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Semi Analytic Approximate Solution Of Nonlinear Partial Differential Equations

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Abstract: Homotopy analysis method (HAM) is a very strong semi analytical method used to solve almost all nonlinear ordinary and partial differential equations. The effects of heat source/ sink of the boundary layer flow on a steady two dimensional flow and heat transfer past a shrinking sheet is studied by Homotopy Analysis Method. The series solution obtained by HAM is shown to be convergent for choosen h value which was obtained by h curve. Region of convergence is obtained by Domb-Sykes plot. We have also applied Pade for the HAM series and were able to identify the singularity and is reflected in the graph. The convergence of Homotopy series solution is obtained by the h curves. We find that HAM gives better approximation to the solutions.

Keywords: Homotopy analysis method, Domb-Sykes plot, Pade approximations, h-curves, Region of convergence.

1. Introduction

Homotopy Analysis Method (HAM) was first proposed by Liao in his Ph.D. thesis [2]. A systematic exposition on HAM is given in [3]. Solution of Non-linear partial differential equations can be solved analytically using HAM by L. N. Achala, S. B. Satyanarayana [4, 5] and many authors [6, 7, 8, 9, 10, 11].The incompressible fluid flow due to a shrinking sheet is gaining responsiveness by the current research scholars due to increase of applications. A Steady flow over a shrinking sheet is not possible because the generated vorticity is not restricted with in the boundary layers and needs an external opposite force at the sheet. Some analytical solution of the flow past a permeable shrinking sheet was studied by L. N. Achala and Sathyanarayana [1]. The effects of heat source or sink on the MHD boundary layer flow and heat transfer over a porous shrinking sheet with mass suction are investigated by K. Bhattacharyya [12]. A. Sami Bataineh, M.S.M. Noorani, I. Hashim [13], studied the approximate analytical solutions of systems of PDEs by HAM. M. Sajida, T. HayatIn [14] studied the application of HAM for MHD viscous flow due to a shrinking sheet. S.Nadeem, Anwar Hussain [15] studied MHD flow of a viscous fluid on a nonlinear porous shrinking sheet with HAM. Hang Xu,Shi-Jun Liao,Xiang-Cheng You [16], studied Analysis of nonlinear fractional partial differential equations with the HAM. In the present paper, heat transfer over a permeable shrinking sheet is studied.The solutions are compared with L. N. Achala and Sathyanarayana [1].

To the best of our knowledge , no one has solved PDE’s directly to flow problems. In all the above references they have studied the problem by converting governing partial differential equations in to ordinary differential equations by using similarity transformations. The convergence of HAM series solution is verified by a well known method called pade approximant and Domb-Sykes plot. The details of Basic idea of HAM can be found in [17, 18, 19, 20, 21].

Pade Approximation

The idea of pade summation is to replace a power series by a sequence of rational functions called pade approximants. The technique was developed by Henry pade. The pade approximant often gives better approximation of the function than truncating its Taylor's series and it may still work where the Taylor series does not converge.

A pade approximation of series is

$$f(z) = \sum_{n=0}^{\infty} a_n Z^n = Zp_M^N = \frac{\sum_{n=0}^{N} A_n Z^n}{\sum_{n=0}^{M} B_n Z^n}. \tag{1}$$

where we choose $B_0 = 1$ with out loss of generality.

Domb-Sykes plot

Domb-Sykes plot is used to estimate the radius of convergence of a power series say

$$\sum_{n=0}^{\infty} C_n Z^n. \tag{2}$$

Domb and Sykes proposed plotting c_n / c_{n-1} against $1/n$, fitting a straight line extrapolation and taking the intercept of this line as z_0, which is an estimate to the reciprocal of the radius of convergence.The distance to the nearest singularity can be determined by estimating the radius of convergence. The radius of convergence can be calculated using the D'Alembert's ratio test,

$$R = \frac{1}{\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|}. \tag{3}$$

2. Basic idea of HAM

In this paper we apply Homotopy analysis method for the governing partial differential equations. In order to discuss the basic idea of HAM, consider the following Non-linear differential equation,

$$N[u(x, y)] = 0. \tag{4}$$

where N is a non-linear operator, x and y denote the independent variables and u is an unknown function. Liao [3] constructs the so called zero order deformation equation

$$(1 - p)L[\varnothing(x, y : p) - u_0(x, y)] = qhH(x, y)N[\varnothing(x, y : p)], \tag{5}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\varnothing(x, y : p)$ is an unknown function, $u_0(x, y)$ is an initial guess of $u(x, y)$ and $H(x, y)$ denotes a nonzero auxiliary function. When the embedding parameter takes $p = 0$ and $p = 1$ then (5) becomes

$$\varnothing(x, y : 0) = u_0(x, y), \varnothing(x, y : 1) = u(x, y), \tag{6}$$

respectively. Thus as p increases from 0 to 1, the solution $\varnothing(x, y : p)$ varies from the initial guess to the solution $u(x, y)$ .Expanding $\varnothing(x, y : p)$ in Taylor series with respect to p ,

$$\varnothing(x,y:p) = u_0(x,y) + \sum_{m=1}^{\infty} u_m(x,y)\,p^m, \tag{7}$$

where

$$u_m(x,y) = \frac{1}{m!}\frac{\partial^m \varnothing(x,y:p)}{\partial p^m} \quad at \quad p = 0. \tag{8}$$

The convergence of the series (7) depends on the auxiliary parameter h .If it is convergent at $p = 1$, one has

$$u(x,y) = u_0(x,y) + \sum_{m=1}^{\infty} u_m(x,y). \tag{9}$$

Differentiating the zeroth-order deformation equation (5) m times with respect to p and then dividing them by $m!$ and finally setting $p = 0$, we get the following m^{ht} order deformation equation:

$$L\left[u_m(x,y) - \chi_m u_{m-1}(x,y)\right] = h\left(R_m u_{m-1}\right), \tag{10}$$

where

$$R_m u_{m-1} = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\varnothing(x,y:p)]}{\partial p^{m-1}} atp = 0, \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{12}$$

It is noted that $u_m(x,y)$ for $m \ge 1$ is governed by the linear equation (5) which can be solved by using Mathematica and Matlab.

3. Formulation of the problem

Consider the Newtonian fluid past a permeable shrinking sheet which is electrically conducting and magnetic field is applied perpendicular to the fluid flow. A boundary layer is formed due to the flow.Heat transfer is due to internal heat absorption or generation.The sheet coincides with x - axis and flow is confined to region $y > 0$. The governing equations for steady two dimensional flow in presence of uniform transverse magnetic field and the energy equations are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{13}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu_m \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_o^2}{\rho} u, \tag{14}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\kappa}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{Q_0}{\rho C_p}(T - T_\infty). \tag{15}$$

Where x and y are distances along and perpendicular to the sheet, u and v are components of the velocity along x and y directions respectively. $\nu_m = \frac{\mu}{\rho}$ is kinematic viscosity, ρ is fluid density, σ is electrical conductivity, B_o is the strength of the magnetic field. T is temperature, T_∞ is free stream temperature, κ is thermal conductivity of the fluid, Q_0 is volumetric rate of heat absorption or generation.

The corresponding boundary conditions are

$$u(x,0) = U_w = -cx, v(x,0) = -v_w, \tag{16}$$

$$u \to 0 \ as \ y \to \infty, \tag{17}$$

$$T(x,0) = T_w, \tag{18}$$

$$T \to T_\infty \ as \ y \to \infty. \tag{19}$$

Where $c > 0$ is the shrinking sheet. T_w is temperature of the sheet, v_w represents the wall mass suction through the porus sheet.

The dimensionless variables for u and T are:

$$u^* = \frac{u}{U_0}, v^* = \frac{v}{U_0}, x^* = \frac{x}{L_0}, y^* = \frac{y}{L_0}, \rho^* = \frac{\rho}{\rho_0}, T^* = \frac{T}{T_0}. \tag{20}$$

Using the dimensionless variables the equations (14) and (15) will take the form,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{21}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{R_e}\frac{\partial^2 u}{\partial y^2} - Mu, \tag{22}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{p_e}\frac{\partial^2 T}{\partial y^2} + \frac{\lambda}{p_e}. \tag{23}$$

Where $R_e = \frac{L\rho V}{\mu}$ is the Reynold's number, $p_e = \frac{cL_0 p_0 \rho_0 U_0}{k_0}$ is the Peclet number, M is the Hartmann number and λ is the heat source.

We apply the homotopy analysis method for (14) and (15) to obtain an approximate analytical solution of the boundary layer flow of a viscous flow over a nonlinearly shrinking sheet in the presence of magnetic field. Comparison of the present solution with the solution obtained by L. N. Achala and Sathyanarayana.[1] is shown through graphs.

4. HAM Solution

We apply Homotopy Analysis Method for (14). Consider the Linear operator as

$$L = \frac{\partial}{\partial y}. \tag{24}$$

On solving the equation $L(u) = 0$, we get the initial approximation as

$$u(x,0) = -cx. \tag{25}$$

Another initial guess can be found by (13)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{26}$$

$$v_i = -\int \frac{\partial u_i}{\partial x} dy + c1, \tag{27}$$

$$v(x,y) = cy - v_w. \tag{28}$$

The Dimensionless Nonlinear operator is

$$N_u = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{1}{R}\left(\frac{\partial^2 u}{\partial y^2}\right) + Mu. \tag{29}$$

Thus,

$$L(u_m - u_{m-1}\chi_m) = hR_m. \tag{30}$$

where,

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{31}$$

$$R_m = \sum_{i=0}^{m-1} u_i \frac{\partial}{\partial x} u_{-i+m-1} + \sum_{i=0}^{m-1} v_i \frac{\partial}{\partial y} u_{-i+m-1} - \frac{1}{R}\left(\frac{\partial^2 u_{m-1}}{\partial y^2}\right) + u_{m-1}M. \tag{32}$$

Using equations (25), (28), (32) along with $m = 1,2,3, \ldots$ in (30) we get

$$\begin{aligned}
u_0 &= -cx, \\
v_0 &= cy - B, \\
u_1 &= y\left(c^2hx - chMx\right) - cx, \\
v_1 &= c\left(\frac{1}{2}chy^2 - \frac{1}{2}hMy^2 - y\right), \\
u_2 &= -chx\left(cy(Bh-3) + My(2-Bh) + \frac{1}{2}c^2hy^2 - chMy^2 + \frac{1}{2}hM^2y^2\right) - cx, \\
v_2 &= \frac{1}{2}chy^2(Bch - BhM - 3c + 2M) + \frac{1}{6}ch^2y^3(c-M)^2 + cy.
\end{aligned} \tag{33}$$

and so on.

The HAM series solution for (14) is given by

$$u = u_0 + u_1 + u_2 + u_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{34}$$

We apply Homotopy Analysis Method for (15). Consider the Linear operator as

$$L = \left(\frac{\partial}{\partial y} + 1\right). \tag{35}$$

On solving the equation $L(T - T_\infty) = 0$, we get the initial approximation as

$$T - T_\infty = c_1 e^{-y}. \tag{36}$$

The Dimensionless Nonlinear operator is

$$N_u = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \frac{1}{p_e}\frac{\partial^2 T}{\partial y^2} - \frac{\lambda}{p_e}(1 - \chi_m). \tag{37}$$

Thus,

$$L(T_m - T_{m-1}\chi_m) = hR_m. \tag{38}$$

where,

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{39}$$

$$R_m = \sum_{i=0}^{m-1} u_i \frac{\partial}{\partial x} T_{-i+m-1} + \sum_{i=0}^{m-1} v_i \frac{\partial}{\partial y} T_{-i+m-1} - \frac{1}{p_e}\left(\frac{\partial^2 T_{m-1}}{\partial y^2}\right) - \frac{\lambda}{p_e}. \tag{40}$$

Using equations (36) and (40)along with $m = 1,2,3, \ldots$ in (38) we get

$$\begin{aligned}
T_0 &= Ae^{-y}, \\
T_1 &= \frac{he^{-y}(Ay(BP-1) - \frac{1}{2}AcPy^2 - \lambda e^y)}{P} + e^{-y}(Ae^{-y} + \frac{h\lambda}{P}), \\
T_2 &= \frac{1}{24P^2}\{he^{-2y}A(P^2(12B^2he^y(y-2)y - 12B(e^y y(ch(y-2)y-2) \\
&\quad + 4) + c(-4he^y y^3(3c - M) + 3che^y y^4 + 48y + 48)) \\
&\quad + 12P(e^y y(-2Bh(y-3) + ch(y^2 - 3y + 2) - 2) + 8) + 12he^y(y \\
&\quad - 4)y) - 12\lambda e^y(hy(-2BP + cPy + 2) + 4Pe^y))\} + e^{-y}\Big(Ae^{-y} \\
&\quad - \frac{h(A(P^2(48c - 48B) + 96P) - 48\lambda P)}{24P^2}\Big)
\end{aligned} \tag{41}$$

and so on.

The HAM series solution for (15) is given by

$$T = T_0 + T_1 + T_2 + T_3 + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \tag{42}$$

The graphs of u and T are drawn and are compared with previous results by different methods.

5. Results and Discussions

Figure 1 and 2 are h curves for velocity and Temperature. The value of h is estimated as $h = 0.1$ through these curves. For this h value velocity and temperature curves are drawn in Figure 3 and 4 for different values of M .These curve exactly matches with the one obtained in L. N. Achala and Sathyanarayana [1]. We observe that the graphs of the solution exhibit singularities for large values of y . In order to identify the singularity, we have applied Pade approximation for (34) with $h = 0.1$ and is given by,

$$\frac{0.11071 \quad y - 0.11}{0.00018278 \quad 1\,y^{5} + 0.00044621 \quad 1\,y^{4} - 0.0289078 \quad y^{3} + 0.286837 \quad y^{2} - 1.26353 \quad y + 1.0}. \tag{43}$$

We observe that there is a singularities at $y = -17.6948$ and $y = 6.61984$ and is also presented in Figure 5.

Pade approximation for (42) with $h = 0.1$ is,

$$\frac{0.7 - 325.9 \quad y}{0.118455 \quad y^{5} + 0.335887 \quad y^{4} + 0.780041 \quad y^{3} + 1.38463 \quad y^{2} + 1.65911 \quad y + 1.00000000 \quad 00}. \tag{44}$$

We observe that there is a singularities at $y = -1.28509$ which is seen in Figure 6. We have also estimated the radius of convergence for HAM solutions by Domb-Sykes plot and is presented in figure 8 and 9.

6. Graphs

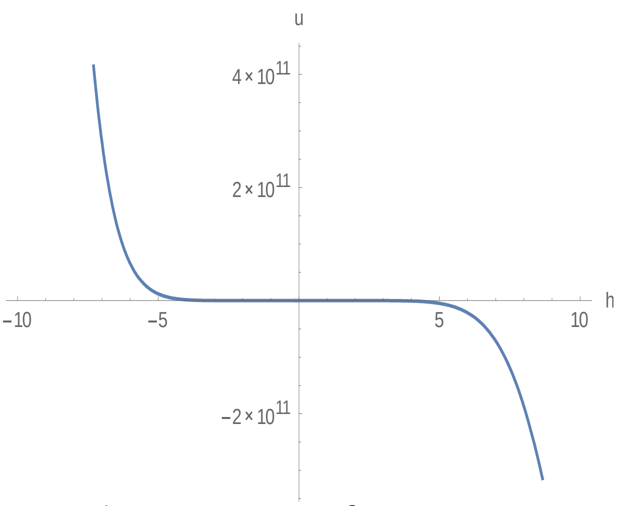

Figure 1 : h **-curve for** u

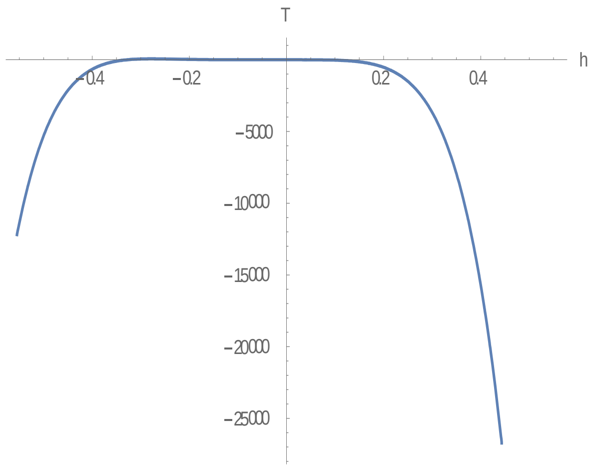

Figure 2 : h **-curve for** T

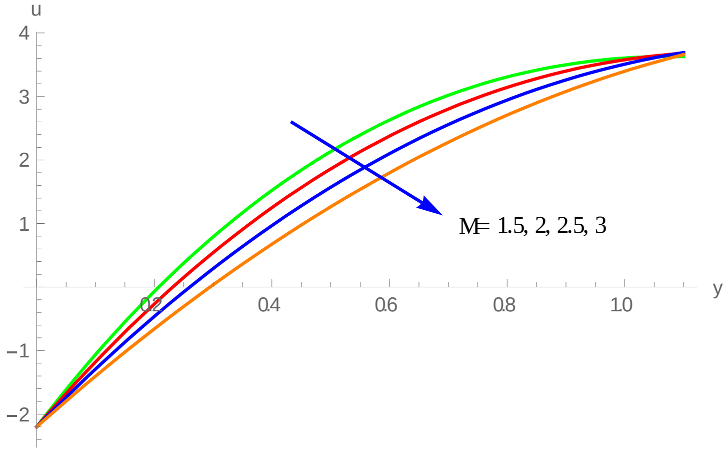

Figure 3 : Velocity Profiles for $M = 1.5, 2, 2.5, 3$

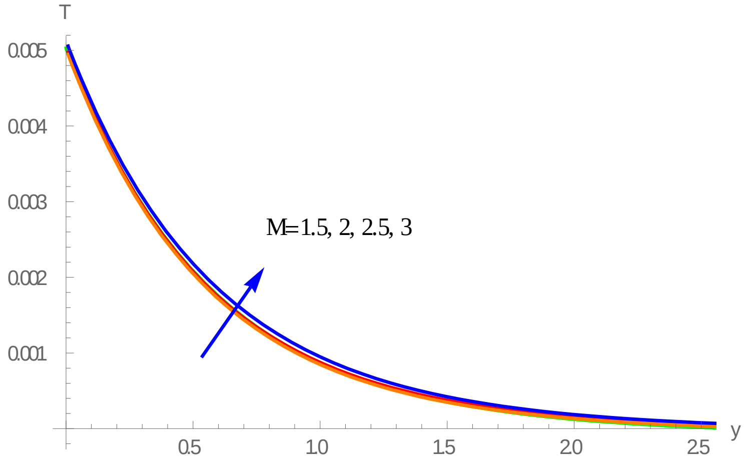

Figure 4 : Temperature Profiles for $M = 1.5, 2, 2.5, 3$

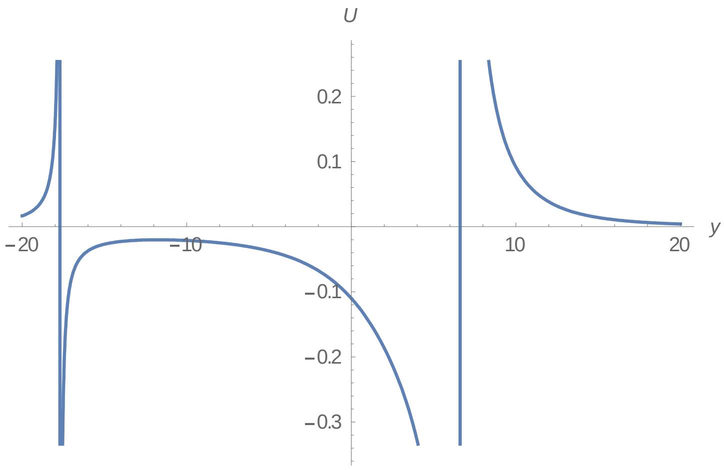

Figure 5 : singularities at $y = -17.6948, 6.61984$

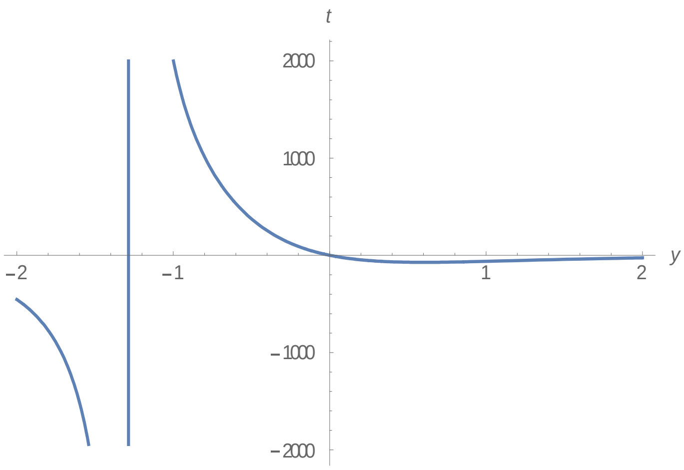

Figure 6 : singularities at $y = -1.28509$

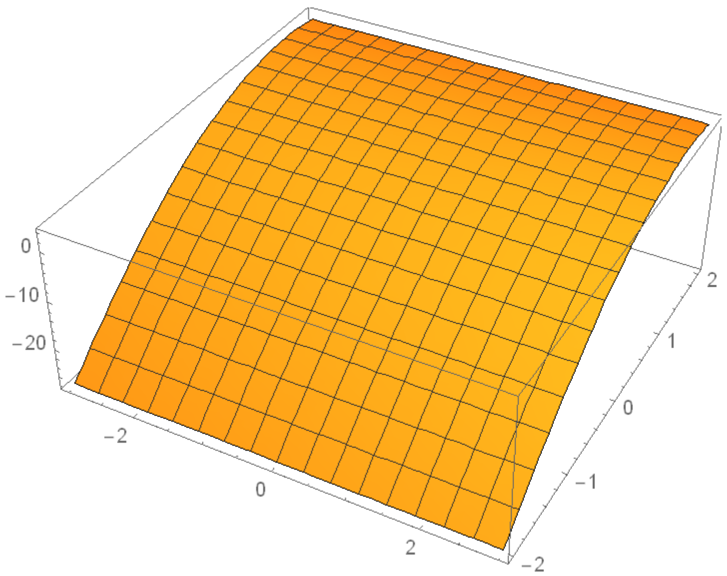

Figure 7 : 3D Plot for Velocity Profile

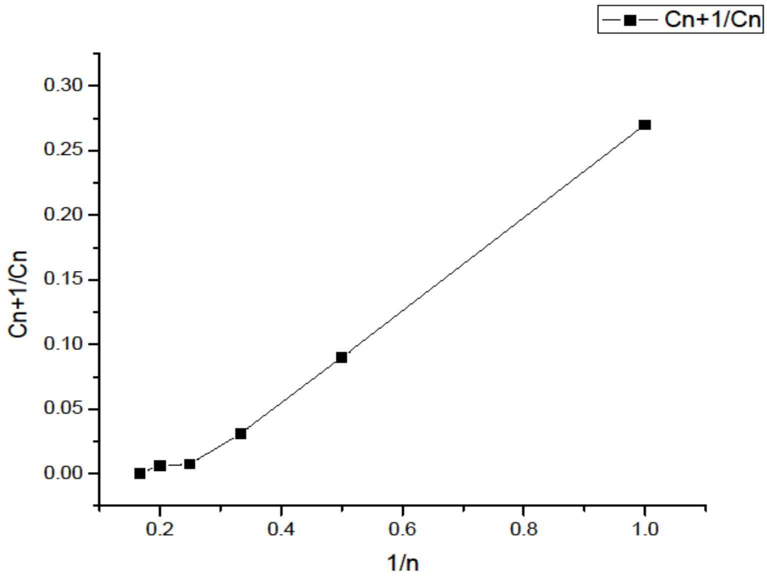

Figure 8 : Domb-Sykes plot for $h = 0.1$ **and** $R = 1/0.00618$

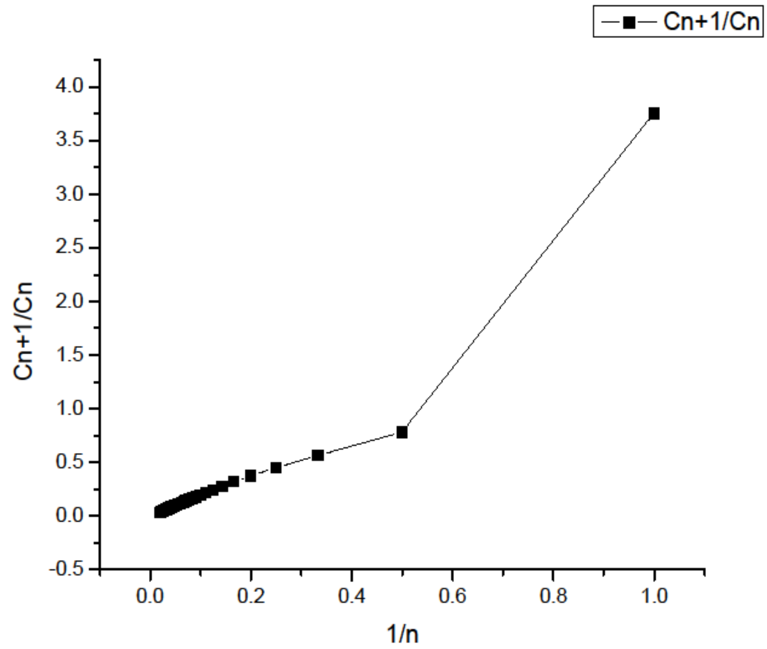

Figure 9 : Domb-Sykes plot for $h = 0.1$ **and** $R = 1/0.034$

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