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Stability Analysis of Mathematical Model and Optimal Control Strategies for Reducing the Covid-19 Spread

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ABSTRACT
The respiratory infectious disease Coronavirus Disease 2019 caused by the SARS-CoV-2 virus has become a world pandemic. This paper aims to develop a mathematical model of the spread of Covid-19 SEIQR (Susceptible-Exposed-Infected-Quarantine-Recovered), then analyse the stability of the model and its optimal control. Based on the local stability analysis using the Routh-Hurwitz criteria obtained two equilibrium points, namely the local asymptotically stable disease-free equilibrium points if reproduction number less than one and the local asymptotically stable endemic equilibrium point if reproduction number more than one. Next, to analyze global stability at the equilibrium points is used the Lyapunov method. Further, four controls, namely vaccination, physical distancing, treatment of infected individuals, and self-prevention (wearing masks and hand sanitisers) are applied to reduce the spread of covid-19. We applied Pontryagin's Maximum Principle to obtain the optimal solution for the control. Finally, based on numerical result is found that the value of reproduction number is 9,44183. By using data from province of East Java, Indonesia, it is obtained that the physical distancing control has a better level effectiveness than the others three controls.

KEYWORDS: Mathematical Modelling, Covid-19, Routh-Hurwitz, Lyapunov, Optimal Control

I. INTRODUCTION

The World Health Organization (WHO), on March 11, 2020, declared that the Covid-19 pandemic had caused various problems ranging from health and economy to social problems [1]. The characteristics of the highly contagious Covid-19 virus have resulted in the rapid and massive spread of the Covid-19 virus. The spread of the Covid-19 virus can occur through direct physical contact, air exposure to droplets of people with Covid-19, to objects exposed to the Covid-19 virus [2].

The Covid-19 case in Indonesia was first recorded on March 2 2020, and until now, the confirmed number of Covid-19 is still increasing and decreasing in line with policies that are able to limit the growth of Covid-19 cases. According to data compiled by the National Covid-19 Task Force of the Republic of Indonesia, confirmed Covid-19 sufferers as of November 01, 2021, reached 4,245,373, positive cases were 11,629, recovered cases were 4,090,287, and death cases were 143,457 [3]. The case data shows that the Covid-19 number is quite large, even though there has been a decrease in confirmed cases of Covid-19 in Indonesia every day.

The branch of mathematics that tries to represent or explain real-world problems into mathematical statements to gain insight into existing problems is mathematical modelling [4]. Mathematical models are, in practice, powerful tools used in describing, controlling, limiting and minimizing the impact of a pandemic. In an optimal control system, optimal control tries to obtain an input function that produces an output function and fulfils certain requirements as much as possible [5].

Various studies discussing the spread of the Covid-19 virus have been carried out, ranging from time series to deterministic. A logistic growth method [6] is used to forecast the increase of covid-19 itself. Research conducted by [7] discusses the modification of the SIR model with the addition of the Q variable so that it becomes the SQIR model and is used to analyse the impact of Covid-19. The construction of the Covid-19 spread model and its stability analysis using the Routh-Hurwitz criteria was also carried out by [8] with a case study in Central Java Province with the STQIR model; it is known that this model can be applied and produces satisfactory results. The optimal control method in [9], which discusses the cost-effectiveness analysis for economic health

costs and expenditures using the optimal control approach in the SEIARB model, uses the Runge Kutta method of order four forward and backwards to solve the differential equation problem.
Research by Yujia et al. [10] discussed the application of optimal control with treatment uncertainty and transmission in the SIR model, then analysed using a stochastic optimal control approach. Furthermore, Gustavo et al. [11] with model SIR added with two control namely minimizing a quantity related to the number of infected individuals and the number of vaccines used during the treatment, it shows that the controls can be used to determine the spread of covid-19. Anwarud et al. [12] applied two control (treatment and isolation) to the model SIQ, concluded that the spread od covid-19 can be decrease by using these control with proper applications. Yang et al. [13] explained the impact of vaccination on the spread of Covid-19 in the SVAIR model with a backward bifurcation analysis; it was found that giving Covid-19 vaccination can control the spread of Covid-19. In order to determine the global stability and equilibrium points, model SEIR [14, 15] using Lyapunov function to determine the global stability and characteristic equation to determine the local stability. Another method to determine the spread of covid-19 also proposed by using fractional order and stochastic models in [16, 17, 18].
This study discussing about the mathematical model SEIQR and its stability. The purpose of this study was to determine the dynamical stability analysis system of SEIQR model by using Routh-Hurwitz criterion and Lyapunov function. In this study we added optimal control analysis by adding four controls namely; Covid-19 vaccination, self-prevention (using masks and hand sanitisers), physical distancing, and treatment of infected individuals. At the end of the study, we give a numerical simulation to give some knowledge about the dynamical spread of Covid-19 in west java, Indonesia.

II. MODEL FORMULATION

The SEIR model based on [19] by adding Quarantine class to see more deep the dynamic of the spread of Covid-19 itself. The total population at time $N(t)$ is divided into five classes namely Susceptible class, Exposed class, Infected class, Quarantine class, and Recovered class. The process of mathematical model of covid-19 is given in Figure 1.

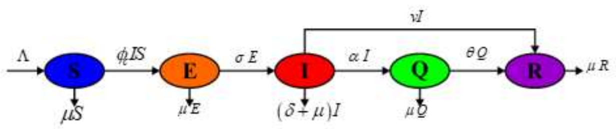

Fig. 1. The process of mathematical modelling of Covid-19

From Figure 1 we can construct a mathematical model is given in

$$\left.\begin{aligned}\frac{dS}{dt}&=\Lambda-\phi_c IS-\mu S\\ \frac{dE}{dt}&=\phi_c IS-(\sigma+\mu)E\\ \frac{dI}{dt}&=\sigma E-(\alpha+v+\delta+\mu)I\\ \frac{dQ}{dt}&=\alpha I-(\theta+\mu)Q\\ \frac{dR}{dt}&=\theta Q-\mu R+vI\end{aligned}\right\} \tag{1}$$

Where S, E, I, Q, R are susceptible, exposed, infected, quarantine, and recovered respectively at time t.

2.1 positivity and boundness of the solution

Theorem 1. Let $S(0)\geq 0, E(0)\geq 0, I(0)\geq 0, Q(0)\geq 0,$ $R(0)\geq 0$, so, the solutions $S(t), E(t), I(t), Q(t), R(t)$ from the model (1) are positive invariant for any $\forall t>0$ in the region $\Omega=\left\{\begin{matrix}S(t),E(t),I(t),Q(t),R(t)\in\mathbb{R}^5_+\mid 0\\ \leq S(t)+E(t)+I(t)+Q(t)+R(t)\leq\dfrac{\Lambda}{\mu}\end{matrix}\right\}$

Proof. From the equations (1), we get for $\frac{dS}{dt}+(\phi_c I-\mu)S\geq\Lambda$ we have $S(t)=S(0)e^{-\int_0^t k(x)dx}+e^{-\int_0^t k(x)dx}\int_0^t\Lambda e^{\int_0^t k(x)dx}dx\geq 0$ for $\frac{dE}{dt}+(\sigma+\mu)E=\phi_c IS$ we have $E(t)=E(0)e^{-\int_0^t m(x)dx}.+e^{-\int_0^t m(x)dx}\int\phi_c ISe^{\int_0^t m(x)dx}dx\geq 0,\forall t>0.$ for $\frac{dI}{dt}+(\alpha+v+\delta+\mu)I=\sigma E$ we have $I(t)=I(0)e^{-\int_0^t n(x)dx}+e^{-\int_0^t n(x)dx}\int_0^t\sigma E(t)e^{\int_0^t n(x)dx}dx\geq 0,\forall t>0$. For $\frac{dQ}{dt}+(\theta+\mu)Q=\alpha I$ we have $Q(t)=Q(0)e^{-\int_0^t j(x)dx}+e^{-\int_0^t j(x)dx}\int_0^t\alpha I(t)e^{\int_0^t j(x)dx}dx\geq 0$. And also, for $\frac{dR}{dt}+\mu R=\theta Q+vI$ we have $R(t)=R(0)e^{-\int_0^t h(x)dx}+e^{-\int_0^t h(x)dx}\int_0^t(\theta Q(t)+vI(t))e^{\int h(x)dx}dx\geq 0.$
Therefore it shows that $S(t), E(t), I(t), Q(t), R(t)$ are positif. □

Thorem 2. All the solutions $S(0), E(0), I(0), Q(0), R(0)$ from the equations (1) is bounded if and only if $\limsup_{t\to\infty} N(t)\leq\frac{\Lambda}{\mu}$, so $N(t)=S(t)+E(t)+I(t)+Q(t)+R(t)$.

Proof. We know that $N(t)=S(t)+E(t)+I(t)+Q(t)+R(t)$, so we get $\frac{dN(t)}{dt}\leq\Lambda-\mu N(t)$ with

$N(t) \le \left(N(0) - \dfrac{\Lambda}{\mu}\right)e^{-\mu t} + \dfrac{\Lambda}{\mu}, t > 0$. It shows that the solutions are positive invariant and its proof that the system is bounded. □

III. STABILITY ANALYSIS

The equilibrium point is a condition where the change in the number of populations in each sub-population over time is zero. So, the system (1) is said to be balance if

$$\frac{dN}{dt} = \frac{dS}{dt} = \frac{dE}{dt} = \frac{dI}{dt} = \frac{dQ}{dt} = \frac{dR}{dt} = 0.$$

The subpopulation *R* has no effect on the system because it is separate from the others. The solution of the system (1) has two equilibrium points, namely $\varepsilon^0(S^0, E^0, I^0, Q^0)$ as the disease-free equilibrium and $\varepsilon^*(S^*, E^*, I^*, Q^*)$ as the endemic equilibrium point.

3.1 Basic Reproduction Number

The basic reproduction number can be obtaining by determine the eigen value from matrix FV^{-1} commonly called Next Matrix Generation (NGM) [20, 21, 22, 23], where

$$F == \begin{pmatrix} 0 & \dfrac{\phi_c \Lambda}{\mu} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} (\sigma+\mu) & 0 & 0 \\ -\sigma & (\alpha+v+\delta+\mu) & 0 \\ 0 & -\alpha & (\theta+\mu) \end{pmatrix}.$$

Therefore, the basic reproduction number in this paper is

$$\Re_0 = \frac{\phi_c \Lambda \sigma}{\mu(\sigma+\mu)(\alpha+v+\delta+\mu)}.$$

3.2 Local stability of the Covid-19 free equilibrium

Theorem 3. if $\Re_0 < 1$ then the Covid-19 free equilibrium in equation (1) is locally asymptotically stable. Otherwise, it is unstable if $\Re_0 > 1$.

Proof. From equation (1) we obtain a matrix Jacobian is,

$$J(\varepsilon^0) = \begin{bmatrix} -\mu & 0 & -\dfrac{\phi_c \Lambda}{\mu} & 0 \\ 0 & -(\sigma+\mu) & \dfrac{\phi_c \Lambda}{\mu} & 0 \\ 0 & \sigma & -(\alpha+v+\delta+\mu) & 0 \\ 0 & 0 & \alpha & -(\theta+\mu) \end{bmatrix} \quad (2)$$

We have the characteristic equation of (2) as follows

$$(X+\mu)P_1(X)P_2(X) = 0$$

With $P_1 = X + a$; $P_2 = X^2 + b_1 + b_2$; $a = \sigma + \mu$; $b_1 = (\sigma + 2\mu + v + \delta + \alpha)X$;

$$b_2 = \frac{-\phi_c \Lambda \sigma}{\mu(\sigma+\mu)(\alpha+v+\delta+\mu)}(\sigma+\mu)(\alpha+v+\delta+\mu) + \frac{1}{\mu}\left(\mu\sigma v + \mu\sigma\alpha + \mu^2\sigma + \mu\sigma\delta + \mu^2\alpha + \mu^3 + \mu^2 v + \mu^2\delta\right)$$

based on Routh-Hurwitz criterion [24], [25] we know that $a, b_1, b_2 > 0$, then it is prove that Covid-19 free equilibrium is locally asymptotically. □

3.3 Local stability of Covid-19 endemic equilibrium

Theorem 4. if $\Re_0 > 1$ then the Covid-19 endemic equilibrium in equation (1) is locally asymptotically stable. Otherwise, it is unstable.

Proof. Based on equation (1) we have Jacobian matrix for endemic equilibrium as follows

$$J(\varepsilon^*) = \begin{bmatrix} -(\phi_c I^* + \mu) & 0 & -\phi_c S^* & 0 \\ \phi_c I^* & -(\sigma+\mu) & \phi_c S^* & 0 \\ 0 & \sigma & -(\alpha+v+\delta+\mu) & 0 \\ 0 & 0 & \alpha & -(\theta+\mu) \end{bmatrix}$$

(3)

Then we will find the eigenvalues of the matrix (3) by calculating $\det(X\mathbf{I} - J(\varepsilon^*)) = 0$. We obtain the characteristic equation of matrix (3) is

$$\left(X + \phi_c I^* + \mu\right)\left(a_0 X^3 + a_1 X^2 + a_2 X + a_3\right) = 0$$

with

$a_0 = 1; a_1 = 3\mu + \sigma + \alpha + v + \delta + \theta; a_2 = -\sigma\phi_c S^* + 3\mu\mu + 2\mu$ $(\alpha + v + \delta + \mu + \sigma + \theta) + \theta(\sigma + \alpha + v + \delta) + \sigma(\alpha + v + \delta)$;

and $a_3 = (\Re_0 - 1)(\sigma+\mu)(\alpha+v+\delta+\mu)(\theta+\mu)$. Based on Routh-Hurwitz criterion [24,25] value of $\det(H_2) = a_1 a_2 - a_0 a_3 > 0$, $a_1 > 0, a_2 > 0$, and its shows the Covid-19 endemic equilibrium is locally asymptotically stable. □

3.4 Global stability of the Covid-19 free equilibrium

Theorem 5. [26, 27] If $\Re_0 < 1$ then the Covid-19 free equilibrium point ε^0 is globally asymptotically stable. Otherwise, it is unstable if $\Re_0 > 1$.

Proof. Let's assume that $\Re_0 < 1$ and the Lyapunov function $\mathrm{L} : P \subset \mathbb{R}^4 \to \mathbb{R}$, we consider the Lyapunov function as follow

$$\mathrm{L}(S, E, I, Q) = \left(S - S^0 - S^0 \ln \frac{S}{S^0}\right) + x_1 E + x_2 I + x_3 Q \quad (4)$$

where x_1, x_2, and x_3 are positive, then we will check the function (4) to know it fulfill the definition Lyapunov [28] and the Theorem 5. The Function $\mathscr{L}$ is called Lyapunov function if only if fulfills the criterion below:

(i) The function $\mathscr{L}$ consists of logarithmic functions, so it is clear that the function is a continuous function on *P*, and that the first partial derivative is also a continuous function on *P*.

(ii) The function $\mathscr{L}$ have a minimum global at ε^0 is related to all points in P. then we will investigate whether $\mathrm{L}(S,E,I,Q)=0$ for $\mathrm{L}(S,E,I,Q)=\mathrm{L}(S^0,E^0,I^0,Q^0)$ and $\mathrm{L}(S,E,I,Q)>0$ for $\mathrm{L}(S,E,I,Q)\neq \mathrm{L}(S^0,E^0,I^0,Q^0)$.

For $\mathrm{L}(S,E,I,Q)\neq \mathrm{L}(S^0,E^0,I^0,Q^0)$ we will prove that $\mathrm{L}(S,E,I,Q)>0$.

$$\begin{aligned}\mathrm{L} &= \left(S-S^0-S^0\ln\frac{S}{S^0}\right)+x_1E+x_2I+x_3Q\\ &=S^0\left(\frac{S}{S^0}-1-\ln\frac{S}{S^0}\right)+x_1E+x_2I+x_3Q\end{aligned}\tag{5}$$

Based on equation (5) it's shows that the function $\mathscr{L}$ will be positive if $S^0\left(\frac{S}{S^0}-1-\ln\frac{S}{S^0}\right)>0$. Suppose that $\frac{S}{S^0}=y$ and $g(y)=\left(\frac{S}{S^0}-1-\ln\frac{S}{S^0}\right)$, then $g(y)=y-1-\ln y$. Function $g(y)$ will reach a maximum at time $y=1$ with $g(1)=0$, since $g(1)=0$ and $g^{''}(y)=\frac{1}{y^2}>0$. Therefore, we obtained $g(y)=\left(\frac{S}{S^0}-1-\ln\frac{S}{S^0}\right)>0$ for $S\neq S^0$. Then we will show that the equilibrium point S^0 is the global minimum point by using the Hessian matrix, we have

$$H\left(S^0\right)=\left[\frac{\partial^2\mathrm{L}}{\partial S^2}\right]=\left[\frac{1}{S^0}\right]\tag{6}$$

Matrix (6) is positive definite because $\det\left(H\left(S^0\right)\right)=\frac{1}{S^0}>0$. Then for $\mathrm{L}(S,E,I,Q)=\mathrm{L}(S^0,E^0,I^0,Q^0)$ we will be shown $\mathrm{L}(S,E,I,Q)=0$ as follows:

$$\begin{aligned}\mathrm{L} &= \left(S-S^0-S^0\ln\frac{S}{S^0}\right)+x_1E+x_2I+x_3Q\\ &=\left(\frac{\Lambda}{\mu}-\frac{\Lambda}{\mu}-\frac{\Lambda}{\mu}\ln 1\right)+0+0+0=0.\end{aligned}$$

So, it's proven that $\mathrm{L}(S,E,I,Q)>0$ when $\mathrm{L}(S,E,I,Q)\neq \mathrm{L}(S^0,E^0,I^0,Q^0)$ with $\mathrm{L}(S,E,I,Q)\in P$, and $\mathrm{L}(S,E,I,Q)=0$ when $\mathrm{L}(S,E,I,Q)=\mathrm{L}(S^0,E^0,I^0,Q^0)$, along with S^0 is the global minimum point.

(iii) The derivative of the function $\mathrm{L}(t)$ namely $\frac{d\mathrm{L}(t)}{dt}$ fulfill $\frac{d\mathrm{L}(t)}{dt}\leq 0$ for all points in *P*.

$$\frac{d\mathrm{L}}{dt}=\frac{\partial \mathrm{L}}{\partial S}\frac{dS}{dt}+\frac{\partial \mathrm{L}}{\partial E}\frac{dE}{dt}+\frac{\partial \mathrm{L}}{\partial I}\frac{dI}{dt}+\frac{\partial \mathrm{L}}{\partial Q}\frac{dQ}{dt}$$

$$\frac{d\mathrm{L}}{dt}=\left(1-\frac{S^0}{S}\right)\frac{dS}{dt}+k_1\frac{dE}{dt}+k_2\frac{dI}{dt}+k_3\frac{dQ}{dt}$$

$$\begin{aligned}\frac{d\mathrm{L}}{dt}&=\left(1-\frac{S^0}{S}\right)\left(\Lambda-\phi_c IS-\mu S\right)+\left(\phi_c IS-(\sigma+\mu)E\right)k_1\\ &+\left(\sigma E-(\alpha+v+\delta+\mu)I\right)k_2+\left(\alpha I-(\theta+\mu)Q\right)k_3.\end{aligned}\tag{7}$$

While $S\leq S^0=\frac{\Lambda}{\mu}$ so the equation (7) can be written to be

$$\begin{aligned}\frac{d\mathrm{L}}{dt}&=\left(\phi_c\frac{\Lambda}{\mu}k_1-(\alpha+v+\delta+\mu)k_2+\alpha k_3\right)I\\ &+\left(\sigma k_2-(\sigma+\mu)k_1\right)E-k_3(\theta+\mu)Q\end{aligned}\tag{8}$$

Suppose that $k_1=1, I=Q=0$ so the equation (8) can be written as follows

$$\frac{d\mathrm{L}}{dt}=(R_0-1)(\sigma+\mu)E.$$

When $\Re_0<1$, then the value of $\frac{d\mathrm{L}(t)}{dt}\leq 0$. So, it can be concluded that the Covid-19 free equilibrium point ε^0 for equation (1) is globally asymptotically stable. □

3.5 Global stability of the Covid-19 endemic equilibrium

Theorem 6. [26, 27] If $\Re_0>1$ then the Covid-19 endemic equilibrium ε^* is globally asymptotically stable. Otherwise, it's unstable.

Proof. To prove the endemic equilibrium point's stability, we can implement the Lyapunov methods. Assume $\Re_0>1$ and use an appropriate Lyapunov function $\mathrm{L}:P\subset\mathbb{R}^4\rightarrow\mathbb{R}$ with the form as follow

$$\Im(t)=\sum_{i=1}^{n}a_i\left(t_i-t_i^*-t_i^*\ln\frac{t_i}{t_i^*}\right),\tag{9}$$

with $t=(S,E,I,Q)$ and

$a_1=(\sigma+\mu), a_2=(\alpha+v+\delta+\mu), a_3=(\theta+\mu)$.

So that the equation (9) can be written as follows:

$$\begin{aligned}\Im(t)&=\left(S-S^*-S^*\ln\frac{S}{S^*}\right)+a_1\left(E-E^*-E^*\ln\frac{E}{E^*}\right)\\ &+a_2\left(I-I^*-I^*\ln\frac{I}{I^*}\right)+a_3\left(Q-Q^*-Q^*\ln\frac{Q}{Q^*}\right)\end{aligned}\tag{10}$$

Then we will investigate whether the Lyapunov function in eq. (10) fulfills the definition [28] and the Theorem 6. The Function $\Im$ is called Lyapunov function if only if fulfills the criterion below:

(i) The function $\Im$ consists of logarithmic functions, so it is clear that the function is a continuous function on *P*, and that the first partial derivative is also a continuous function on *P*.

(ii) The function $\Im$ have a minimum global at ε^* is related to all points in *P*. then we will investigate whether

$\Im(S,E,I,Q)=0$ for $\Im(S,E,I,Q)=\Im(S^*,E^*,I^*,Q^*)$ and $\Im(S,E,I,Q)>0$ for $\Im(S,E,I,Q)\neq\Im(S^*,E^*,I^*,Q^*)$.

For $\Im(S,E,I,Q)\neq\Im(S^*,E^*,I^*,Q^*)$ we will show that $\Im(S,E,I,Q)>0$, so we have:

$$\Im=\left(S-S^*-S^*\ln\frac{S}{S^*}\right)+\left(E-E^*-E^*\ln\frac{E}{E^*}\right)$$
$$+\left(I-I^*-I^*\ln\frac{I}{I^*}\right)+\left(Q-Q^*-Q^*\ln\frac{Q}{Q^*}\right)$$
$$\Im=S^*\left(\frac{S}{S^*}-1-S^*\ln\frac{S}{S^*}\right)+E^*\left(\frac{E}{E^*}-1-E^*\ln\frac{E}{E^*}\right)$$
$$+I^*\left(\frac{I}{I^*}-1-I^*\ln\frac{I}{I^*}\right)+Q^*\left(\frac{Q}{Q^*}-1-Q^*\ln\frac{Q}{Q^*}\right)$$

It shown that the function $\Im$ will be positive if $S^*\left(\frac{S}{S^*}-1-S^*\ln\frac{S}{S^*}\right)>0$, $E^*\left(\frac{E}{E^*}-1-E^*\ln\frac{E}{E^*}\right)>0$, $I^*\left(\frac{I}{I^*}-1-I^*\ln\frac{I}{I^*}\right)>0$, and $Q^*\left(\frac{Q}{Q^*}-1-Q^*\ln\frac{Q}{Q^*}\right)>0$.

Suppose that $\frac{S}{S^*}=b,\frac{E}{E^*}=c,\frac{I}{I^*}=d,\frac{Q}{Q^*}=e$, then there are functions such as $g(b)=b-1-\ln b$, $g(c)=c-1-\ln c$, $g(d)=d-1-\ln d$, and $g(e)=e-1-\ln e$. The functions $g(b),g(c),g(d),g(e)$ will reach a minimum at time $b=c=d=e=1$ with $g(1)=0$, while $g(1)=0$ and $g^{"}(b)=\frac{1}{b^2}>0,\ g^{"}(c)=\frac{1}{c^2}>0,\ g^{"}(d)=\frac{1}{d^2}>0.$ there was we obtained $g(b)=\left(\frac{S}{S^*}-1-S^*\ln\frac{S}{S^*}\right)>0$ for $S\neq S^*$, $g(c)=\left(\frac{E}{E^*}-1-E^*\ln\frac{E}{E^*}\right)>0$ for $E\neq E^*$, $g(d)=\left(\frac{I}{I^*}-1-I^*\ln\frac{I}{I^*}\right)>0$ for $I\neq I^*$, and $g(e)=\left(\frac{Q}{Q^*}-1-Q^*\ln\frac{Q}{Q^*}\right)>0$ for $Q\neq Q^*$. Next, we will show that the equilibrium point ε^* is minimum global point by using Hessian matrix, below:

$$H(S^*,E^*,I^*,Q^*)=\begin{bmatrix}\frac{1}{S^*}&0&0&0\\0&\frac{1}{E^*}&0&0\\0&0&\frac{1}{I^*}&0\\0&0&0&\frac{1}{Q^*}\end{bmatrix}\tag{11}$$

The matrix (11) is definite positive because $\det H\left((S^*,E^*,I^*,Q^*)\right)=\frac{1}{S^*E^*I^*Q^*}>0$.

For $\Im(S,E,I,Q)=\Im(S^0,E^0,I^0,Q^0)$ we will show that $\Im(S,E,I,Q)=0$, as follows

$$\Im=\left(0-S^*\ln1\right)+\left(0-E^*\ln1\right)+\left(0-I^*\ln1\right)+\left(0-Q^*\ln1\right)=0$$
$$\Im=-S^*\ln1-E^*\ln1-I^*\ln1-Q^*\ln1=0.$$

So, it's proven that $\Im(S,E,I,Q)>0$ while $\Im(S,E,I,Q)\neq\Im(S^*,E^*,I^*,Q^*)$ with $\Im(S,E,I,Q)\in P$, and $\Im(S,E,I,Q)=0$ while $\Im(S,E,I,Q)=\Im(S^*,E^*,I^*,Q^*)$, along with ε^* is the minimum global point.

(iii) the derivative function of $\Im(t)$ is $\frac{d\Im(t)}{dt}$ fulfill $\frac{d\Im(t)}{dt}\leq 0$ for all points in P.

$$\frac{d\Im(t)}{dt}=\frac{\partial\Im}{\partial S}\frac{dS}{dt}+\frac{\partial\Im}{\partial E}\frac{dE}{dt}+\frac{\partial\Im}{\partial I}\frac{dI}{dt}+\frac{\partial\Im}{\partial Q}\frac{dQ}{dt}$$
$$=\left(1-\frac{S^*}{S}\right)\frac{dS}{dt}+\left(1-\frac{E^*}{E}\right)\frac{dE}{dt}+\left(1-\frac{I^*}{I}\right)\frac{dI}{dt}+\left(1-\frac{Q^*}{Q}\right)\frac{dQ}{dt}$$
$$=\left(1-\frac{S^*}{S}\right)\left(\Lambda-\phi_c IS-\mu S\right)+\left(1-\frac{E^*}{E}\right)\left(\phi_c IS-a_1E\right)$$
$$+\left(1-\frac{I^*}{I}\right)\left(\sigma E-a_2I\right)+\left(1-\frac{Q^*}{Q}\right)\left(\alpha I-a_3Q\right)\tag{12}$$

by substitute the equation (9) into eq. (12) we have:

$$\frac{d\Im(t)}{dt}=\phi_cI^*S^*\left(2-\frac{S^*}{S}\right)+\mu S^*\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)$$
$$+\phi_cIS^*\left(1-\frac{S}{S^*}\frac{E^*}{E}\right)+\sigma E^*\left(1-\frac{I^*}{I}\frac{E}{E^*}+\frac{E}{E^*}\right)$$
$$+\alpha I\left(1-\frac{Q^*}{Q}\right)+a_3Q^*\left(1-\frac{Q}{Q^*}\right)-a_1E-a_2I$$

Suppose that $E^*=E$ and $I^*=I$, we get

$$\frac{d\Im(t)}{dt}=\phi_cI^*S^*\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)-\mu\left(\frac{S-S^*}{S}\right)^2$$
$$+\alpha I^*\left(2-\frac{Q^*}{Q}-\frac{Q}{Q^*}\right).$$

Based on Arithmetic Mean-Geometric Mean (AM-GM) Theorem [29] we obtained:

$$\frac{Q^*}{Q}-\frac{Q}{Q^*}\geq 2\sqrt{\frac{Q^*}{Q}\frac{Q}{Q^*}}$$
$$\frac{Q^*}{Q}-\frac{Q}{Q^*}\geq 2.$$

So, we obtained that $\frac{d\Im(t)}{dt} \le 0$, it can be concluded that the covid-19 endemic equilibrium point ε^* for the equation (1) is globally asymptotically stable. □

I. Optimal Control Analysis

The spread of Covid-19 that is out of control as it is today requires prevention efforts so that the virus can be controlled [30, 31, 32]. In this paper we propose four controls to add into model (1), namely Covid-19 vaccination $(u_1(t))$, self-prevention such as wearing masks and hand sanitisers $(u_2(t))$, treatment for individual who positive Covid-19 $(u_3(t))$, and physical distancing $(u_4(t))$. Suppose that resources are limited and cost function in a quadratic form, so we have Covid-19 vaccination as $(1-u_1(t))$, and for self-prevention, physical distancing, and treatment for individual who positive Covid-19 respectively can be written as $(u_2(t))$, $(u_3(t))$, $(u_4(t))$, So, the model (1) with optimal control can be written as

$$\left.\begin{aligned}
\frac{dS}{dt} &= \Lambda - \phi_c IS(1-u_1(t)) - \mu S(1-u_1(t)) \\
\frac{dE}{dt} &= \phi_c IS(1-u_1(t)) - \left(u_2(t)+\mu\right)E \\
\frac{dI}{dt} &= u_2(t)E - \left(\alpha + u_3(t) + \delta + \mu\right)I \\
\frac{dQ}{dt} &= \alpha I - \left(u_4(t)+\mu\right)Q \\
\frac{dR}{dt} &= u_4(t)Q - \mu R + u_3(t)I
\end{aligned}\right\} \qquad (13)$$

With the initial conditions $\dot{S}(0)>0, \dot{E}(0)>0, \dot{I}(0)>0, \dot{Q}(0)>0, \dot{R}(0)>0$. To find an optimal control of u_1^*, u_2^*, u_3^* and u_4^* then apply $J(u_1^*,u_2^*,u_3^*,u_4^*) = \min(J(u_1,u_2,u_3,u_4), u_1,u_2,u_3,u_4 \in U)$ where $U = \left\{(u_1,u_2,u_3,u_4) \mid 0 \le u_i(t) \le 1, i=1,2,3,4, t \in (o,T_r)\right\}$.

We define an objective function for the optimal control in this paper as follows as

$$\min J = \int_0^T \begin{bmatrix} A_1S(t) + A_2E(t) + A_3I(t) + A_4Q(t) \\ +\frac{1}{2}\left(w_1u_1^2(t) + w_2u_2^2(t) + w_3u_3^2(t) + w_4u_4^2(t)\right) \end{bmatrix} dt$$

where A_1, A_2, A_3, and A_4 are positive weight constants of susceptible, exposed, infected, and quarantine respectively. w_1, w_2, w_3, and w_4 are positive weight constants for control variables $u_1(t), u_2(t), u_3(t)$, and $u_4(t)$. For the quadratic costs $w_1u_1^2(t) + w_2u_2^2(t) + w_3u_3^2(t) + w_4u_4^2(t)$ are associated with four controls respectively vaccination, self-prevention such as wearing masks and hand sanitisers, treatment for individual who positive Covid-19, and physical distancing.

Then we can construct the Hamiltonian function as

$$\begin{aligned}
\mathrm{H} = f &+ \varphi_1\left(\Lambda - \phi_c IS(1-u_1(t)) - \mu S(1-u_1(t))\right) \\
&+ \varphi_2\left(\phi_c IS(1-u_1(t)) - \left(u_2(t)+\mu\right)E\right) + \varphi_3\left(u_2(t)E - \left(\alpha + u_3(t) + \delta + \right.\right. \\
&+ \varphi_4\left(\alpha I - \left(u_4(t)+\mu\right)Q\right) + \varphi_5\left(u_4(t)Q - \mu R + u_3(t)I\right)
\end{aligned}$$

the optimal control problems can be solved by using Pontryagin’s Maximum Principle [33,34] as follows as:

Theorem 7. Consider optimal control variables of $u_1^*, u_2^*, u_3^*, u_4^*$ and solutions S^*, E^*, I^*, Q^*, R^* of the system (1) for minimizing $J(u_1,u_2,u_3,u_4)$ over U. Then there exist adjoint variables $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ satisfied

$$\begin{aligned}
\frac{d\varphi_1}{dt} &= -A_1 + \varphi_1\left(\phi_c I(1-u_1(t)) + \mu(1-u_1(t))\right) - \varphi_2\left(\phi_c I(1-u_1(t))\right) \\
\frac{d\varphi_2}{dt} &= -A_2 - \varphi_2\left(u_2(t)+\mu\right) - \varphi_3 u_2(t) \\
\frac{d\varphi_3}{dt} &= -A_3 + \varphi_1\left(\phi_c S(1-u_1(t))\right) - \varphi_2\left(\phi_c S(1-u_1(t))\right) \\
&\quad + \varphi_3\left(\alpha + u_3(t) + \delta + \mu\right) - \varphi_4\alpha + \varphi_5 u_3(t) \\
\frac{d\varphi_4}{dt} &= -A_4 + \varphi_4\left(u_4(t)+\mu\right) - \varphi_5 u_4(t) \\
\frac{d\varphi_5}{dt} &= \varphi_5\mu
\end{aligned}$$

with conditions

$\varphi_1(T_r) = \varphi_2(T_r) = \varphi_3(T_r) = \varphi_4(T_r) = \varphi_5(T_r) = \varphi(T_r) = 0$

We have stationer conditions for control variables u_1, u_2, u_3, u_4 as

$$u_1 = -\frac{1}{w_1}\left(\varphi_1\left(\phi_c IS + \mu S\right) + \varphi_2\left(\phi_c IS\right)\right)$$

$$u_2 = \frac{1}{w_2}\left(\varphi_2 E - \varphi_3 E\right)$$

$$u_3 = \frac{1}{w_3}\left(\varphi_3 I - \varphi_5 I\right)$$

$$u_4 = \frac{1}{w_4}\left(\varphi_4 Q - \varphi_5 Q\right)$$

The boundary conditions are met, because of $0 \le u_i(t) \le 1; 0 \le t \le T_r$, so we have:

$$u_1^*(t)=\begin{cases}0, \text{ jika } u_1<0\\ u_1, \text{ jika } 0<u_1<1\\ 1, \text{ jika } u_1\geq 1\end{cases}$$

$$=\min\left\{u_{1\max}, \max\left(0, -\frac{1}{w_1}\left(\varphi_1\left(\phi_c IS+\mu S\right)+\varphi_2\left(\phi_c IS\right)\right)\right)\right\}$$

$$u_2^*(t)=\begin{cases}0, \text{ jika } u_2<0\\ u_2, \text{ jika } 0<u_2<1\\ 1, \text{ jika } u_2\geq 1\end{cases}$$

$$=\min\left\{u_{2\max}, \max\left(0, \frac{1}{w_2}\left(\varphi_2 E-\varphi_3 E\right)\right)\right\}$$

$$u_3^*(t)=\begin{cases}0, \text{ jika } u_3<0\\ u_3, \text{ jika } 0<u_3<1\\ 1, \text{ jika } u_3\geq 1\end{cases}$$

$$=\min\left\{u_{3\max}, \max\left(0, \frac{1}{w_3}\left(\varphi_3 I-\varphi_5 I\right)\right)\right\}$$

$$u_4^*(t)=\begin{cases}0, \text{ jika } u_4<0\\ u_4, \text{ jika } 0<u_4<1\\ 1, \text{ jika } u_4\geq 1\end{cases}$$

$$=\min\left\{u_{4\max}, \max\left(0, \frac{1}{w_4}\left(\varphi_4 Q-\varphi_5 Q\right)\right)\right\}$$

IV. NUMERICAL SIMULATIONS

Numerical simulations are used to verify the proposed dynamical model of the Covid-19 spread. Based on data of the Covid-19 transmission in the East Java province, Indonesia from March 1 until May 31, 2022 and by using nonlinear least square method, we found the parameters of the model eq. (1) in Table 1.

Table 1. Value of parameters in this model

Parameter	*Value*	*Unit*	*Source*
Λ	492,4869	*Individual.day*$^{-1}$	Estimated
ϕ_c	1,033x10^{-5}	*Individual.day*$^{-1}$	Estimated
σ	0,01076	*day*$^{-1}$	Estimated
μ	0,00004	*day*$^{-1}$	[35]
α	0,38146	*day*$^{-1}$	Estimated
δ	0,0913	*day*$^{-1}$	Estimated
θ	0,125	*day*$^{-1}$	Estimated
ν	0,100	*day*$^{-1}$	Estimated

The initial conditions are *S(0)*=11.114.198, *E(0)*=2.367.770, *I(0)*=536.173, *Q(0)*=70.250, and *R(0)*=479.668. Using Runga Kutta four order method [36] for solving the differential equation (1), Pontryagin's Maximum Principle for solving the optimal control (13), and using the Matlab 2016a software package, we presented numerical simulation of the dynamical behavior model COVID-19 transmission;

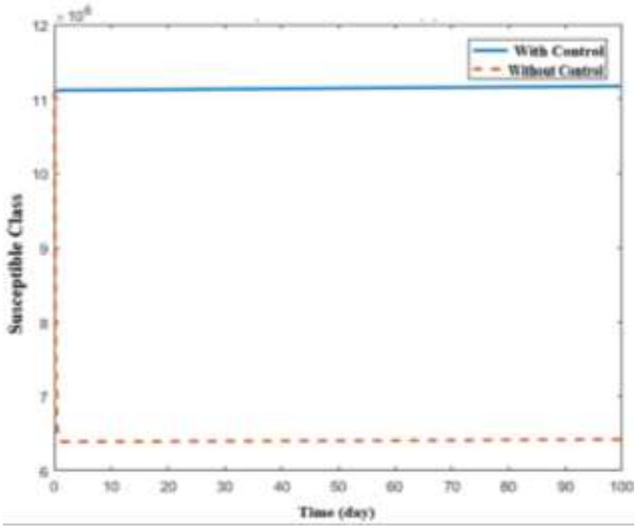

Fig. 2. The simulation of Covid-19 with control and without control in susceptible population

In Figure 1 it can be seen that the effect of giving Covid-19 vaccination control for susceptible individuals over a period of 60 days there is no significant impact. the susceptible populations with controls tend to be stable at their initial values due to the magnitude of the recruitment parameter compared to the reduction value itself, so the graphs for susceptible population with control tend to be stable at their initial values.

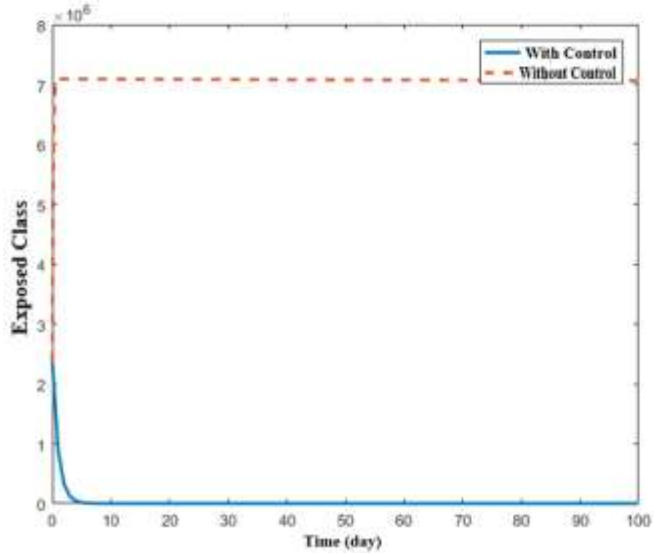

Fig. 3. The simulation of Covid-19 with control and without control in exposed population

Figure 2 shows that the provision of self-prevention controls such as wearing masks and hand sanitizers to exposed subpopulations shows that between the period of 0-5 days it has decreased towards its equilibrium point until the 60th day and will enter the infected subpopulation. Meanwhile, for the exposed subpopulation without control, it shows that the number of exposed subpopulations tends to increase over the same time period compared to the control.

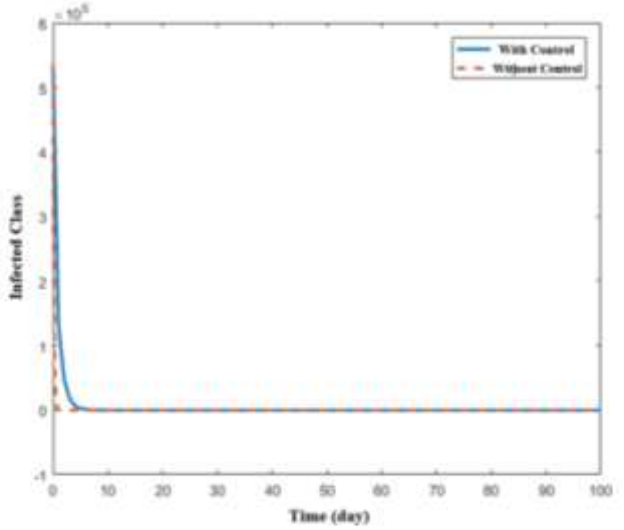

Fig. 4. The simulation of Covid-19 with control and without control in infected population

In Figure 3 it can be seen that the graph with control and without control has decreased cases. The reduction in infected cases with treatment control takes a little longer than without control because there are a number of exposed subpopulations that enter the infected stage, then the number of infected subpopulations will reach its equilibrium point. Furthermore, the infected sub-population will enter the quarantine stage before being declared cured of Covid-19 and some are immediately declared cured without having to carry out quarantine first.

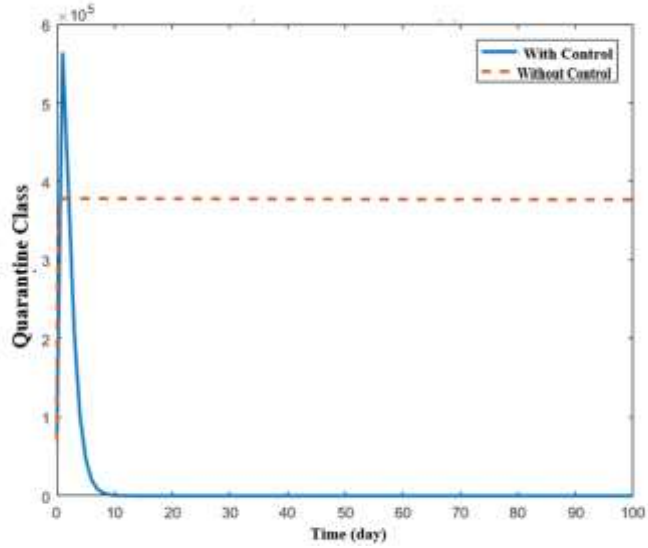

Fig. 5. The simulation of Covid-19 with control and without control in quarantine population

In Figure 4 it can be seen that there was an increase at the beginning of the graph, this happened because the infection rate was high on the same day so that infected individuals were quarantined. The provision of physical distancing controls or physical distancing for the quarantine sub-population shows that there has been a decrease in the quarantine sub-population and will then enter a period of recovery from Covid-19. Whereas for those without control, it can be seen that during the same period the quarantine subpopulation tends to increase and is stable at its initial value, so it takes a long time to reach its equilibrium point.

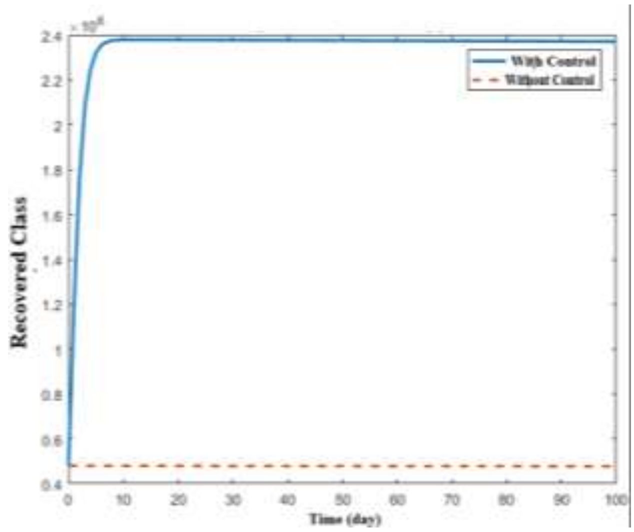

Fig. 6. The simulation of Covid-19 with control and without control in recovered population

Figure 5 shows that there has been an increase in individuals recovering from Covid-19. This is due to the effectiveness of providing the four controls, namely Covid-19 vaccination, self-prevention, treatment of Covid-19 sufferers, and physical distancing. Compared with no control, it can be seen that the recovered population did not increase in the same period.

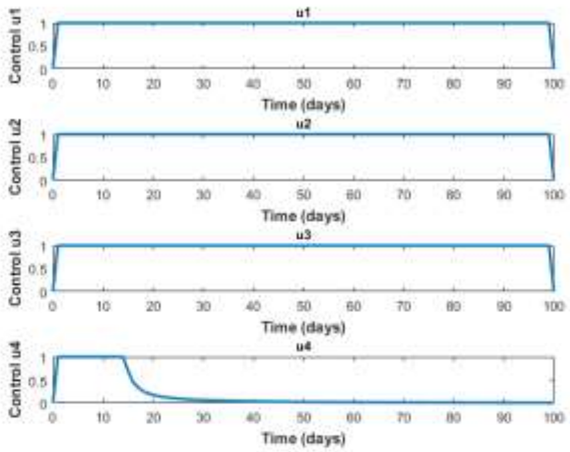

Fig. 7. Graph of control effectiveness level against time

In Figure 6, the first graph shows that the administration of Covid-19 vaccination control within 100 days has not shown satisfactory results because the administration of the vaccine must be gradual, starting from dose 1 to dose 3 or boosters, so it takes more than 100 days to be effective. reduce the number of Covid-19 cases. In Figure 6 the second graph shows that the application of self-prevention controls such as wearing masks and hand sanitizers correctly and correctly will be able to reduce the number of infected with Covid-19 after the 100th day.

In Figure 6, the third graph shows that giving optimal control of treatment to positive Covid-19 individuals is able to reduce positive cases of Covid-19 after the 100th day. In Figure 6, the fourth graph shows that the control of maintaining physical distance or physical distancing for the quarantined subpopulation can be seen that after 15 days there has been a decrease in Covid-19 cases.

V. CONCLUSION

The basic reproduction number was determined using the NGM (Next Matrix Generation) method and it was found that the value of the basic reproduction number was more than one, this means that the asymptotically stable model equation at the endemic equilibrium point or the spread of Covid-19 occurs, so it is necessary policies that can reduce the spread of Covid-19. In this case, the application of four controls has been discussed to control the spread of COVID-19 i.e. vaccination, self-prevention, treatment of Covid-19 sufferers, and physical distancing.

Numerical solving by using the fourth order Runge Kutta method with MATLAB software. Based on the simulation results, these indicates that the controls application of vaccination, self-prevention such as wearing masks and hand-sanitizers, treatment of positive individuals with Covid-19, and maintaining physical distancing is within the period 0 - 100 days will be able to reduce infections caused by Covid-19. Further, based on the effectiveness of each control it is obtained that the implementation of physical distancing controls has a better level of effectiveness than the vaccination, self-prevention, and treatment.

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