* a complex endomorphism of M_*. M_* is said to be strongly hopfian, if for any endomorphism f_* of M_*, $(kerf_*^k)$ is stationary.

Théorème 3.1

Let M be a graded left A-module. Then we have the following equivalences :

1. *All nonempty set of submodules of M, contains a maximal element ;*
2. *all increase sequence of submodules of M is stationary.*

Proof

Assume that all nonempty set of submodules of M, contains a maximal element.
We consider the increase sequence $N_1 \subset N_2 \subset N_3 \subset \ldots$ of submodules of M. Then the family $(N_n)_{n>0}$ contains a maximal element N_p, so, for all $n \geq$, we have $N_s = N_p$, hence $(N_n)_{n>1}$ is stationary.
Reciprocally, assume that all increase sequence of submodule of M is stationary, we consider Λ nonempty set of submodules of M. Suppose that, by contradiction Λ doesn't contain a maximal element, thus if $N \in \Lambda$, then it exists $N' \in \Lambda$ such that $N \subset N'$. hence, we can build an increase sequence $N_1 \subset N_2 \subset N_3 \ldots$ of submodules of Λ non stationarily of Λ. This is a contradiction.

Définitions 3.4

Let M be a graded left A-module with verifies one of conditions of Theorem(3.1) is said to be noetherian

Théorème 3.2

Let M a graded left A-module. Then we have the following equivalences :

1. *All nonempty set of submodules of M contains a minimal element.*
2. *All decrease sequence of submodules of M is stationary.*

Définitions 3.5
Let M a graded left A-module which one of conditions of Theorem(3.2) is said to be artinian.

Définitions 3.6
Let M be graded left module, M_* the associate graded complex of M. Then M_* is said to be noetherian (respectively artinian) if for all $n \in \mathbb{Z}$, $M(n)$ is noetherian (respectively artinian).

Théorème 3.3
Let M a graded left module, M_* the associate graded complex of M. Then M_* is noetherian if, and only, for all $n \in \mathbb{Z}$, all submodule of $M(n)$ is finitely generated.

Proof
M_* is noetherian, if for all $n \in \mathbb{Z}$, $M(n)$ is noetherian. Since $M(n)$ noetherian is equivalent to $M(n)$ is finitely generated is verified in the category $AGr(A - Mod)$.

Proposition 3.3
Let M a graded left A-module and M_* the associated complex à M. Then :

1. *If M_* is noetherian (resp. artinian) then M_* is strongly hopfian ;*
2. *If M_* is strongly hopfian, then M_* is hopfian.*

Proof

1. Suppose that M_* is noetherian (resp. artinian), then for all $n \in \mathbb{Z}, M(n)$ is noetherian (resp. artinian) so, it exists $n \geq 1$ such that $ker f^k(n) = ker f^{k+1}(n)\ \forall\ n \in \mathbb{Z}$, hence M_* is strongly hopfian.
2. Suppose that M_* is strongly hopfian, let f_* an epimorphism of M_*. Then, it exists $k \geq 1$ and $n \in \mathbb{Z}$ such that $ker^k(n) = kef^{k+1}(n)$. Let $x \in M(n)$ such that $f(n)(x) = 0$; since $f(n)$ is an epimorphism, then $f^k(n)$ is an epimorphism, therefore it exists $y \in M(n)$ such that $x = f^k(n)(y) \Longrightarrow 0 = f^{k+1}(n)(y)$. Thus, $y \in ker f^{n+1} = ker f^k(n)$. Then $x = 0$ and $f(n)$ is a monomorphism for all $n \in \mathbb{Z}$, so f_* is an automorphism, M_* is hopfian.

Proposition 3.4
Let M a graded left A-module and M_* the associate complex of M and N a submodule of M et N_* invariant fully in M_*. Then : If N_* and M_*/N_* are strongly hopfian, then M_* is strongly hopfian.

Proof
Let f_* an endomorphism of M_* and N_* a fully invariant subcomplex of M_*. Then, for all $n \in \mathbb{Z}$, $f(n)(N) \subset M(n)$ and $f(n)$ induces two endomorphisms $f_1(n)$ of $N(n)$ and $\overline{f}(n)$ of $M(n)/N(n)$ such that the following diagram commutes :

$$\begin{array}{ccccc} N(n) & \xrightarrow{i(n)} & M(n) & \xrightarrow{\pi(n)} & M(n)/N(n) \\ \downarrow{\scriptstyle f_1(n)} & & \downarrow{\scriptstyle f(n)} & & \downarrow{\scriptstyle \overline{f}(n)} \\ N(n) & \xrightarrow{i(n)} & M(n) & \xrightarrow{\pi(n)} & M(n)/N(n) \end{array}$$

If N_* and M_*/N_* are strongly hopfian, then it exists $s \geq 1$, for all $n \in \mathbb{Z}$ such that $ker\overline{f}^k(n) = ker\overline{f}^s(n)$ and that $ker f_1(n) = ker f_1(s)$ for all $k \geq s$. Let $x \in ker f^{2s+1}(n)$, then $f^{2s+1}(n+1)(x) = 0 \in N(n)$ thus $\overline{f}^{2s+1}(n)(\bar{x}) = \bar{0}$ this implies that $\bar{x} \in ker\overline{f}^{2s+1}(n) = ker\overline{f}^{n}(s)$, and so $\overline{f}^{s}(n)(\bar{x}) = \bar{0}$, ie, $y = f^s(n)(x) \in N(n)$.. Since $f^{s+1}(n)(y) = 0$, therefore $y \in kerf_1^{s+1}(n) = kerf_1^{2s}(n)$. Hence, $kerf^{2s+1}(n) = kerf^{2s}(n)$ for all $n \in \mathbb{Z}$. Thus M_* is strongly hopfian.

Proposition 3.5

Let M a graded left A-module, N a submodule and M_* the associate complex to M. if M_*/N_* is strongly hopfian, then M_* is strongly hopfian.

Proof

Assume on the contrary that M_* is not strongly hopfian, then for all $n \in \mathbb{Z}$ it exists $f \in end(M(n))$ such that $kerf^k(n) \neq kerf^{k+1}(n+1)$ for all $k \geq 1$. In particular, $f(n)$ is not a monomorphism, thus $N = kerf(n) \neq 0$. $f(n)$ induces an endomorphism $\overline{f}(n)$ of $M(n)/N(n)$ such that $\overline{f}(n)(\bar{x}) = \overline{f(n)(x)}$. By hypothesis, $M(n)/N(n)$ is strongly hopfian for all $n \in \mathbb{Z}$, so it exists $m \geq 1$ such that $ker\overline{f}^{m}(n) = ker\overline{f}^{k}(n)$ for all $k \geq m$. Let $x \in kerf^{m+2}(n)$, then $\bar{x} \in ker\overline{f}^{m+2}(n) = ker\overline{f}^{m}(n)$ this implies that $f^m(n)(x) \in N(n)$. Hence $f(n)(f^m(n)(x) = 0$ ie., $x \in kerf^{m+1}(n)$. Therefore, we have $kerf^{m+1}(n) \neq kerf^{m+2}(n)$ this contradicting the no stationary of the sequence $kerf(n) \subset kerf^2(n) \subset \ldots$. We deduce that $M(n)$ is strongly hopfian, so M_* is strongly hopfian.

4 Strongly cohopfian objects in the catégories $AGr(A - Mod)$ and $COMP(AGr(A - Mod))$

Définitions 4.1

Let M_* be an object of $COMP(AGr(A-Mod))$ and f_* a complex endomorphism of M_* :

$$\begin{array}{ccccccccc} M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \\ \downarrow f_* & & \downarrow f(n+1) & & \downarrow f(n) & & \downarrow f(n-1) & & \\ M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \end{array}$$

Then the chain complex

$$(Imf_*^k) : \ldots \to \ Imf^k(n+1) \overset{\delta^k_{n+1}}{\to} Imf^k(n) \overset{\delta^k_n}{\to} Imf^k(n-1) \overset{\delta^k_{n-1}}{\to} \ldots$$

is stationary if, and only if, it exists $k_0 \in \mathbb{N}^*$ such that $(Imf_*^{k_0}) = (Imf_*^{k_0+s})$ for all $s \in \mathbb{N}$.

Proposition 4.1

Let M_* be an object of $COMP(AGr(A-Mod))$ and f_* a complex endomorphism of M_* :

$$\begin{array}{ccccccccc} M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \\ \downarrow f_* & & \downarrow f(n+1) & & \downarrow f(n) & & \downarrow f(n-1) & & \\ M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \end{array}$$

Given $\delta^k(n+1) : Imf^k(n+1) \longrightarrow Imf^k(n)$
$x \longmapsto d_{n+1}(x)$
where $f^k(n+1) = f(n+1) \circ f(n+1) \circ \ldots \circ f(n+1)$, with k factors, then $\delta^k(n+1)$ is the induced morphism by $\delta^{k+1}(n+1)$.

Proof

Considering $\delta^k(n+1) : Imf^k(n+1) \longrightarrow Imf^k(n)$

$x \longmapsto d_{n+1}(x)$

and $\delta^k(n+1) : Imf^{k+1}(n+1) \longrightarrow Imf^{k+1}(n)$

$x \longmapsto d_{n+1}(x)$

We obtain $Imf^{k+1}(n+1) \subset Imf^k(n+1)$ and $Imf^{k+1}(n) \subset Imf^k(n)$, therefore $\delta^{k+1}(n+1)$ is the morphism induced by δ^k_{n+1}.

Proposition 4.2

Let M_* be an object of $COMP(AGr(A-Mod))$ and f_* a complex endomorphism of M_* :

$$\begin{array}{ccccccccc}
M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots \\
\downarrow f_* & & \downarrow f(n+1) & & \downarrow f(n) & & \downarrow f(n-1) & & \\
M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots
\end{array}$$

Then (Imf^k_*) is stationary if, and only if, $(Imf^k(n))$ stabilizes for all $n \in \mathbb{Z}$

Proof

If (Imf^k_*) is stationary, then it exists $k_0 \in \mathbb{N}^*$ such that $(Imf^k_*) = (Imf^{k+s}_*)$, for all $s \in \mathbb{N}$, hence $Imf^{k_0}(n) = Imf^{k_0+s}(n)$, for all $n \in \mathbb{Z}$. This implies that $(Imf^k(n))$ stabilizes for all $n \in \mathbb{Z}$.

Reciprocally, assume that $(Imf^k(n))$ for all $n \in \mathbb{Z}$, then it exists $k_0 \in \mathbb{N}^*$ such that for all $s \in \mathbb{N}$ then $Imf^{k_0}(n) = Imf^{k_0+s}(n)$, for all $n \in \mathbb{Z}$.

So (Imf^k_*) is stationary for all $k \in \mathbb{N}$

Définitions 4.2

Let M_* be an object of $COMP(AGr(A-Mod))$ and f_* a complex endomorphism of M_*. M_* is said to be strongly cohopfian, if for any endomorphism f_* of M_*, (Imf^k_*) is stationary.

Proposition 4.3

Let M a graded left A-module and M_* the associated complex à M. Then :

1. *If M_* is noetherian (resp. artinian) then M_* is strongly cohopfian ;*
2. *If M_* is strongly cohopfian, then M_* is cohopfian.*

Proof

1. Suppose that M_* is noetherian (resp. artinian), then for all $n \in \mathbb{Z}, M(n)$ is noetherian (resp. artinian) so, it exists $n \geq 1$ such that $Imf^k(n) = Imf^{k+1}(n)\ \forall\ n \in \mathbb{Z}$, hence M_* is strongly cohopfian.
2. Suppose that M_* is strongly cohopfian, let f_* a monomorphism of M_*. Then, it exists $k \geq 1$ and $n \in \mathbb{Z}$ such that $Imr^k(n) = Imf^{k+1}(n)$, since $f(n)$ is injective, then $f^k(n)$ is injective. Let $x \in M(n)$, it exists $y \in M(n)$ such that $f^k(n)(x) = f^{k+1}(n)(y)$. Thus, $x - f(n)(y) \in kerf^k(n) = 0$. Hence $x = f(n)(y)$, therefore $f(n) \in Aut(M(n))$, this equivalent to f_* is an automorphism of M_*.

Proposition 4.4

Let M a graded left A-module and M_* the associate complex of M and N a submodule of M et N_* invariant fully in M_*. Then : If N_* and M_*/N_* are strongly cohopfians, then M_* is strongly cohopfian.

Prove

Let f_* an endomorphism of M_* and N_* a fully invariant subcomplex of M_*. Then, for all $n \in \mathbb{Z}$, $f(n)(N) \subset M(n)$ and $f(n)$ induces two endomorphisms $f_1(n)$ of $N(n)$ and $\overline{f}(n)$ of $M(n)/N(n)$ such that the following diagram commutes :

$$\begin{array}{ccccc} N(n) & \xrightarrow{i(n)} & M(n) & \xrightarrow{\pi(n)} & M(n)/N(n) \\ \downarrow{\scriptstyle f_1(n)} & & \downarrow{\scriptstyle f(n)} & & \downarrow{\scriptstyle \overline{f}(n)} \\ N(n) & \xrightarrow{i(n)} & M(n) & \xrightarrow{\pi(n)} & M(n)/N(n) \end{array}$$

If N_* and M_*/N_* are strongly cohopfian, then it exists $s \geq 1$, for all $n \in \mathbb{Z}$ such that $\overline{f}^k(n)(M(n)/N(n) = \overline{f}^s(n)(M(n)/N(n)$. Again, it exists $m \geq 1$ such that $f_1^k(n)(N(n)) = f_1^m(n)(N(n))$ for all $k \geq m$. Put $p = s + m$. For each $x \in M(n)$, we have $\overline{f}^s(n)(\bar{x}) = \overline{f}^{s+1}(n)(\bar{y})$ for some $y \in M(n)$. This implies $t = f^s(n)(x) - f^{s+1}(n)(y) \in N(n)$. Hence $f^m(n)(t) = f_1^{p+1}(z)$, for some $z \in N(n)$. It becomes that $f^p(n)(x) = f^{p+1}(n)(y+z)$, hence $Imf^p(n) = Imf^{p+1}(n)$. Therefore $M(n)$ is strongly cohopfian for all $n \in \mathbb{Z}$, so M_* is cohopfian.

Proposition 4.5

Let M a graded leftA-module, N a submodule and M_* the associate complex to M. If all subcomplex of M_* is cohopfian, then M_* is cohopfian.

Proof

Suppose that M_* is not strongly cohopfian, then for all $n \in \mathbb{Z}$ it exists an endomorphism $f(n) \in End(M(n))$ such that $Imf^k(n) \neq Imf$ for all $k \geq 1$. In particular, $f(n)$ is not surjective, thus $N(n) = Imf(n)$ is an own submodule of $M(n)$. By hypothesis, $N(n)$ is strongly cohopfian, thus the endomorphism $g(n) = f(n)|_{N(n)}$ of $N(n)$, it exists $m \geq$ such that $Img^k(n) = Img^m(n)$ for all $n \geq m$. Let $x \in Imf^{m+1}(n)$, then it exists $y \in M(n)$ such that $x = f^{m+1}(n)(y) = f^m(n)(f(n)(y))$. Since $f(n)(y) \in N(n)$, thus $x \in Img^m(n) = Img^{2(m+1)}$, ie, it exists $z \in N(n)$ such that $x = g^{2(m+1)}(n)(z) = f^{2(m+1)}(n)(z)$. Therefore $Imf^{m+1}(n) = Imf^{2(m+1)}(n)$, contradicting the strongly cohopfian nature of M_*.

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