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Dynamic Comparison of Variations of Newton's Methods with Different Types of Means for Solving Nonlinear Equations

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ARTICLE INFO	ABSTRACT
Published Online: 19 November 2022 Corresponding Author: **M. Imran**	This article discusses dynamic comparison of variations of Newton's methods with different types of means for solving nonlinear equations. There are two factors that are considered to affect the shape of the basin of attraction of a method namely the size of the determined convergence area and number of partitions. The computation results of some functions show that harmonic mean Newton's method (HMN) has small divergence area. On the other hand, contra harmonic mean Newton's method (CMN) has the largest divergence area and is considered to be the least effective method.

KEYWORDS: Basin of attraction, variations of Newton method, arithmetic mean, harmonic mean, geometry mean, heronian mean, root mean square, contra harmonic mean.

I. INTRODUCTION

Finding the solution of nonlinear equation $f(x)=0$ is one of the significant topics in mathematics and is widely used in other area of science such as in engineering, natural sciences, and economics. Some improvements of iterative methods are usually focused on number of iterations needed to fulfill the convergence criteria, number of function evaluation, CPU time and efficiency index.

In the advancement, some researchers have succeeded in modifying Newton's method and increasing the speed of convergence of the modified method such as Halley [7], Argyros [1], and Weerakoon et.al. [18]. Weerakoon had derived a modified Newton's method using arithmetic mean. Other variations of Newton's method are harmonic mean Newton's method by Özban [12], geometric mean Newton's method by Lukić et.al. [17], root mean square Newton's square by Syamsudhuha et.al. [16], Newton's method based on Heronian mean by Imran [8] and Newton's method based on contra harmonic mean by Ababneh [11].

Analyzing basin of attraction is one of the newly developed approaches in the subject of iterative method for solving nonlinear equations. Through this analysis one can observe the behavior of the convergence of the iteration for some initial values. It helps to provide answers to some of the questions regarding problem of finding solutions of nonlinear equations such as how close the initial guess with the root? Is there a particular way of choosing the initial guess? How to make a comparison between the iterative methods for solving nonlinear equations based on the initial guess? Should all the initial guesses converge to the root if different iterative methods is applied?

II. VARIATIONS OF NEWTON'S METHOD

Newton's method is one of the most known methods for solving nonlinear equations numerically. Atkinson et.al. in [2] described the basic idea by using tangent line to approximate root α of a nonlinear equation $f(x)=0$. Formula of Newton's method is defined by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},\quad n=0,1,2,\ldots \tag{1}$$

where $f'(x_n)\neq 0$. The method convergent quadratically and it needs two function evaluations in each iteration. The efficiency index of this method is 1.414.

Researchers have developed some modifications on Newton's method where one of the purposes was to increase the speed of convergence. Weerakoon et.al. in [18] introduced a variation of Newton's method with arithmetic mean (AMN) that was derived from integrating the Newton's method

$$f(x)=f(x_n)+\int_{x_n}^{x} f'(\mu)d\mu \tag{2}$$

By employing the trapezoid to approximate the area of the integral in equation (2) it yields

$$x_{n+1}=x_n-\frac{2f(x_n)}{f'(x_n)+f'(y_n)},\quad n=0,1,2,\ldots \tag{3}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$. This method has third order of convergence and takes three function evaluations in each iteration with efficiency index 1.442 which is better than Newton's method.

Some researchers have developed iterative methods by using different types of means. The following methods are of third order of convergence and have efficiency index 1.442. Özban [12] constructed harmonic mean Newton's method (HMN),

$$x_{n+1} = x_n - \frac{f(x_n)(f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)}, \quad n = 0,1,2,\ldots \quad (4)$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$.

Furthermore, Lukić in [17] proposed geometric mean Newton's method (GMN),

$$x_{n+1} = x_n - \frac{f(x_n)}{sign(f'(x_0))\sqrt{f'(x_n)f'(y_n)}}, n = 0,1,2,\ldots \quad (5)$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$.

Imran in [8] constructed Heronian mean iteration Newton's method (HeMN)

$$x_{n+1} = x_n - \frac{3f(x_n)}{f'(x_n) + f'(y_n) + sign(f'(x_0))\sqrt{f'(x_n)f'(y_n)}}, \quad (6)$$

$n = 0,1,2,\ldots$ where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$.

In addition, Syamsudhuha et.al. in [16] suggested a square root mean Newton's method (RMN)

$$x_{n+1} = x_n - \frac{\sqrt{2}f(x_n)}{sign(f'(x_0))\sqrt{f'^2(x_n) + f'^2(y_n)}}, n = 0,1,2,\ldots \quad (7)$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$.

Finally, Ababneh in [11] employed contra harmonic mean to get a contra harmonic Newton's method (CMN)

$$x_{n+1} = x_n - \frac{f(x_n)(f'(x_n) + f'(y_n))}{f'^2(x_n) + f'^2(y_n)}, \quad n = 0,1,2,\ldots \quad (8)$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$.

III. NUMERICAL COMPUTATION TEST

In this section, some computational tests are presented to observe number of iterations (n) that are resulted by Newton's method and its variations. To study the behavior of the methods, consider the following transcendental functions

- $f_1(x) = x^3 + 4x^2 - 10$, with $\alpha = 1.365230013414097$
- $f_2(x) = \sin^2(x) - x^2 + 1$, with $\alpha = 1.404491648215341$
- $f_3(x) = x^2 - e^2 - 3x + 2$, with $\alpha = 0.257530285439861$
- $f_4(x) = x^3 - 10$, with $\alpha = 2.154434690031884$
- $f_5(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$, with $\alpha = -1.207647827130919$
- $f_6(x) = x^2\sin^2(x) + e^{x^2\cos(x)\sin(x)} - 28$, with $\alpha = 3.437471743421766$
- $f_7(x) = e^{x^2+7x-30} - 1$, with $\alpha = 3.000000000000$

The stopping criteria in the program is when $|\alpha - x_n| < tolerance$ or the maximum iteration calculated is reached. Tolerance is set to be $1e-10$ and maximum iteration is 100. The possible outcomes are: total number of iterations if the method converges, div if the method diverges and NA if the method is not applicable. Computational results are presented in Table 1 below.

Table 1. Comparison of number of iterations of iterative methods with several initial values

Nonlinear Equatio	x_0	Number of iterations						
		MN	AMN	HMN	GMN	HeMN	RMN	CMN
$f_1(x)=0$	1.2	4	3	2	2	3	3	3
	3.0	6	4	3	4	4	4	4
	5.0	7	4	4	4	4	5	5
$f_2(x)=0$	1.2	4	3	2	3	3	3	3
	3.0	5	3	3	3	3	3	4
	5.0	6	4	4	4	4	5	5
$f_3(x)=0$	0.0	3	2	2	2	2	2	2
	2.0	4	3	3	3	3	4	4
	4.0	6	5	4	5	5	5	5
$f_4(x)=0$	1.5	5	3	3	3	3	4	4
	2.0	4	3	2	2	3	3	3
	5.0	6	4	4	4	4	4	5
$f_5(x)=0$	-2.0	7	5	4	5	5	5	6
	-1.0	5	3	3	3	3	4	4
	1.0	7	55	5	7	31	Div	Div
$f_6(x)=0$	3.5	5	3	3	3	3	3	3
	3.9	8	5	4	5	5	6	6
	5.0	9	Div	Div	NA	NA	Div	Div
$f_7(x)=0$	3.2	7	5	4	4	5	5	5
	3.5	11	8	6	7	7	8	9
	4.0	18	13	10	11	12	14	15

In general, from Table 1 it can be said that for all initial values close to the roots, all methods can find the expected approximate roots. However, for the nonlinear equation $f_6(x) = 0$ with an initial value of $x_0 = 5.0$ which is located quite far from the root, only Newton's method succeeded in finding the expected approximation root while other methods failed. Likewise, in the nonlinear equation $f_5(x) = 0$ with an initial value of $x_0 = 1.0$ which is also a bit far from the root, the root mean square newton and contra harmonic newton methods also fail to find the expected approximation root.

Comparing the computational results among the methods with different types of means, it can be concluded that by selecting an initial value that is quite close to the root, all variations of Newton's method with different types of means have fewer iterations. Among these methods, the

Harmonic Mean Newton (HMN) method has relatively smaller iterations than the others.

IV. ANALYSIS OF THE BASINS OF ATTRACTION

Basin of attraction, was first introduced by Cayley [3], is a collection of points of a dynamics system that automatically move to some attractors. In the case of iterative method, the attractors are the roots of a nonlinear equation. It maps the convergence of a set of initial values. Through this method, one can compare iterative methods based on their areas of convergence regions. It means that the larger the area of convergence, the better the method. For this purpose, the area of convergence region is defined as total number of points of convergence to the root of $f(x)=0$ on some interval.

Stewart [15] used the idea of basin of attraction to compare Newton's method and iterative method proposed by Halley [7]. In the case of multiple roots with known multiplicity, many researchers have compared methods with different orders of convergence by observing their basins of attraction, as in Scott et. al. [14], Neta et.al. [10]. and Jamaludin et.al. [9]. Chun et.al. [4] proposed this approach for several methods with third order of convergence. Geum [6] developed basin of attraction-based method with optimal third order of convergence. Cordero et.al [5] proposed basins of attraction method of iterative methods similar to Steffensen's iterative method. Chun et.al [4] compared several iterative methods with eighth order of convergence by observing their basins of attraction. Said Solaiman et.al [13] suggested comparison of several optimal and non-optimal iterative methods with sixteenth order of convergence by analyzing their basins of attractions.

A. Figures of Basins of Attraction

This section discusses the dynamic of the six iterative methods to solve nonlinear equations $g(z)=0$ where $g: C \to C$ is complex plane.

There is a particular colour assigned for every $z_0 \in D \subset C$ if the method converges. Black colour is assigned if the method does not converge to any root. Aside from producing basin of attraction, each program will also produce non-convergent points (black dots) and computation time (CPU time).

Example 1. Consider the following polynomial

$$g_1(z) = z^4 - \tfrac{5}{4}z^2 + \tfrac{1}{4}.$$

The roots are $\{1, -\frac{1}{2}, \frac{1}{2}, -1\}$. The figures of the basins of attractions will be constructed by taking a square plane $[-1,1]\times[-1,1] \subset C$ which is divided into 1000 partitions or 1000000 cells and are given in Figure 1. It shows the region of convergence of the four roots of $g_1(z)=0$.

The comparison of the number of non-converging points and the CPU time of each iterative method is shown in Table 2. In this table shows that HMN method does not have divergent region which mean that there is always a root for any z_0 taken. On the other hand, HMN takes computation time relatively longer than the other methods. Meanwhile, CMN method can be considered as the worst method of all the six methods since it has the largest divergent area (4.7368%) with computation time also cannot compete with other methods.

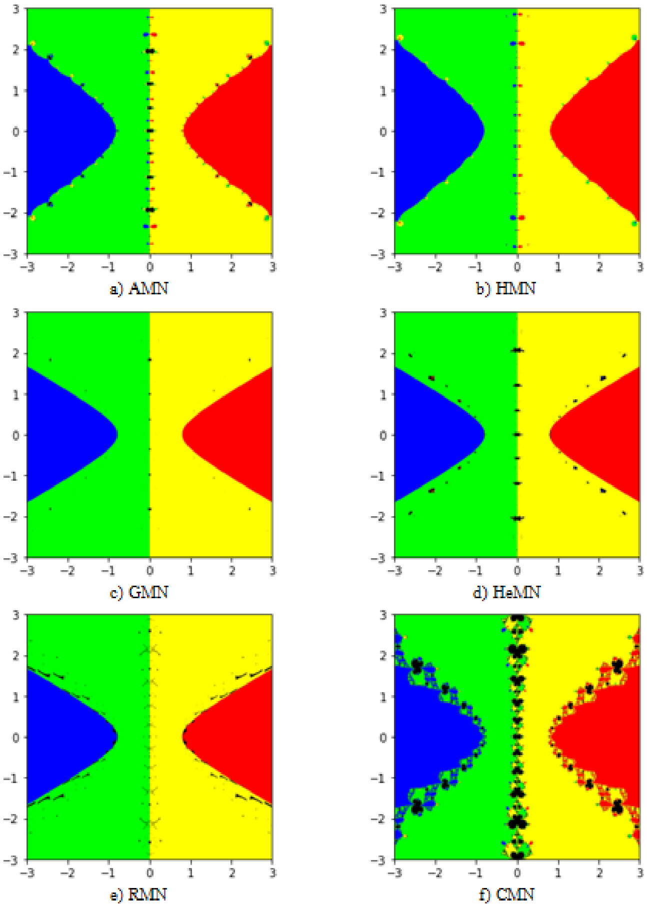

Figure 1. Basins of attraction of iterative methods for $g_1(z) = z^4 - \frac{5}{4}z^2 + \frac{1}{4}$.

Table 2. Comparison of number of non-convergent points and CPU time of each iterative method in solving $g_1(z)=0$ in complex plane.

Methods	Divergent points		Computation Time (second)
	Total number	%	
AMN	4252	0.4252	48.319
HMN	0	0	60.307
GMN	784	0.0784	50.833
HeMN	5532	0.5532	66.264
RMN	11144	1.1144	58.573
CMN	47368	4.7368	61.929

Example 2. Consider the following polynomial

$$g_2(z) = z^4 - 1.$$

The roots are $\{1, -1, i, -i\}$. The figures of the basins of attractions will be constructed by taking a square plane $[-3,3] \times [-3,3] \subset C$ which is divided into 1000 partitions or 1000000 cells and are given in Figure 2 below. It shows the region of convergence of the four roots of $g_2(z) = 0$.

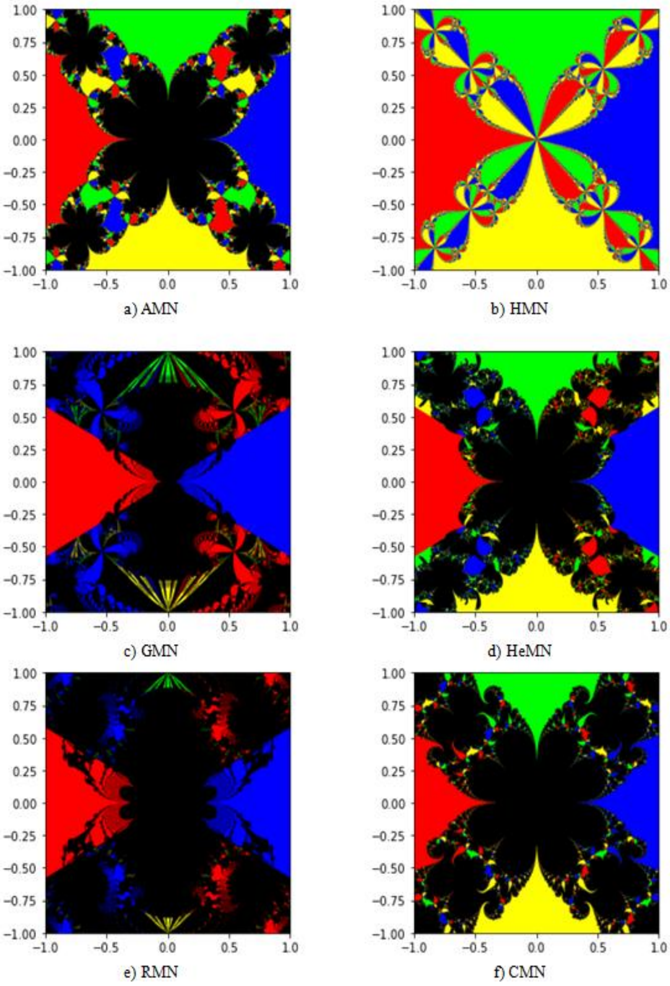

Figure 2. Basins of attraction of iterative methods for $g_2(z) = z^4 - 1$.

In the following Table 3, shows that HMN has the smallest divergent area (0.1598%) of all the area of convergent region and requires computation time longer than others. From this observation, CMN and RMN can be considered as the worst methods since they have divergent region area the largest (66.4688% and 74.3971% respectively) with computation time longer than others (45.670 seconds dan 43.759 seconds respectively). It means that it would be relatively difficult to choose z_0 for these methods in order for them to be convergent.

Table 3. Comparison of number of non-convergent points and CPU time of each iterative method in solving $g_2(z) = 0$ in complex plane

Methods	Divergent points		Computation Time (second)
	Total number	%	
AMN	441596	44.1596	33.591
HMN	1598	0.1598	41.299
GMN	588789	58.8789	40.503
HeMN	538220	53.8220	49.082
RMN	743971	74.3971	43.759
CMN	664688	66.4688	45.670

B. Analysis of Basins of Attraction

In this section, analysis of basins of attraction is used to answer the following questions: (1) What are the factors that can affect basin of attraction of an iterative method? (2) Can basin of attraction be influenced by number of steps of the method? (3) If basin of attraction of an iterative method is better than other basin of attractions in a case, can it be considered to be the best of all? (4) Based on the basin attraction of iterative methods with similar shapes, is the current efficiency index sufficient to make comparisons between the iterative methods?

Two factors that influence the shape of the basin of attraction of a method are the size of fixed area of convergence region and number of partitions used. Basins of attraction with various number of partitions and different sizes of convergent region can be seen in Figure 3 and Figure 4.

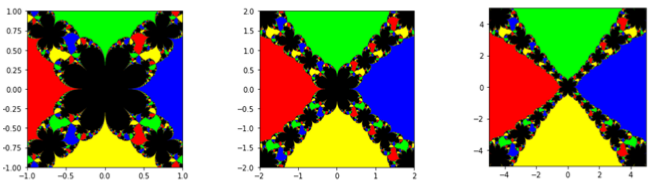

Figure 3. Basin of attraction with various of number of partitions

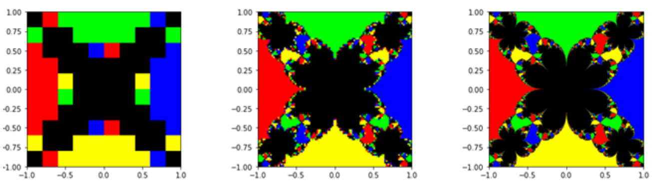

Figure 4. Basin of attraction with various size of area of convergent region

Number of colors of a basin of attraction depends on the number of the roots of the nonlinear equations. The shape of the color depends on the convergence of the method with fixed initial guesses.

With the fact above, it can be inferred that basin of attraction of the method does not depend on the number of steps.

From Figure 1 and 3a it can be seen that basin of attraction of AMN method is better than Figure 2. This deduction aligns with the results from Table 2 and Table 3 where AMN method in Table 2 has fewer black dots (0.4252%) compared to AMN method presented in Table 3 (44.1596%). Therefore, it can be said that if basin of attraction of a method is better than other methods for a function it does not guarantee that it would be better for other functions.

From Figure 3 and Figure 4 it can be seen that iterative methods with the same order convergence and total number of function evaluations can give different shapes of basins of attraction. The size of the convergence region is significant as the divider. On the other hand, the efficiency index of an iterative method is not sufficient enough to be used as comparison parameter among iterative methods which have the same order of convergence and number of function evaluations.

V. CONCLUSION

This article discusses the analysis of the basins of attraction of variations of Newton's methods with several means such as arithmetic mean, geometric mean, harmonic mean, heronian mean, root mean square and contra harmonic mean.

Two factors that influence the shape of a basin of attraction are the size of the area of the convergence region prepared and number of partitions used. Each color in the basin of attraction shows distinct area of convergence of each initial guess. It has been shown that not all initial guesses contribute to the convergence region. If the initial guess does not converge then it is marked by black color. By this approach, it is easy to identify the size of the convergence area of an iterative method and to pick an initial guess.

Another fact that can be drawn is that iterative methods with the same order of convergence and number of function evaluations give different basins of attraction. In the correlation with efficiency index of an iterative method, it can be concluded that efficiency index is not sufficient enough to make a comparison among the iterative methods.

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