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More Results on Multiring

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ABSTRACT

In this paper we extended the study of S. Debnath and A. Debnath(Debnath and Debnath, (2019)) on the study of ring structure from multiset context. We study more of the operations of multiset on the ring theory, where we discover that the raising to an arithmetic power of a multiring is again a multiring. So also is the composition of multirings. We also critically analyze the rootsets of multirings and introduces the concept multiring with unity, multiring and zero divisor among other results.

KEYWORDS: Multiset, Rings,Multiring, Rootsets and Mset operation.

1. INTRODUCTION

The theory of multiset is the generalization of the classical set theory which has emerge by violating a basic property of classical set that an element can belong to a set only once. This term multiset(mset for short) as Knuth noted in [1] first suggested by [2] in a private communication. Owing to its aptness it has replace a variety of terms viz; list, heap, bunch, bags, sample, weighted set, occurrence set and fireset (finitely repeated element set) used in different context but conveying synonymity with mset. The mset theory has various applications in mathematics, computer science. Other researchers study the mset theory from the perspective. For example: Tella and Daniel in [5] study the group theory in the perspective of mset, Nazmul et al in [6] extended the work of Tella and Daniel by adding two axioms and studying other aspect of the group theory.Ejegwa and Ibrahim[15 and 17] study the homomorphic nature of multigroups and the Abelian fuzzy multi groups. Girish and John in [7] wrote no Multiset Topology, where they lay a foundation of their studies on defining the power multiset, power whole multisets among others. S. Debnath and A. Debnath [18] introduces the study of rings from the multiset perspective. This paper now seeks to extend the research work on multirings. In addition to this section, section two of this paper gives the preliminary definitions and notations. Section three would contain the main results of the work, while section four would summarize and conclude.

2. PRELIMINARIES AND NOTATIONS

Definition 2.1[1]**.** A multiset(mset for short) A over the set X can be defined as a function $C_A: X \to \mathbb{N} = \{0,1,2, \dots\}$ where the value $C_A(x)$ denote the number of times or multiplicity or count function of x in A . For example, Let $A = [x, x, x, y, y, y, z, z]$, then $C_A(x) = 3, C_A(y) = 3, C_A(z) = 2$. $[C_A(x) = 0 \Leftrightarrow x \notin A]$. The mset M over the set X is said to be empty if $C_M(x) = 0\, for\ all\, x \in X$. We denote the empty mset by $\emptyset$. Then $C_\emptyset(x) = 0, \forall\, x \in X. if\ C_A(x) > 0, then\ x \in A$. We denote the class of all finite msets M over the set X by $M(X)$ throughout the study. Also, elements of mset A can belong exactly n many times denoted as $x \in^n A$. If $C_A(x) = n$ then the membership of x in A can be denoted by $x \in^n A$, meaning x belong to A exactly n times.

Definition 2.2[1]**:** The cardinality of a mset$M$ denoted $|M|$ or $card(M)$ is the sum of all the multiplicities of its elements given by the expression $|M| = \sum_{x \in X} c_A(x)$.

Note 1: Presentation of mset on paper work became a challenged as every researcher has his thought in that aspect . However the use of square brackets was adopted in ([1],[9],[11]) to represent an mset and ever since then it has become a standard. For example if the multiplicity of elements x, y and z in an mset M are 2,3 and 2 respectively, then the mset M can be represented as $M = [x, x, y, y, y, z, z,]$, others may put it like $[x, y, z]_{2,3,2}$ or $[x^2, y^3, z^2]$ or $[x2, y3, z2]$ or $[2/x\,, 3/y\,, 2/z]$

depending on one's taste or expediencies. But for conveniences sake, curly bracket can be used instead of the square bracket.

Definition2.3[2]: Let M be an mset drawn from a set X. The support set of M denoted byM^* is a subset of X given by $M^* = \{x \epsilon X : C_M(x) > 0\}$ that is M^* is an ordinary set. M^* is also called root set.

Definition 2.4[1]: Equal msets. Two msets$A, B \in M(X)$ are said to be equal, denoted $A = B$ if and only if for any objects $x \in X, C_A(x) = C_B(x)$. This is to say that $A = B$ if the multiplicity of every element in A is equal to its multiplicity in B and conversely. Clearly, $A = B \Longrightarrow A^* = B^*$, though the converse need not hold. For example, let $A = [a, a, b, b, c]$ and $B = [a, a, b. b, b, c, c]$ $where$ $A^* = B^* = \{a, b, c\}$ but $A \neq B$.

Definition 2.5[1]:Submultiset space. Let X be a set and let A and B be msets over X. A is a submultiset(submset for short) of B, denoted by $A \subseteq B$ or $B \supseteq A$, if $C_A(x) \leq C_B(x)$ $for\ all$ $x \in X$. Also if $A \subseteq B$ and $A \neq B$, then A is called proper submset of B denoted by $A \subset B$. In other words $A \subset B$ if $A \subseteq B$ $and\ there\ exist\ at\ least\ one$ $x \in X$ such that $C_A(x) < C_B(x)$. We assert that a mset B is called the parent mset in relation to the mset A.

Note 2: That: For any two msets$A, B \in M(X)$, $A = B$ if and only if $A \subseteq B$ and $B \supseteq A$.

Definition**.** 2.6 [1]: Regular or Constant mset: A mset$A$ over the set X is called regular or constant if all its elements are of the same multiplicities, i.e for any $x, y \in A$ $such\ that$ $x \neq y, C_A(x) = C_A(y)$.

Definition 2.7: [6]. The notations ⋀ and ⋁: The notations ⋀ and ⋁ denote the minimum and maximum operator respectively for instance

$C_A(x) \wedge C_A(y) = min\{C_A(x), C_A(y)\}$ and $C_A(x) \vee C_A(y) = max\{C_A(x), C_A(y)\}$.

Definition 2.8 [17](Power mset): Let $A \in M(X)$. The power mset of A, denoted $\wp(A)$, is defined as the mset of all submsets of A i.e $\wp(A) = \{m/p \mid p \subseteq A$ and $p \in^n \wp(A)\}$. For instance if $A = [x, y]_{2,1} = [x, x, y]$. Then $\wp(A) = [\emptyset, \{x\}, \{x\}, \{x\}_2, \{y\}, \{x, y\}, \{x, y\}, [x, y]_{2,1}]$.

In this case the cardinality of $\wp(A)$ is given by $Card(\wp(\text{A})) = 2^{Card(A)} = 2^3 = 8$, for any mset A.

For any $N \subseteq M$ such that $N \neq \emptyset$.

Now $N \in^k \wp(M)$ $if\ and\ only\ if$ $k = \prod_z \binom{|M|_z}{|N|_z}$. Where $\prod_z$ is the product taken over distinct elements z of the mset N . $|M|_z = m$ iff $z \in^m M$ and $|N|_z = n$ iff $z \in^n N$.

Note that $\binom{|M|_z}{|N|_z} = \binom{m}{n} = \frac{m!}{n!(m-n)!}$.

We denote the root set of $\wp(M)$ by $\wp^*(M)$.

Definition 2.9[17](Power set of an mset): Let $M \in M(X)$, the power set of M is just the root set $\wp^*(M)$.

Example 2.10: Let $M = \{6/x, 3/y\}$ be an mset and let $\wp(M)$ denote the power mset, if $\{3/x\}$ is a member of $\wp$(M), then $\{3/x\}$ repeats $k = \binom{6}{3} = 20$ $times$. Also, if $\{4/x, 2/y\}$ is a member of $\wp$(M), then $\{4/x, 2/y\}$ repeats $k = \binom{6}{4}\binom{3}{2} = 45$ $times$.

Theorem 2.11[17](Cardinality of power set): Let $M \in M(X)$ such that $M = \{{}^{m_1}/_{x_1}, {}^{m_2}/_{x_2}, \ldots, {}^{m_n}/_{x_n}\}$, then $Card(\wp^*(M)) = \prod_{i=1}^{n}(1 + m_i)$.

Definition 2.12[17](Whole submset): A submset$N$ of M is a whole submset of M

if $C_N(x) = C_M(x)\ \forall\ x \in N$.

Definition 2.13[17](Partial Whole Submset): A submset$N$ of M is a partial whole submset of M if there exist an element $x \in N$ such that $C_N(x) = C_M(x)$.

Definition 2.14[17](Full Submset): A submset$N$ of M is full submset if $M^* = N^*$

Example 2.15: Let $M = \{2/x, 3/y, 5/z\}$ be an mset. The following are some of the submset which are whole submsets, partial whole submset and full submets.

(a) A submset$\{2/x, 3/y\}$ is a whole submset and partial whole submset of M but it is not full submset of M.
(b) A submset$\{1/x, 3/y, 2/z\}$ is a partial whole submset and full submset of M but it is not a whole submet of M.
(c) A submset$\{1/x, 3/y\}$ is a partial whole submset of M which is neither a whole submet nor full submset of M.

Definition 2.16[17, 15] (Power whole mset): Let $M \in M(X)$ be an mset. The power whole mset of M denoted by $PW(M)$ is defined as the set of all whole submsets of M. The cardinality of the support set $PW(M)$ is 2^n where n is the cardinality of the support set M^*, i.e $n = \mid M^* \mid$.

Definition 2.17[17] (Power full mset): Let $M \in M(X)$ be an mset. Then the power full mset of M denoted, $PF(M)$, is defined as the set of all full submsets of M. The cardinality of $PF(M)$ is the product of the counts of the elements in M.

That is $PF(M) = \{y / y \subseteq M\}$.

Examples 2.18: Let $M = \{2/x, 3/y\}$ be a mset. Then $PW(M) = \{\{2/x\}, \{3/y\}, M, \emptyset\}$ and $PF(M) = \begin{Bmatrix} \{2/x, 1/y\}, \{2/x, 2/y\}, \{2/x, 3/y\}, \{1/x, 1/y\}, \\ \{1/x, 2/y\}, \{1/x, 3/y\}, \end{Bmatrix}$.

Definition 2.19 [9]: (⋀ and ⋁ notations): The notations ⋀ and ⋁ denote the minimum and maximum operator respectively, for instance;

$C_A(x) \wedge C_A(y) = min\{C_A(x), C_A(y)\}$ and $C_A(x) \vee C_A(y) = max\{C_A(x), C_A(y)\}$.

Operations on msets.

Definition 2.20[9]: Union (∪) of msets. Let $A, B \in M(X)$. The union of $A\ and\ B$ denoted by $A \cup B$is the mset defined by $C_{A\cup B}(x) = max\{C_A(x), C_B(x)\}$,

That is if object x occurs a times in A and b times in B. Then it occurs $a \vee b$ times in $A \cup B$, (maximum should exist for all finite msets).

Definition 2.21[9]: Intersection (∩) of msets. Let$A, B \in M(X)$.The intersection of two mset A and B denoted by $A \cap B$, is the mset for which

$C_{A\cap B}(x) = min\{C_A(x), C_B(x)\}$for all $x \in X$.

In other words, $A \cap B$ is the smallest mset which is contained in both A and B. That is an objects x occurring a times in A and b times in B, occurs $a \wedge b$ times in $A \cap B$.

Definition 2.22[9]: Addition or sum ofmset. Let $A, B \in M(X)$. The direct sum or arithmetic addition of A and B denoted by A+B or $A \uplus B$ is the mset defined by

$C_{A+B}(x) = C_A(x) + C_B(x)$ for all $x \in X$.

That is, an object x occurring a times in A and b times in B, occurs $a + b$ times in $A \uplus B$.

Note[9]: That $|A \uplus B|$= $|A \cup B|$ + $|A \cap B|$.

Definition 2.23[9]: Difference of msets. Let $A, B \in M(X)$, then the difference of B from A, denoted by $A - B$ is the mset such that $C_{A-B}(x) = max\{C_A(x) - C_B(x), 0\}$for all $x \in X$. If $B \subseteq A$, then $C_{A-B}(x) = C_A(x) - C_B(x)$.

It is sometimes called the arithmetic difference of B from A. If $B \nsubseteq A$ this definition still holds. It follows that the deletion of an element$x$ from an mset A give rise to a new mset$A' = A - x$ such that $C_{A'}(x) = max\{C_A(x) - 1, 0\}$.

Definition 2.24[8]: Symmetric Difference. Let X be a set and $A, B \in M(X)$Then the symmetric difference, denoted $A \Delta B$, is defined by $C_{A\Delta B}(x) = |C_A(x) - C_B(x)|$.

Note 3:: That it can easily be proved that $A \Delta B = (A - B) \cup (B - A)$.

Definition 2.25[8]: Complement in msets: Let $G = \{A_1, A_2, \ldots\}$ be a family of finite msets generated from the set X. Then, the maximum mset Z is defined by $C_Z(x) = max_{A\in G} C_A(x)$ for all $A \in G$ and $x \in X$. The Complement of an mset A, denoted by $\bar{A}$, and defined by

$\bar{A} = Z - A\ and\ C_{\bar{A}}(x) = \{C_Z(x) - C_A(x), for\ all\ x \in X\}$

where Z is a universal mset.

Note that $A_i \subseteq Z\ for\ all\ i$.

Definition 2.26[8]: Multiplication by Scalar. Let $A, B \in M(X)$, then the scalar multiplication denoted by $b.A$ is defined as $C_{b.A}(x) = b.C_A(x),\ and\ b = \{1,2,3, \ldots\}$.

Definition 2.27[8]: Arithmetic Multiplication. Let $A, B \in M(X)$, then the Arithmetic Multiplication denoted by $A.B$ is defined as $C_{A.B}(x) = C_A(x).C_B(x)\ \ \forall\ x \in X$.

Definition 2.28[7]: Raising to an Arithmetic Power. Let $A \in M(X)$, then A raised to power n denoted by A^n**is** defined:

$C_{A^n}(x) = \left(C_A(x)\right)^n\ for\ n = \{0,1,2,3, \ldots\}$ and $C_A(x) > 0$.

Proposition 2.29[19]: Let X be a set and let $A \in M(X)$. Then $A^* = A^0$.

Note 4:: That $A^n . A^m = A^{n+m}$, and $(A.B)^n = A^n . B^n\ for\ any\ n, m = \{0,1,2, \ldots\}$

Theorem 2.30 [11]: Let $M, N \in M(X)$, $M \subseteq N \Rightarrow M^* \subseteq N^*$

Definition 2.31[6]; Composition of msets. Let $A, B \in MG(X)$, then we call

(i) $A \circ B$ as the composition between two msets defined as

$C_{A\circ B}(x) = \vee\{C_A(y) \wedge C_B(z) : y, z \in X \ni yz = x\}$

Definition 2.32[19]: Let X be a non empty set and let $A, B \in M(X)$. We defined the mset function $f: A \to B$ as just the function $f: A^* \to B^*$ such that for any $x \in X$, $C_{f(A)}\left(f(x)\right) = C_A(x)$. The image of an mset $A \in M(X)$ under an mset function $f$denoted by $f(A)$ is given by

$f(A) = \left\{\frac{m_i}{f(x_i)} : x \in A,\ m_i = C_{f(A)}\left(f(x_i)\right) = C_A(x_i)\right\}$.

Definition 2.32[18]: Let X be a ring. A mset A over X is said to be a multiring over X if the count function of A i.e C_A satisfies the following condition:

(i) $C_A(x + y) \geq \min\{C_A(x), C_A(y)\}\ and\ C_A(x.y) \geq \min\{C_A(x), C_A(y)\}\ \forall x, y \in X$

(ii) $C_A(-x) \geq C_A(x)\ \forall\ x \in X$

The set of all multiring over $X$is denoted as $MR(X)$.

Definition 2.33[18]: Let $A \in MR(X)$, then A is said to be commutative iff $C_A(x.y) = C_A(y.x)$

Note 5: Non commutative ring may be commutative under multiring.

Theorem 2.34[18]: Let $A \in MR(X)$, then

(i) $C_A(0) \geq C_A(x)\ \forall\ x \in X$

(ii) $C_A(nx) \geq C_A(x)\ \forall\ x \in X$

(iii) $C_A(-x) = C_A(x)\ \forall\ x \in X$

Theorem 2.35[18]: Let $A \in MR(X)$. If $C_A(x) \leq C_A(y)$ for some $x, y \in X$then $C_A(x + y) = C_A(x)$.

Theorem 2.36[18] Let $A \in MR(X)$, then $C_A(x - y) = 0$ implies $C_A(x) = C_A(y)$.

Theorem 2.37[18] Let A be a mset. Then $A \in MR(X)$ iff $C_A(x - y) \geq \min\{C_A(x), C_A(y)\}$ and $C_A(x.y) \geq \min\{C_A(x), C_A(y)\}$.

Theorem 2.38[18]: Let $A \in MR(X)$, then $A_n, n \in N$ are subring of X, where $A_n = \{x \in X, C_A(x) \geq n, n \in N\}$.

Theorem 2.39[18]: If $A_n, n \in N$ are subrings of X, then the mset is A is a mring over X.

Theorem 2.40[18] Let $A, B \in MR(X)$, then $A \cap B \in MR(X)$.

Theorem 2.41[18] Let $A, B \in MR(X)$, then $A \cup B \notin MR(X)$.

Remark 1: If $A_i, i \in I$ be family of mrimg over X, then their intersection $\bigcap_{i \in I} A_i$ is mring.

3. RELATED RESULTS

Some mset operations on mring

Proposition 3.1: Let $A, B \in MR(X)$, then $A + B \in MR(X)$.

Proof: Since A and B are multiring then A satisfies $C_A(x-y) \geq \min\{C_A(x), C_A(y)\}$ *and* $C_A(x.y) \geq \min\{C_A(x), C_A(y)\} \forall x, y \in X$ and B saisfies $C_B(x-y) \geq \min\{C_B(x), C_B(y)\}$ *and* $C_B(x.y) \geq \min\{C_B(x), C_B(y)\} \forall x, y \in X$

Now $C_{A+B}(x-y) = C_A(x-y) + C_B(x-y)$ *by hypothesis*

$$\geq C_A(x) \wedge C_A(y) + C_B(x) \wedge C_B(y) \text{ by definition}$$
$$= [C_A(x) \wedge C_B(x)] + [C_A(y) \wedge C_B(y)]$$
$$= C_{A+B}(x) \wedge C_{A+B}(y)$$

And $C_{A+B}(x.y) = C_A(x.y) + C_B(x.y)$ *by hypothesis*

$$\geq C_A(x) \wedge C_A(y) + C_B(x) \wedge C_B(y) \text{ by definition}$$
$$= [C_A(x) \wedge C_B(x)] + [C_A(y) \wedge C_B(y)]$$
$$= C_{A+B}(x) \wedge C_{A+B}(y)$$

Hence $A + B \in MR(X)$

Remark 2: Let $A, B \in MR(X)$, then $A - B \notin MR(X)$.

Example 3.2.7:For example: Let $X = \{e, a, b, c\}$ be a groupoid with $a^2 = b^2 = c^2 = e^2 = e$ *and* $ab = ba = c, ac = ca = b, bc = cb = a$.

If $A = \{e, a, b, c\}_{5,3,2,2}$ and $B = \{e, a, b, c\}_{3,2,2,1}$. Then $A - B = \{e, a, b, c\}_{2,1,0,1}$.

Now

$$C_{A-B}(ac) = C_{A-B}(b) = 0 \ngeq C_{A-B}(a) \wedge C_{A-B}(c)$$
$$= \min\{C_{A-B}(a), C_{A-B}(c)\} = \min\{2.1\}$$
$$= 1$$

Showing that $A - B \notin MGP(X)$.

Proposition 3.2: Let $A, B \in MR(X)$, then $A.B \in MR(X)$.

Proof: Since A and B are multiring then A satisfies $C_A(x-y) \geq \min\{C_A(x), C_A(y)\}$ *and* $C_A(x.y) \geq \min\{C_A(x), C_A(y)\} \forall x, y \in X$ and B saisfies $C_B(x-y) \geq \min\{C_B(x), C_B(y)\}$ *and* $C_B(x.y) \geq \min\{C_B(x), C_B(y)\} \forall x, y \in X$

Now $C_{A.B}(x-y) = C_A(x-y).C_B(x-y)$ *by hypothesis*

$$\geq C_A(x) \wedge C_A(y).C_B(x) \wedge C_B(y) \text{ by definition}$$
$$= [C_A(x) \wedge C_B(x)].[C_A(y) \wedge C_B(y)]$$
$$= C_{A.B}(x) \wedge C_{A.B}(y)$$

And $C_{A.B}(x.y) = C_A(x.y).C_B(x.y)$ *by hypothesis*

$$\geq C_A(x) \wedge C_A(y).C_B(x) \wedge C_B(y) \text{ by definition}$$
$$= [C_A(x) \wedge C_B(x)].[C_A(y) \wedge C_B(y)]$$
$$= C_{A.B}(x) \wedge C_{A.B}(y)$$

Hence $A.B \in MR(X)$.

Proposition 3.3: Let $A \in MR(X)$ then the scalar multiplication $b.A \in MR(X), b > 0$.

Proof: Since A and B are multiring then A satisfies

$$C_A(x-y) \geq \min\{C_A(x), C_A(y)\} \text{ and } C_A(x.y) \geq \min\{C_A(x), C_A(y)\} \forall x, y \in X$$

Let $x, y \in X$ and $b \in \mathbb{N}$ (the set of natural numbers). Let $A \in MR(X)$. We want to show that $b.A \in MR(X)$.

Now

$$C_{b.A}(x-y) = b.C_A(x-y)$$
$$\geq b.[C_A(x) \wedge C_A(y)] \text{ (by hypothesis)}$$
$$= C_{b.A}(x) \wedge C_{b.A}(y)$$

Thus $C_{b.A}(xy) \geq C_{b.A}(x) \wedge C_{b.A}(y)$

$$C_{b.A}(xy) = b.C_A(xy)$$
$$\geq b.[C_A(x) \wedge C_A(y)] \text{ (by hypothesis)}$$
$$= C_{b.A}(x) \wedge C_{b.A}(y)$$

Thus $C_{b.A}(xy) \geq C_{b.A}(x) \wedge C_{b.A}(y)$

Hence $b.A \in MR(X)$.

Proposition 3.4: Let X be a semi-group and let $A \in MR(X)$, then $A^n \in MR(X)$ For any $n \in \{1,2,3, \dots\}$.

Proof: Since A and B are multiring then A satisfies $C_A(x-y) \geq \min\{C_A(x), C_A(y)\}$ *and* $C_A(x.y) \geq \min\{C_A(x), C_A(y)\} \forall x, y \in X$ and B saisfies $C_B(x-y) \geq \min\{C_B(x), C_B(y)\}$ *and* $C_B(x.y) \geq \min\{C_B(x), C_B(y)\} \forall x, y \in X$.

Let $x, y \in X$ and let $A \in MR(X)$. We want to show that $A^n \in MR(X)$.

Since $C_{A^n}(x-y) = (C_A(x-y))^n \geq [C_A(x) \wedge C_A(y)]^n$

$$= [C_A(x)]^n \wedge [C_A(y)]^n$$
$$= C_{A^n}(x) \wedge C_{A^n}(y)$$

Thus $C_{A^n}(xy) \geq C_{A^n}(x) \wedge C_{A^n}(y)$.

$$C_{A^n}(xy) = (C_A(xy))^n \geq [C_A(x) \wedge C_A(y)]^n$$
$$= [C_A(x)]^n \wedge [C_A(y)]^n$$
$$= C_{A^n}(x) \wedge C_{A^n}(y)$$

Thus $C_{A^n}(xy) \geq C_{A^n}(x) \wedge C_{A^n}(y)$.

Hence $A^n \in MR(X)$.

Proposition 3.5: Let $A, B \in MR(X)$. Then the direct product or the Cartesian product $A \times B \in MR(X \times X), \forall (x, y) \in (X \times X)$.

Proof: Since A and B are multiring then A satisfies $C_A(x-y) \geq \min\{C_A(x), C_A(y)\}$ *and* $C_A(x.y) \geq \min\{C_A(x), C_A(y)\} \forall x, y \in X$ and B saisfies $C_B(x-y) \geq \min\{C_B(x), C_B(y)\}$ *and* $C_B(x.y) \geq \min\{C_B(x), C_B(y)\} \forall x, y \in X$

Let $A, B \in MR(X \times X)$. If $x = (a, b), y = (c, d)$ *where* $a, c \in A$ *and* $b, d \in B$, such that the pair can be express as $xy = (a, b)(c, d) = (ac, bd)$ (by definition).

Now, $C_{A \times B}(x-y) = C_{A \times B}[(a, b) - (c, d)] = C_{A \times B}[a-c, b-d] = C_A(a-c).C_B(b-d)$

$$\geq [C_A(a) \wedge C_A(c)].[C_B(b) \wedge C_B(d)]$$
$$= [C_A(a).C_B(b)] \wedge [C_A(c).C_B(d)]$$
$$= [C_{A \times B}(a, b) \wedge C_{A \times B}(c, d)]$$
$$= [C_{A \times B}(x) \wedge C_{A \times B}(y)]$$

And $C_{A\times B}(xy) = C_{A\times B}[(a,b)(c,d)] = C_{A\times B}[ac,bd] = C_A(ac).C_B(bd)$

$$\geq [C_A(a)\wedge C_A(c)].[C_B(b)\wedge C_B(d)]$$
$$= [C_A(a).C_B(b)]\wedge[C_A(c).C_B(d)]$$
$$= [C_{A\times B}(a,b)\wedge C_{A\times B}(c,d)]$$
$$= [C_{A\times B}(x)\wedge C_{A\times B}(y)]$$

Thus $C_{A\times B}(xy) \geq [C_{A\times B}(x)\wedge C_{A\times B}(y)]$.

Hence $A\times B \in MR(X\times X)$.

Proposition 3.6: Let $A_i \in MGP(X\times X\times \cdots \times X)\ n-times, for\ i = 1,\dots,n$. Then raising to the direct $power\ (\times A)^n \in MR(X\times X\times\cdots\times X), n-times.$

Proof: Since A and B are multiring then A satisfies $C_A(x-y)\geq \min\{C_A(x),C_A(y)\}\ and\ C_A(x.y)\geq \min\{C_A(x),C_A(y)\}\forall x,y\in X$ and B saisfies

$$C_B(x-y)\geq \min\{C_B(x),C_B(y)\}\ and\ C_B(x.y) \geq \min\{C_B(x),C_B(y)\}\forall x,y\in X$$

And $since\ C_{(\times A)^n}(x-y) = \left[C_{(\times A)}(x-y)\right]^n\ by\ hypothesis.$

$$\geq \left[C_{(\times A)}(x)\wedge C_{(\times A)}(y)\right]^n by\ definition$$
$$= \left[C_{(\times A)^n}(x)\wedge C_{(\times A)^n}(y)\right]$$
$$= \left[\left(C_A(x_1).C_A(x_2).,\dots,.C_A(x_n)\right)\bigwedge\ \left(C_A(y_1).C_A(y_2).,\dots,.C_A(y_n)\right)\right]$$
$$\forall\ x_i,y_i \in A, i = i,\dots,n$$

Also $C_{(\times A)^n}(xy) = \left[C_{(\times A)}(xy)\right]^n\ by\ hypothesis.$

$$\geq \left[C_{(\times A)}(x)\wedge C_{(\times A)}(y)\right]^n by\ definition$$
$$= \left[C_{(\times A)^n}(x)\wedge C_{(\times A)^n}(y)\right]$$
$$= \left[\left(C_A(x_1).C_A(x_2).,\dots,.C_A(x_n)\right)\bigwedge\ \left(C_A(y_1).C_A(y_2).,\dots,.C_A(y_n)\right)\right]$$
$$\forall\ x_i,y_i \in A, i = i,\dots,n$$

Hence $(\times A)^n \in MR(X)$.

Proposition 3.7: Let $x,y\in X$. Let $A\in MR(X)$. Then $A^n.A^m = A^{n+m}\in MR(X), for\ n,m\in \mathbb{N}$.

Proof: Since $C_{A^n.A^m}(x-y) = C_{A^{n+m}}(x-y) = [C_A(x-y)]^{n+m}(by\ definition)$

$$\geq [C_A(x)\wedge C_A(y)]^{n+m}\ (by\ hypothesis)$$
$$= C_{A^{n+m}}(x)\wedge C_{A^{n+m}}(y)$$

Also, $C_{A^n.A^m}(xy) = C_{A^{n+m}}(xy) = [C_A(xy)]^{n+m}(by\ definition)$

$$\geq [C_A(x)\wedge C_A(y)]^{n+m}\ (by\ hypothesis)$$
$$= C_{A^{n+m}}(x)\wedge C_{A^{n+m}}(y)$$

So $C_{A^n.A^m}(xy)\geq C_{A^{n+m}}(x)\wedge C_{A^{n+m}}(y)$.

Thus $A^n.A^m = A^{n+m}\in MR(X)$.

Proposition 3.8: Let $x,y\in X$. Let $A,B\in MR(X)$. Then $(A.B)^n = A^n.B^n \in MR(X), for\ n\in\mathbb{N}$.

Proof: Since A and B are multiring then A satisfies $C_A(x-y)\geq \min\{C_A(x),C_A(y)\}\ and\ C_A(x.y)\geq \min\{C_A(x),C_A(y)\}\forall x,y\in X$ and B saisfies

$$C_B(x-y)\geq \min\{C_B(x),C_B(y)\}\ and\ C_B(x.y) \geq \min\{C_B(x),C_B(y)\}\forall x,y\in X$$

So, $C_{(A.B)^n}(x-y) = [C_{A.B}(x-y)]^n = [[C_A(x-y)].[C_B(x-y)]]^n$

$$= [C_A(x-y)]^n.[C_B(x-y)]^n$$
$$\geq [C_A(x)\wedge C_A(y)]^n.[C_B(x)\wedge C_B(y)]^n$$
$$= C_{A^n}(x)\wedge C_{A^n}(x).C_{B^n}(y)\wedge C_{B^n}(y)$$
$$= [C_{A^n}(x)\wedge C_{B^n}(x)].[C_{A^n}(y)\wedge C_{B^n}(y)]$$
$$= C_{A^n.B^n}(x)\wedge C_{A^n.B^n}(y)$$

i.e $C_{(A.B)^n}(x-y)\geq C_{A^n.B^n}(x)\wedge C_{A^n.B^n}(y)$

Also, $C_{(A.B)^n}(xy) = [C_{A.B}(xy)]^n = [[C_A(xy)].[C_B(xy)]]^n$

$$= [C_A(xy)]^n.[C_B(xy)]^n$$
$$\geq [C_A(x)\wedge C_A(y)]^n.[C_B(x)\wedge C_B(y)]^n$$
$$= C_{A^n}(x)\wedge C_{A^n}(x).C_{B^n}(y)\wedge C_{B^n}(y)$$
$$= [C_{A^n}(x)\wedge C_{B^n}(x)].[C_{A^n}(y)\wedge C_{B^n}(y)]$$
$$= C_{A^n.B^n}(x)\wedge C_{A^n.B^n}(y)$$

Thus, $C_{(A.B)^n}(x-y)\geq C_{A^n.B^n}(x)\wedge C_{A^n.B^n}(y)$

Hence $(A.B)^n = A^n.B^n\in MR(X)$

Proposition 3.9: Let $A,B\in MR(X)$, then $A\circ B\in MR(X)$

Proof: Since A and B are multiring then A satisfies $C_A(x-y)\geq \min\{C_A(x),C_A(y)\}\ and\ C_A(x.y)\geq \min\{C_A(x),C_A(y)\}\forall x,y\in X$ and B saisfies

$$C_B(x-y)\geq \min\{C_B(x),C_B(y)\}\ and\ C_B(x.y) \geq \min\{C_B(x),C_B(y)\}\forall x,y\in X$$

And so, let $x,y\in X$. Let $A,B\in MR(X)$. We show that $AoB\in MR(X)$.

Now, $C_{AoB}(x-y) = \vee[C_A(a-b)\wedge C_B(c-d); a,b,c,d\in X,(ac)(bd) = x+y]$

$$\geq \vee[[C_A(a)\wedge C_A(b)]\wedge[C_B(c)\wedge C_B(d)]; a,b,c,d\in X,(ac)(bd) = xy]$$
$$= \vee[[C_A(a)\wedge C_B(c)]; a,c\in X,(ac) = x]\wedge[C_A(b)\wedge C_B(d)]; b,d\in X,(bd) = y]]$$
$$= [\vee[C_A(a)\wedge C_B(c)]; a,c\in X,(ac) = x]\wedge[\vee[C_A(b)\wedge C_B(d)]; b,d\in X,(bd) = y]]]$$
$$= C_{AoB}(x)\wedge C_{AoB}(y)$$

Thus $C_{AoB}(x-y)\geq C_{AoB}(x)\wedge C_{AoB}(y)$.

Also, $C_{AoB}(xy) = \vee[C_A(ab)\wedge C_B(cd); a,b,c,d\in X,(ac)(bd) = xy]$

$$\geq \vee[[C_A(a)\wedge C_A(b)]\wedge[C_B(c)\wedge C_B(d)]; a,b,c,d\in X,(ac)(bd) = xy]$$
$$= \vee[[C_A(a)\wedge C_B(c)]; a,c\in X,(ac) = x]\wedge[C_A(b)\wedge C_B(d)]; b,d\in X,(bd) = y]]$$
$$= [\vee[C_A(a)\wedge C_B(c)]; a,c\in X,(ac) = x]\wedge[\vee[C_A(b)\wedge C_B(d)]; b,d\in X,(bd) = y]]]$$
$$= C_{AoB}(x)\wedge C_{AoB}(y)$$

Thus $C_{AoB}(xy)\geq C_{AoB}(x)\wedge C_{AoB}(y)$.

Hence $AoB\in MGR(X)$.

Definition 3.10: Multiring with unity.

Let $A \in MR(X)$. We defined A to be a mring with unity if A^* is a ring with unity.

Definition 3.11: Zero divisor of mring.

Let $A \in MR(X)$. We defined A to be a mring with zero divisor if A^* is a ring with a zero divisor.

Proposition 3.12: Let $A \in MR(X)$. Then A^* is a sub ring of X.

Proof: Supposing $A \in MGP(X)$. Let $x, y \in A^*$, then $C_A(x), C_A(x) > 0(by\ definition\ 2.3)$.

In particular $C_A(x) \wedge C_A(y) > 0$. But $A \in MR(X)$ implies

$$C_A(xy) \geq C_A(x) \wedge C_A(y) > 0 \ and \ C_A(x-y) \geq C_A(x) \wedge C_A(y) \forall\, x, y \in X$$

Thus $C_A(xy)\ and\ C_A(x-y) > 0$ i.e $xy\ and\ x - y \in A^*$.

In particular A^* is a sub subring of X (since $A^* \subseteq X$)

Proposition 3.13: For any $A \in MR(X)$, $A^* \in MR(X)$.

Proof: Let $x, y \in A^*$. We want to show that

$$C_{A^*}(xy) \geq C_{A^*}(x) \wedge C_{A^*}(y), \forall\, x, y \in X \quad (*)$$

$$C_{A^*}(x-y) \geq C_{A^*}(x) \wedge C_{A^*}(y), \forall\, x, y \in X \quad (**)$$

From the following possibilities:

(i) $x, y \in A^* \Rightarrow xy \in A^*$ (from proposition 3.12)

(ii) $x \in A^*$and$y \notin A^* \Rightarrow xy \in A^*$or $xy \notin A^*$

(iii) $x \notin A^*$and$y \in A^* \Rightarrow xy \in A^*$or $xy \notin A^*$

(iv) $x \notin A^*$and$y \notin A^* \Rightarrow xy \in A^*$or $xy \notin A^*$ and

(v) $x, y \in A^* \Rightarrow x - y \in A^*$ (from proposition 3.12)

(vi) $x \in A^*$and$y \notin A^* \Rightarrow x - y \in A^*$or $x - y \notin A^*$

(vii) $x \notin A^*$and$y \in A^* \Rightarrow x - y \in A^*$or $x - y \notin A^*$

(viii) $x \notin A^*$and$y \notin A^* \Rightarrow x - y \in A^*$or $x - y \notin A^*$

the inequality $(*)$ and $(**)$ is valid, from (i) - (viii) above.

Thus $C_{A^*}(xy) \geq C_{A^*}(x) \wedge C_{A^*}(y)\ and\ C_{A^*}(x-y) \geq C_{A^*}(x) \wedge C_{A^*}(y)\ \forall\, x, y \in X$

In particular, $A^* \in MGP(X)$.

Proposition 3.14: Let $A \in MR(X)$, then Let $A^{-1} = A\ and - A = A$.

Proof: Since $C_{A^{-1}}(x) = C_A(x)^{-1} = C_A(x)$. Thus $A^{-1} = A$. And $C_{-A}(x) = C_A(-x) = C_A(x)$ that is $-A = A$.

4. CONCLUSION

We have extended the work of S. Debnath and A. Debnath (Debnath and Debnath, (2019)) on the study of ring structure from multiset context. In the study, we have established the closure of some multiset operations (over the class of finite mring). Such as union, intersection, arithmetic multiplication, scalar multiplication, raising to an arithmetic power among others. We have established that the root set of an mring is a subring and sub mring. Multi ring with unity and multi ring with zero divisor is introduced and studied.

Further Directions: Other aspect of the ring theory such as the integral domain, theory of field can be exploited among so many others.

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