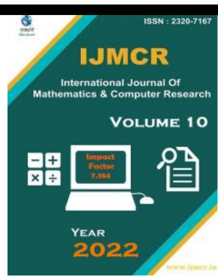
International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 10 Issue 07 July 2022, Page no. – 2789-2796

Index Copernicus ICV: 57.55, Impact Factor: 7.362

DOI: 10.47191/ijmcr/v10i7.02



A Novel Analytical Approach for Solving Time-Fractional Diffusion Equations

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ARTICLE INFO	**ABSTRACT**
Published Online: 04 July 2022 Corresponding Author: **R.K.Bairwa**	This article investigates the approximate analytical solutions of the time-fractional diffusion equations using a novel analytical approach, namely the Sumudu transform iterative method. The time-fractional derivatives are considered in the Caputo sense. The analytical solutions are found in closed form, in terms of Mittag-Leffler functions. Furthermore, the findings are shown graphically, and the solution graphs demonstrate a strong relationship between the approximate and exact solutions.

KEYWORDS: Sumudu transform, iterative method, diffusion equations, and Caputo fractional derivative, fractional differential equations.

I. INTRODUCTION

Fractional calculus has attracted growing interest in the last two decades, and it has been successfully tested and implemented in a range of real-world challenges in science and industry [11, 13, 19, 20, 21, 25]. Furthermore, it has been the focus of several studies in a variety of fields, including signal processing, random walks, Levy statistics, chaos, porous media, electromagnetic flux, thermodynamics, circuit theory, optical fibre, solid state physics, and other fields of science. There are a wide range of approaches used by researchers to solve differential equations of fractional orders, such as the Adomian decomposition method (ADM) [22], the Natural transform method (NTM) [23], the Natural transform decomposition method (NTDM) [24], the variational iteration method (VIM) [9], the generalized difference transform method (GDTM)[6], the Homotopy analysis method (HAM)[16],the Laplace decomposition method (LDM) [12], the Homotopy perturbation transform method (HPTM) [15] , the iterative Laplace transform method (ILTM) [2], etc. The methods described above provide numerical approximate solutions as well as analytical solutions to linear and nonlinear fractional differential equations in immediate and visible symbolic terms. In 2006, Daftardar-Gejji and Jafari [8] proposed the iterative method for numerically solving nonlinear functional equations. Since then, the iterative method has been used to solve a variety of nonlinear differential equations of integer and fractional order [5], as well as fractional boundary value problems [7]. Recently, Wang and Liu [26] developed the Sumudu transform iterative method (STIM) by combining the Sumudu transform with an iterative technique to find approximate analytical solutions to time-fractional Cauchy reaction-diffusion equations. The Sumudu transform iterative method was used to successfully solve a variety of time and space fractional partial differential equations as well as their systems [14] and fractional Fokker-Planck equations [1].

In this article, we will investigate the following time-fractional diffusion equations :

(i) The one-dimensional time-fractional diffusion equation of the form

$$D_t^{\alpha} u - \frac{\partial^2 u}{\partial x^2} - \frac{\partial (xu)}{\partial x} = 0,\ \ 0 < \alpha \le 1,$$
$$u(x,0) = g(x). \tag{1}$$

(ii) The two-dimensional time-fractional diffusion equation of the form

$$D_t^{\alpha} u - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,\ \ 0 < \alpha \le 1,$$
$$u(x,y,0) = \phi(x,y). \tag{2}$$

(iii)The three-dimensional time-fractional diffusion equation of the form

$$D_t^{\alpha} u - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0,\ \ 0 < \alpha \le 1,$$
$$u(x,y,z,0) = \psi(x,y,z). \tag{3}$$

The main objective of this paper is to expand the application of the Sumudu transform iterative technique for constructing approximate analytical solutions to diffusion equations with time-fractional derivatives.

II. PRELIMINARIES

This section introduces the fundamental definitions, notations, and properties of fractional calculus and Sumudu transform theory that will be used throughout the article.

Definition 1. In Caputo's sense, the fractional derivative of a function $u(x,t)$ is defined as [17]

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} u^{(m)}(x,\xi)\, d\xi, \quad m-1 < \alpha \le m,\ m \in \mathbb{N}, \quad = I_t^{m-\alpha} D^m u(x,t). \tag{4}$$

Here $D^m \equiv \dfrac{d^m}{dt^m}$ and I_t^α stands for the Riemann-Liouville fractional integral operator of order $\alpha > 0$, defined as [17]

$$I_t^\alpha u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} u(x,\xi)\, d\xi, \quad \xi > 0. \tag{5}$$

Definition 2. The Sumudu transform is defined over the set of functions

$$\left\{ f(t) \,|\, \exists M, \rho_1 > 0, \rho_2 > 0, |f(t)| < M e^{|t|/\rho_j} \ if \ t \in (-1)^j \times [0,\infty) \right\}$$

by the following formula [3, 27]

$$S[f(t)] = F(\omega) = \int_0^\infty e^{-t} f(\omega t)\, dt, \quad \omega \in (-\rho_1, \rho_2). \tag{6}$$

Definition 3. The Sumudu transform of Caputo fractional derivative is defined in following manner [10, 26]

$$S\left[D_t^\alpha u(x,t)\right] = \omega^{-\alpha} S[u(x,t)] - \sum_{k=o}^{m-1} \omega^{-\alpha+k} u^{(k)}(x,0), \quad m-1 < \alpha \le m,\ m \in \mathbb{N}, \tag{7}$$

where $u^{(k)}(x,0)$ is the k-order derivative of $u(x,t)$ with respect to t at $t = 0$.

Definition 4. The Mittag-Leffler function, a generalization of the exponential function, is defined as follows [18]

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+1)}, \alpha \in C, \mathrm{Re}(\alpha) > 0. \tag{8}$$

A further generalization of equation (8) is as follows [28]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n+\beta)}, \alpha, \beta \in C, R(\alpha) > 0, (\beta) > 0, \tag{9}$$

where $\Gamma(.)$ is the well-known Gamma function.

III. BASIC IDEA OF SUMUDU TRANSFORM ITERATIVE METHOD

To demonstrate the fundamental concept of this method [26], we consider the following general fractional partial differential equation with initial conditions of the form

$$D_t^\alpha u(x,t) + R u(x,t) + N u(x,t) = g(x,t), \quad m-1 < \alpha \le m,\ m \in \mathbb{N}, \tag{10}$$

$$u^{(k)}(x,0) = h_k(x),\ k = 0,1,2,\ldots,m-1, \tag{11}$$

where $D_t^\alpha u(x,t)$ is the Caputo fractional derivative of order α, $m-1 < \alpha \le m$, $m \in \mathbb{N}$, defined by equation (4), R is a linear operator and may include other fractional derivatives of order less than α, N is a non-linear operator which may include other fractional derivatives of order less than α and $g(x,t)$ is a known function.

Applying the Sumudu transform on both sides of equation (11), we have

$$S\left[D_t^\alpha u(x,t)\right] + S[R u(x,t) + N u(x,t)] = S[g(x,t)]. \tag{12}$$

By using the equation (7), we get

$$S[u(x,t)] = \sum_{k=0}^{m-1} \omega^k u^{(k)}(x,0) + \omega^\alpha S[g(x,t)] - \omega^\alpha S[R u(x,t) + N u(x,t)]. \tag{13}$$

On taking inverse Sumudu transform on equation (13), we have

$$u(x,t) = S^{-1}\left[\omega^\alpha \left(\sum_{k=0}^{m-1} \omega^{-\alpha+k} u^{(k)}(x,0) + S[g(x,t)]\right)\right] - S^{-1}\left[\omega^\alpha S[Ru(x,t) + Nu(x,t)]\right]. \tag{14}$$

Furthermore, we employ the iterative method proposed by Daftardar-Gejji and Jafari [8], which represents a solution in an infinite series of components as

$$u(x,t) = \sum_{i=0}^\infty u_i(x,t). \tag{15}$$

As R is a linear operator, so we have

$$R\left(\sum_{i=0}^\infty u_i(x,t)\right) = \sum_{i=0}^\infty R[u_i(x,t)], \tag{16}$$

and the non-linear operator N is decomposed as follows

$$N\left(\sum_{i=0}^\infty u_i(x,t)\right) = N[u_0(x,t)] + \sum_{i=1}^\infty \left\{ N\left(\sum_{j=0}^{i} u_j(x,t)\right) - N\left(\sum_{j=0}^{i-1} u_j(x,t)\right) \right\}. \tag{17}$$

Substituting the results given by equations from (15) to (17) in the equation (14), we get

$$\sum_{i=0}^{\infty} u_i(x,t) = S^{-1}\left[\omega^{\alpha}\left(\sum_{k=0}^{m-1}\omega^{-\alpha+k}u^{(k)}(x,0)+S[g(x,t)]\right)\right]$$
$$-S^{-1}\left[\omega^{\alpha}S\left[\sum_{i=0}^{\infty}R[u_i(x,t)]+N[u_0(x,t)]\right.\right. \tag{18}$$
$$\left.\left.+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j(x,t)\right)-N\left(\sum_{j=0}^{i-1}u_j(x,t)\right)\right\}\right]\right].$$

We have defined the recurrence formulae as

$$u_0(x,t) = S^{-1}\left[\omega^{\alpha}\left(\sum_{k=0}^{m-1}\omega^{-\alpha+k}u^{(k)}(x,0)+S\left(g(x,t)\right)\right)\right]$$
$$u_1(x,t) = -S^{-1}\left[\omega^{\alpha}S\left[R\left(u_0(x,t)\right)+N\left(u_0(x,t)\right)\right]\right]$$
$$u_{m+1}(x,t) = -S^{-1}\left[\omega^{\alpha}S\left[R\left(u_m(x,t)\right)-\left\{N\left(\sum_{j=0}^{m}u_j(x,t)\right)\right.\right.\right.$$
$$\left.\left.\left.-N\left(\sum_{j=0}^{m-1}u_j(x,t)\right)\right\}\right]\right], m \ge 1 \tag{19}$$

Therefore, the approximate analytical solution of equations (10) and (11) in truncated series form is given by

$$u(x,t) \cong \lim_{\square\to\infty}\sum_{m=0}^{\square}u_m(x,t). \tag{20}$$

In general, the solutions in the above series converge quickly. The classical approach to convergence of this type of series has been presented by Bhalekar and Daftardar-Gejji [4] and Daftardar-Gejji and Jafari [8].

IV. SOLUTION OF THE TIME-FRACTIONAL DIFFUSION EQUATIONS

In this section, we apply the Sumudu transform iterative approach (STIM) to derive the solutions of different dimensional diffusion equations with time-fractional derivatives.

Example 4.1 We consider the following one dimensional time-fractional diffusion equation [15]

$$D_t^{\alpha}u = \frac{\partial^2 u}{\partial x^2}+\frac{\partial(xu)}{\partial x}, \quad 0<\alpha\le 1, \tag{21}$$

with the initial condition

$$u(x,0) = x. \tag{22}$$

Taking the Sumudu transform on the both sides of equation (21), and making use of the result given by equation (22), we have

$$S\left[u(x,t)\right] = x+\omega^{\alpha}S\left[u_{xx}+(xu)_x\right]. \tag{23}$$

Operating with the inverse Sumudu transform on both sides of equation (23) gives

$$u(x,t) = x+S^{-1}\left[\omega^{\alpha}S\left[\,u_{xx}+(xu)_x\right]\right]. \tag{24}$$

Substituting the results from equations (15) to (17) in the equation (24) and applying the equation (19), we determine the components of the solution as follows

$$u_0(x,t) = u(x,0) = x, \tag{25}$$

$$u_1(x,t) = S^{-1}\left[\omega^{\alpha}S\left[(u_0)_{xx}+(xu_0)_x\right]\right] = \frac{x\left(2t^{\alpha}\right)}{\Gamma(\alpha+1)}, \tag{26}$$

$$u_2(x,t) = S^{-1}\left[\omega^{\alpha}S\left[\left(u_0+u_1\right)_{xx}+\left(x\left(u_0+u_1\right)\right)_x\right]\right]$$
$$-S^{-1}\left[\omega^{\alpha}S\left[\left(u_0\right)_{xx}+\left(xu_0\right)_x\right]\right] = \frac{x\left(2t^{\alpha}\right)^2}{\Gamma(2\alpha+1)}, \tag{27}$$

$$u_3(x,t) = S^{-1}\left[\omega^{\alpha}S\left[\left(u_0+u_1+u_2\right)_{xx}+\left(x\left(u_0+u_1+u_2\right)\right)_x\right]\right]$$
$$-S^{-1}\left[\omega^{\alpha}S\left[\left(u_0+u_1\right)_{xx}+\left(x\left(u_0+u_1\right)\right)_x\right]\right] \tag{28}$$
$$= \frac{x\left(2t^{\alpha}\right)^3}{\Gamma(3\alpha+1)},$$

and so on. The other components can be found accordingly. Therefore, the approximate analytical solution in the series form can be obtained as

$$u(x,t) \cong \lim_{\square\to\infty}\sum_{m=0}^{\square}u_m(x,t)$$
$$= u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+,\ldots,$$
$$= x\left[1+\frac{\left(2t^{\alpha}\right)}{\Gamma(\alpha+1)}+\frac{\left(2t^{\alpha}\right)^2}{\Gamma(2\alpha+1)}+\frac{\left(3t^{\alpha}\right)^3}{\Gamma(3\alpha+1)}+,\ldots,\right]$$
$$= x\sum_{n=0}^{\infty}\frac{\left(2t^{\alpha}\right)^n}{\Gamma(n\alpha+1)}.$$

Hence, the exact solution can be written as

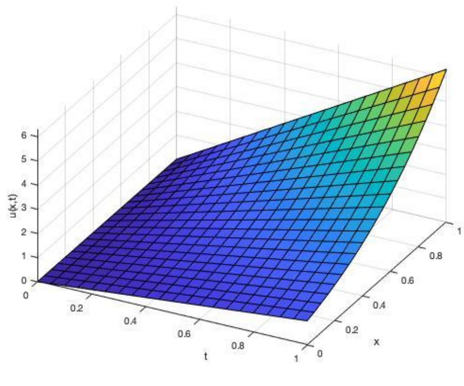
$$u(x,t) = x\,E_{\alpha}\left[2t^{\alpha}\right]. \tag{29}$$

The same result was obtained by Kumar *et al.* [15] using HPTM, Çetinkaya and Kıymaz [6] using GDTM, and Shah *et al.* [23] using NTM.

If we put $\alpha = 1$, in equation (29), we have

$$u(x,t) = x\,e^{2t}. \tag{30}$$

Which is the exactly the same solution obtained by earlier Kumar *et al.* [19] using HPTM method.



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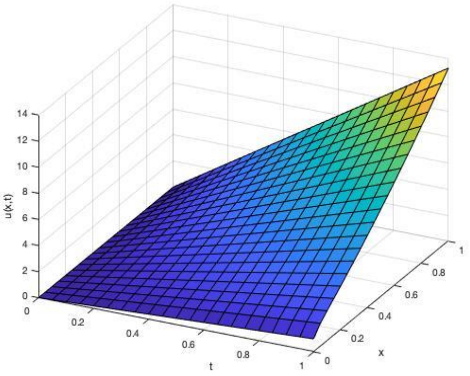

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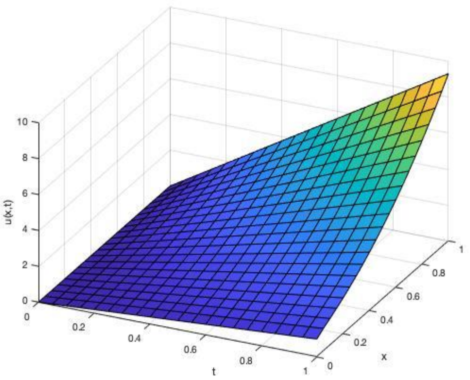

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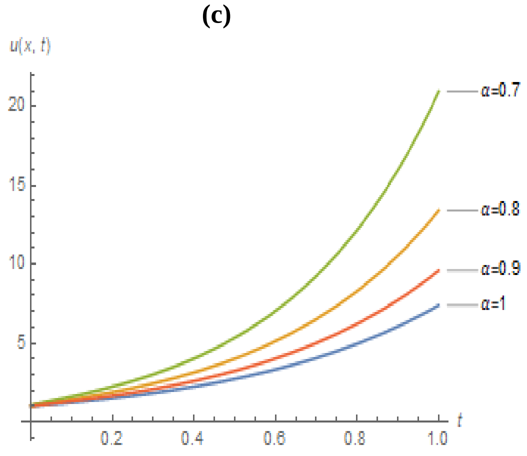


(d)

Figure 4.1. Surface showing the solution $u(x,t)$: when (a) The exact solution for $\alpha=1$, (b) The approximate solution for $\alpha=0.50$, (c) The approximate solution for $\alpha=0.75$, (d) Comparison for different values of α, when $x=1$.

Example 4.2. We consider the following two-dimensional time-fractional diffusion equation [15]

$$D_t^{\alpha}u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},\ \ 0<\alpha\leq 1, \tag{31}$$

with the initial condition

$$u(x,y,0)=\sin x\sin y, \tag{32}$$

Taking the Sumudu transform on the both sides of equation (31), and making use of the result given by equation (32), we have

$$S\left[u(x,y,t)\right]=\sin x\sin y+\omega^{\alpha}S\left[u_{xx}+u_{yy}\right]. \tag{33}$$

Operating with the inverse Sumudu transform on both sides of equation (33) gives

$$u(x,y,t)=\sin x\sin y+S^{-1}\left[\omega^{\alpha}S\left[u_{xx}+u_{yy}\right]\right]. \tag{34}$$

Substituting the results from equations (15) to (17) in the equation (34) and applying the equation (19), we determine the components of the solution as follows

$$u_0(x,y,t)=u(x,y,0)=\sin x\sin y\,, \tag{35}$$

$$\begin{aligned}u_1(x,y,t)&=S^{-1}\left[\omega^{\alpha}S\left[(u_0)_{xx}+(u_0)_{yy}\right]\right]\\&=(-2)^1\sin x\sin y\frac{t^{\alpha}}{\Gamma(\alpha+1)},\end{aligned} \tag{36}$$

$$\begin{aligned}u_2(x,y,t)&=S^{-1}\left[\omega^{\alpha}S\left[(u_0+u_1)_{xx}+(u_0+u_1)_{yy}\right]\right]\\&\quad-S^{-1}\left[\omega^{\alpha}S\left[(u_0)_{xx}+(u_0)_{yy}\right]\right]\\&=(-2)^2\sin x\sin y\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\end{aligned} \tag{37}$$

$$\begin{aligned}u_3(x,y,t)&=S^{-1}\left[\omega^{\alpha}S\left[(u_0+u_1+u_2)_{xx}+(u_0+u_1+u_2)_{yy}\right]\right]\\&\quad-S^{-1}\left[\omega^{\alpha}S\left[(u_0+u_1)_{xx}+(u_0+u_1)_{yy}\right]\right]\\&=(-2)^3\sin x\sin y\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\end{aligned} \tag{38}$$

and so on. The other components can be found accordingly. Therefore, the approximate analytical solution in the series form can be obtained as

$$\begin{aligned}u(x,y,t)&\cong\lim_{n\to\infty}\sum_{m=0}^{n}u_m(x,y,t)\\&=u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+u_3(x,y,t)+,...,\\&=\sin x\sin y\left(1+\frac{(-2)^1t^{\alpha}}{\Gamma(\alpha+1)}+\frac{(-2)^2t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{(-2)^3t^{3\alpha}}{\Gamma(3\alpha+1)}+...,\right)\\&=\sin x\sin y\ \sum_{n=0}^{\infty}\frac{\left(-2t^{\alpha}\right)^n}{\Gamma(n\alpha+1)}.\end{aligned}$$

Hence, the exact solution can be written as

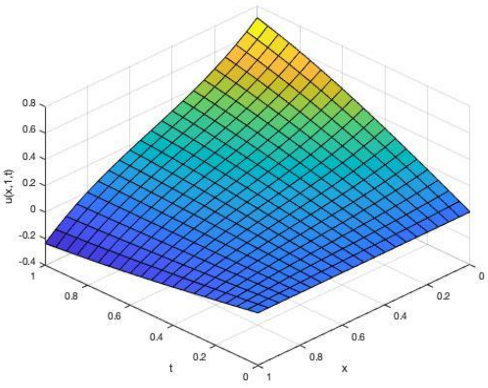
$$u(x,y,t)=(\sin x\sin y)\,E_{\alpha}\left[-2t^{\alpha}\right]. \tag{39}$$

The same result was obtained by Kumar *et al.* [15] using HPTM and Shah *et al.* [23] using NTM.

If we put $\alpha=1$, in equation (39), we have

$$u(x,y,t)=(\sin x\sin y)\,e^{-2t}\,. \tag{40}$$

Which is the exactly the same solution obtained by earlier Shah *et al.* [23] using NTM method.



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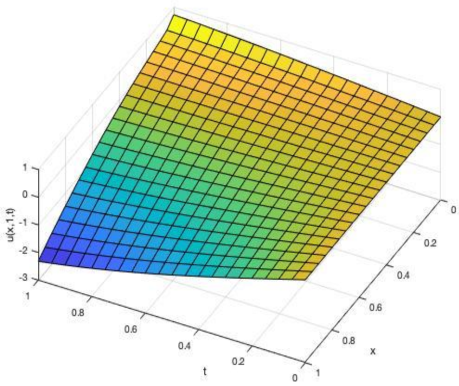

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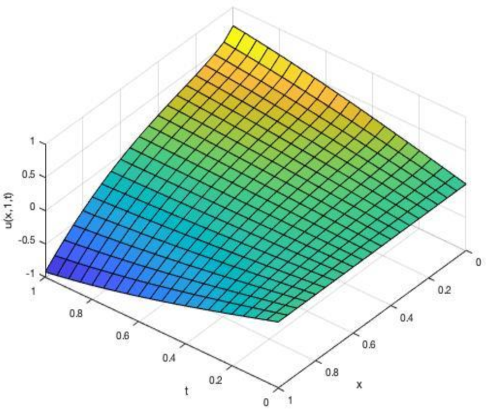

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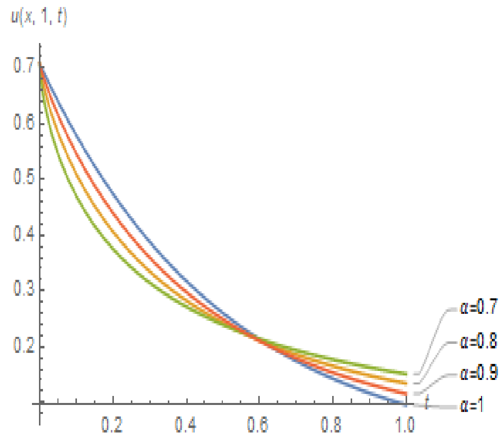


(d)

Figure 4.2. Surface showing the solution graph $u(x, y, t)$: when $y = 1$, (a) The exact solution for $\alpha = 1$,(b) The approximate solution for $\alpha = 0.50$, (c) The approximate solution for $\alpha = 0.75$, (d) Comparison for different values of α ,when $x = 1$.

Example 4.3. We consider the following three dimensional time-fractional diffusion equation [24]

$$D_t^\alpha u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad 0 < \alpha \le 1, \tag{41}$$

with the initial condition

$$u(x, y, z, 0) = \sin x \sin y \sin z, \tag{42}$$

Taking the Sumudu transform on the both sides of equation (41), and making use of the result given by equation (42), we have

$$S\left[u(x, y, z, t)\right] = \sin x \sin y \sin z + \omega^\alpha S\left[u_{xx} + u_{yy} + u_{zz}\right]. \tag{43}$$

Operating with the inverse Sumudu transform on both sides of equation (43) gives

$$\begin{aligned} u(x, y, z, t) &= \sin x \sin y \sin z \\ &+ S^{-1}\left[\omega^\alpha S\left[u_{xx} + u_{yy} + u_{zz}\right]\right]. \end{aligned} \tag{44}$$

Substituting the results from equations (15) to (17) in the equation (44) and applying the equation (19), we determine the components of the solution as follows

$$u_0(x, y, z, t) = u(x, y, z, 0) = \sin x \sin y \sin z\,, \tag{45}$$

$$\begin{aligned} u_1(x, y, z, t) &= S^{-1}\left[\omega^\alpha S\left[(u_0)_{xx} + (u_0)_{yy} + (u_0)_{zz}\right]\right] \\ &= (-3)^1 \sin x \sin y \sin z \,\frac{t^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{46}$$

$$u_2(x,y,z,t) = S^{-1}\left[\omega^{\alpha} S\left[(u_0+u_1)_{xx} + (u_0+u_1)_{yy} + (u_0+u_1)_{zz}\right]\right]$$
$$- S^{-1}\left[\omega^{\alpha} S\left[(u_0)_{xx} + (u_0)_{yy} + (u_0)_{zz}\right]\right]$$
$$= (-3)^2 \sin x \sin y \sin z \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{47}$$

$$u_3(x,y,z,t) = S^{-1}\left[\omega^{\alpha}\left[S(u_0+u_1+u_2)_{xx}\right.\right.$$
$$\left.\left. + (u_0+u_1+u_2)_{yy} + (u_0+u_1+u_2)_{zz}\right]\right]$$
$$- S^{-1}\left[\omega^{\alpha} S\left[(u_0+u_1)_{xx} + (u_0+u_1)_{yy} + (u_0+u_1)_{zz}\right]\right]$$
$$= (-3)^3 \sin x \sin y \sin z \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{48}$$

and so on. The other components can be found accordingly. Therefore, the approximate analytical solution in the series form can be obtained as

$$u(x,y,z,t) \cong \lim_{n\to\infty} \sum_{m=0}^{n} u_m(x,y,z,t)$$
$$= u_0(x,y,z,t) + u_1(x,y,z,t) + u_2(x,y,z,t) + u_3(x,y,z,t) +, \ldots,$$
$$= \sin x \sin y \sin z \left[1 + \frac{(-3)^1 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(-3)^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(-3)^3 t^{3\alpha}}{\Gamma(3\alpha+1)} +, \ldots, \right]$$
$$= \sin x \sin y \sin z \sum_{n=0}^{\infty} \frac{(-3)^n t^{3\alpha}}{\Gamma(n\alpha+1)}.$$

Hence, the exact solution can be written as

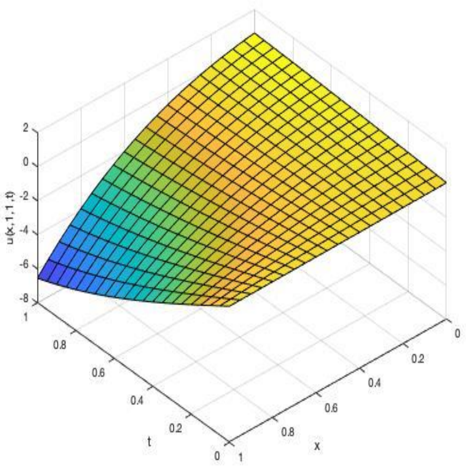
$$u(x,y,z,t) = (\sin x \sin y \sin z) E_{\alpha}\left[-3t^{\alpha}\right]. \tag{49}$$

The same result was obtained by Shah *et al.* [24] using NTDM.

If we put $\alpha = 1$, in equation (49), we have

$$u(x,y,z,t) = (\sin x \sin y \sin z) e^{-3t}. \tag{50}$$

Which is the exactly the same solution obtained by earlier Shah *et al.* [23] using NTM method.



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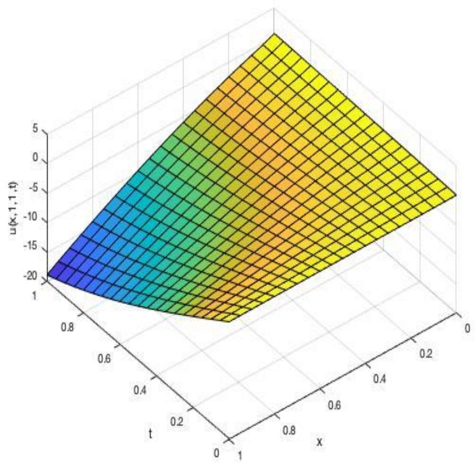

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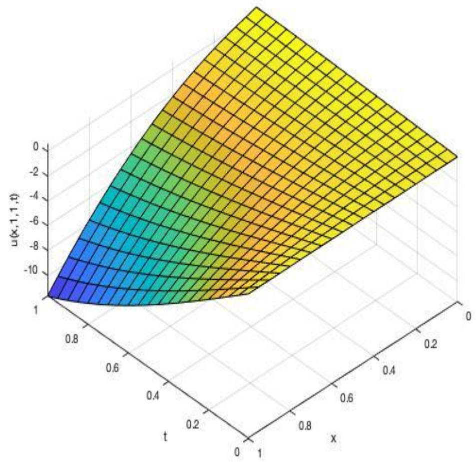

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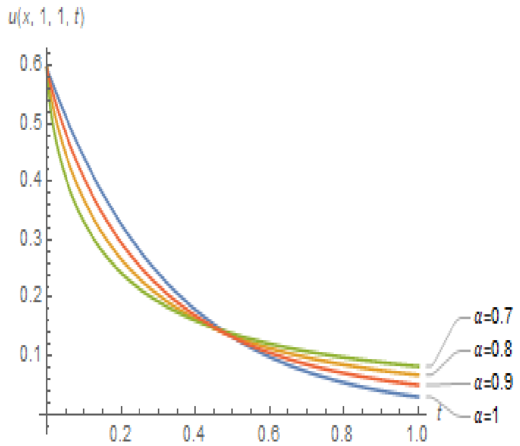

(d)

Figur 4.3. Surface showing the solution graph $u(x,y,z,t)$: when $y=1, z=1$, (a) The exact solution for $\alpha=1$, (b) The approximate solution for $\alpha=0.50$, (c) The approximate solution for $\alpha=0.75$, (d) Comparison for different values of α ,when $x=1$.

V. CONCLUSION

In this paper, we have employed the Sumudu transform iterative approach to construct approximate analytical solutions to various dimension diffusion equations concerning time-fractional derivatives in terms of Mittag-Leffler functions. The obtained solutions are in the form of a series that rapidly converges into a closed exact formula with easily computed terms. The findings are broad in scope and include a number of other studies that have been done in the earlier.

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