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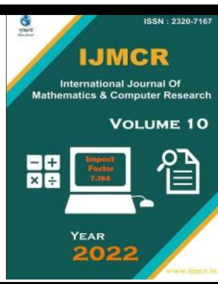

Absolutely Convergence of the Dirichlet Series of the Mobius Function

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ABSTRACT

Wigner distribution is a tool for signal processing to obtain instantaneous spectrum of a signal. From which, another representation of the Euler product can be obtained for Dirichlet series of the Mobius function. From which, we can give the proof of the absolutely convergence of the Dirichlet series on the Mobius function which leads to the proof of the Riemann hypothesis.

KEYWORDS: Wigner-Ville distribution, Mobius function, Euler product, Absolute convergence, Riemann hypothesis

2010 Mathematical Subject Classification; 11M06, 11M26, 11M41, 44A15

1. INTRODUCTION

The Wigner-Ville distribution is a transform technique used in both the time and frequency domain for the signal processing theory. The main characteristics of this transform is that it is not limited by the uncertainty relation of time and frequency. It was originally proposed by E.Wigner in the context of quantum mechanics in 1932 [1] and later J.Ville introduced it for signal analysis in 1948 [2]. The Wigner-Ville distribution (abbreviated Wigner distribution hereafter) is defined by the combination of the Fourier transform and correlation calculation as

$$W_x(\omega,t)=\int_{-\infty}^{+\infty} x(t+\tau/2)\bar{x}(t-\tau/2)e^{-i\omega\tau}d\tau\,,$$

where $\bar{x}(t)$ is a conjugate of $x(t)$.

This transformation has the advantage of high resolution of signals compared with the Fourier transform and it is often utilized as a tool to obtain instantaneous spectrum of signals.

In this paper, the author tries to give another proof of the prime number theorem by using the Euler products for the Dirichlet series of the Mobius function obtained from the Wigner distribution analysis.

2. EULER PRODUCT OF THE DIRICHLET SERIES BY THE WIGNER DISTRIBUTION ANALYSIS

For the Dirichlet series given by

$z(s)=\sum_{n=1}^{\infty} a(n)/n^s$, we define the Wigner distribution function $W_z(\omega,t)$ shown as

$$W_z(\omega,t)=\int_{-\infty}^{+\infty} z(\sigma-i[t+\tau/2])\cdot\bar{z}(\sigma-i[t-\tau/2])e^{-i\omega\tau}d\tau$$

where s is a complex number given by $s=\sigma+it$. As $z(s)$ can be rewritten as

$z(\sigma - it) = \sum_{n=1}^{\infty} \frac{a(n)e^{it\log n}}{n^{\sigma}}$ by real parameters σ and t, then $W_z(\omega,t)$ can be given by

$$W_z(\omega,t) = \int_{-\infty}^{+\infty} \sum_{k=1}^{\infty} \frac{a(k)}{k^{\sigma}} \exp[i(t+\tau/2)\log k] \sum_{l=1}^{\infty} \frac{a(l)}{l^{\sigma}} \exp[-i(t-\tau/2)\log l] e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{+\infty} \sum_{k,l}^{\infty} \frac{a(k)a(l)}{(kl)^{\sigma}} \exp[i\log(k/l)t] \exp[i\log(kl)\tau/2] e^{-i\omega\tau} d\tau$$

Put $n = kl$ and rearranging the equation, we have

$$W_z(\omega,t) = \sum_{k,l}^{\infty} \frac{a(k)a(l)}{n^{\sigma}} \exp[i\log(k/l)t] \int_{-\infty}^{+\infty} \exp(i\log n \cdot \tau/2) e^{-i\omega\tau} d\tau$$

$$= 2\pi \sum_{k,l}^{\infty} \frac{a(k)a(l)}{n^{\sigma}} \exp[i\log(k/l)t] \delta\left(\omega - \frac{\log n}{2}\right)$$

$$= 2\pi \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \delta\left(\omega - \frac{\log n}{2}\right) \cdot \sum_{n=kl} a(k)a(l)\exp[i\log(k/l)t]$$

where $\delta(\omega)$ is a Dirac delta function.

We let $b(n,t) = \sum_{n=kl} a(k)a(l)\exp[i\log(k/l)t]$, the Wigner distribution function of the Dirichlet series $z(s)$ becomes

$$W_z(\omega,t) = 2\pi \sum_{n=1}^{\infty} \frac{b(n,t)}{n^{\sigma}} \delta\left(\omega - \frac{\log n}{2}\right).$$

To obtain the Euler product by the Wigner distribution analysis, we have to prove following Lemmas at first.

Lemma.1. *Let* $\tau = t/2$ *and* $s' = \sigma + i\tau$, *we have* $z(s) = z(\sigma)^{-1} \sum_{n=1}^{\infty} \frac{b(n,\tau)}{n^{s'}}$.

Proof; We utilize the property of the Wigner distribution shown as [3]

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} W_f(\omega,t/2) e^{i\omega t} d\omega = \bar{f}(0) f(t).$$

As the left side integral of this equation yields

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} W_z(\omega,t/2) e^{i\omega t} dt = \sum_{n=1}^{\infty} \frac{b(n,t/2)}{n^{\sigma}} \int_{-\infty}^{+\infty} \delta\left(\omega - \frac{\log n}{2}\right) e^{i\omega t} d\omega = \sum_{n=1}^{\infty} \frac{b(n,t/2)}{n^{\sigma}} \exp\left(i\frac{t}{2}\log n\right), \quad (1)$$

then we have

$$z(\sigma - it) = z(\sigma)^{-1} \sum_{n=1}^{\infty} \frac{b(n,t/2)}{n^{\sigma}} \exp\left(i\frac{t}{2}\log n\right).$$

From the definition of $b(n,t)$, we can see $b(n,t) = b(n,-t)$, then Eq.(1) can be rewritten as

$$z(\sigma + it) = z(\sigma)^{-1} \sum_{n=1}^{\infty} \frac{b(n,t/2)}{n^{\sigma}} \exp\left(-i\frac{t}{2}\log n\right) = z(\sigma)^{-1} \sum_{n=1}^{\infty} \frac{b(n,t/2)}{n^{\sigma+it/2}}$$

We let $\tau = t/2$ and $s' = \sigma + i\tau$, Lemma.1 can be obtained. (QED)

Lemma.2. *Let* $a(n)$ *be a multiplicative function, then* $b(n,t)$ *is a multiplicative function satisfying* $b(mn,t) = b(m,t) \cdot b(n,t)$ *.when* $(m,n) = 1$.

Proof; From the definition of $b(n,t)$, we have

$$b(n,t) = \sum_{d|n} a(d)a(n/d)\exp[it\log(d^2/n)].$$

Put $n = n_1 n_2$,and $d = d_1 d_2$, which satisfy $(n_1 n_2) = 1$, $(d_1 d_2) = 1$, then we can write

$$\sum_{d_1 d_2 | n_1 n_2} a(d_1 d_2) a(n_1 n_2 /(d_1 d_2)) \exp[it \log((d_1 d_2)^2 /(n_1 n_2))]$$
$$= \sum_{d_1 | n_2} a(d_1) a(d_1 / n_1) \exp[it \log(d_1^2 / n_1)] \sum_{d_2 | n_2} a(d_2) a(d_2 / n_2) \exp[it \log(d_2^2 / n_2)]$$

.Hence, it can be seen that $b(n,t)$ is a multiplicative function. (QED)

From which, we can obtain the Euler product of the Dirichlet series as follows.

Theorem.1. *The Dirichlet series* $z(s)$ *consisted of a multiplicative function gives the following Euler product.*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{s'}} = z(\sigma)^{-1} \prod_p \left(1 + \sum_{r=1}^{\infty} \frac{b(p^r, \tau)}{p^{rs'}}\right) , \tag{2}$$

where $s' = \sigma + i\tau$.

Proof; If we let $g(n)$ be a multiplicative function, we have

$$\sum_{n=1}^{\infty} g(n) = \prod_p \left(1 + g(p) + g(p^2) + g(p^3) + \cdots\right),$$

if satisfying $\sum_{p,r} \left|g(p^r)\right| < +\infty$ [4].

From Lemma.2, $b(n,\tau)$ is a multiplicative function, thus we have

$$\sum_{n=1}^{\infty} \frac{b(n,\tau)}{n^{s'}} = \prod_p \left(1 + \sum_{r=1}^{\infty} \frac{b(p^r, \tau)}{p^{rs'}}\right) .$$

Then we can obtain Eq.(2) from Lemma.1. (QED)

3. EULER PRODUCT OF THE DIRICHLET SERIES OF THE MOBIUS FUNCTION

We try to obtain Euler products of the Dirichlet series of an absolute value of the Mobius function shown as follows;

Theorem.2. *Let* $s = \sigma + it$ *and* $s' = \sigma + i\tau$, *where* $\tau = t/2$, *the Dirichlet series of the Mobius function has the Euler product given by*

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{1}{m(\sigma)} \prod_p (1 + 2p^{-s'} \cos(\tau \log p) + p^{-2s'}) , \tag{3}$$

where $m(\sigma) = \prod_p (1 + p^{-\sigma})$,

.**Proof;** From Lemma.2, $b(n,\tau)$ is a multiplicative function because it can be given by

$$b(n,\tau)=\sum_{n=kl}\mu(k)\mu(l)\exp[i\log(k/l)\tau]=\sum_{d|n}\mu(d)\mu(n/d)\exp[i\tau\log(d^2/n)].$$

From which, we have

$b(p,\tau)=2\cos(\tau\log p)$, $b(p^2,\tau)=1$ and $b(p^r,\tau)=0$ $(r\geq 3)$,

then we obtain $1+\sum_{r=1}^{\infty}\frac{b(p^r,\tau)}{p^{rs'}}=1+2p^{-s'}\cos(\tau\log p)+p^{-2s'}$.

When we let $m(s')=\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^{s'}}$, we have $\prod_p(1+2p^{-s'}\cos(\tau\log p)+p^{-2s'})=\prod_p(1+2p^{-\sigma}+p^{-2\sigma})$ at $s'=\sigma$, then

$$m(\sigma)=\prod_p(1+p^{-\sigma})=\prod_p\frac{1-p^{-2\sigma}}{1-p^{-\sigma}}.,$$

Then we have

$$\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s}=\frac{1}{m(\sigma)}\prod_p(1+2p^{-s'}\cos(\tau\log p)+p^{-2s'})$$ from Theorem.1. (QED)

4. ABSOLUTELY CONVERGENCE OF THE DIRICHLET SERIES OF THE MOBIUS FUNCTION

At first, we prove the following equation.

$$\left|1+2p^{-s'}\cos(\tau\log p)+p^{-2s'}\right|=\sqrt{1+p^{-4\sigma}-2p^{-2\sigma}+(4p^{-\sigma}+8p^{-2\sigma}+4p^{-3\sigma})\cos^2(\tau\log p)}, \quad (4)$$

Proof:

$$\left|1+2p^{-s}\cos(t\log p)+p^{-2s}\right|$$

$$=\left|1+2p^{-\sigma}p^{-it}\cos(t\log p)+p^{-2\sigma}p^{-it}\right|$$

$$=\left|1+2p^{-\sigma}[\cos(t\log p)-i\sin(t\log p)]\cos(t\log p)+p^{-2\sigma}[\cos(t\log p)-i\sin(2t\log p)]\right|$$

$$=\{[1+2p^{-\sigma}\cos^2(t\log p)+p^{-2\sigma}\cos(2t\log p)]^2+[-2p^{-\sigma}\sin(t\log p)\cos(t\log p)-p^{-2\sigma}\sin(2t\log p)]^2\}^{1/2}$$

$$=\sqrt{1+p^{-4\sigma}+4p^{-\sigma}\cos^2(t\log p)+8p^{-2\sigma}\cos^2(t\log p)-2p^{-2\sigma}+4p^{-3\sigma}\cos^2(t\log p)}.$$

(QED)

From Eq.(4), we have

$$\frac{1}{m(\sigma)}\left|1-p^{-s'}\cos(\tau\log p)+p^{-2s'}\right|=\frac{1-p^{-\sigma}}{1-p^{-2\omega}}\sqrt{1+p^{-4\sigma}-2p^{-\sigma}+(4p^{-\sigma}+8p^{-2\sigma}+4p^{-3\sigma})\cos^2(\tau\log p)}$$

$$=(1-p^{-\sigma})\sqrt{1+\frac{4p^{-\sigma}(1+p^{-\sigma})^2}{(1-p^{-2\sigma})^2}\cos^2(\tau\log p)}$$

$\left|1+2p^{-s}\cos(t\log p)+p^{-2s}\right|$ has a maximum value at $\cos(t\log p)=1$.

From which, we have the maximum value shown as

$$(1-p^{-\sigma})\sqrt{1+4p^{-\sigma}\left(\frac{1+p^{-\sigma}}{1-p^{-2\sigma}}\right)^2}\ , \tag{5}$$

Lemma.3: $\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s}$ *reaches a certain value including an infinity for* $\sigma>0$.

Proof:;

In regarding ε, there is a number N, which satisfies

$$\left|\frac{|\mu(n)|}{n^s}-\frac{|\mu(m)|}{m^s}\right|=\left|\frac{|\mu(n)|}{n^\sigma}\exp(-it\log n)-\frac{|\mu(m)|}{m^\sigma}\exp(-it\log m)\right|$$

$$=\sqrt{\frac{|\mu(n)|^2}{n^{2\sigma}}+\frac{|\mu(m)|^2}{m^{2\sigma}}+2\frac{|\mu(m)|}{m^\sigma}\frac{|\mu(n)|}{n^\sigma}\cos(t\log(m/n))}<\varepsilon\ \ (\sigma>0)$$

for $m,n>N$ and this is a Cauchy sequence. Hence $\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s}$ does not become an indefinite value, but it reaches to a certain value including an infinity for $\sigma>0$.

(QED)

Theorem.3:

$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$ *is absolutely convergent for* $1/2<\sigma\le 1$.

Proof:

From Eq.(5), we have

$$\left|\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^{s'}}\right|\le\prod_p(1-p^{-\sigma})\cdot\prod_p\sqrt{1+4p^{-\sigma}\left(\frac{1+p^{-\sigma}}{1-p^{-2\sigma}}\right)^2}\ , \tag{6}$$

As $(1+p^{-\sigma})/(1-p^{-2\sigma})>1$ and $1+4p^{-\sigma}>1+p^{-\sigma}$,

$$\prod_p\left[1+4p^{-\sigma}\left(\frac{1+p^{-\sigma}}{1-p^{-2\sigma}}\right)^2\right]>\prod_p(1+p^{-\sigma})=\prod_p\frac{1-p^{-2\sigma}}{1-p^{-\sigma}}=\frac{\zeta(\sigma)}{\zeta(2\sigma)}$$

When we let ω is a transfinite number and ε is an infinitesimal, we have

$\prod_p(1-p^{-\sigma})\to\varepsilon$ for $0<\sigma\le 1$ and $\prod_p\left[1+4p^{-\sigma}\left(\frac{1+p^{-\sigma}}{1-p^{-2\sigma}}\right)^2\right]\to\omega$ for $1/2<\sigma\le 1$ because $\zeta(2\sigma)$ is convergent for $\sigma>1/2$.

We have $|\varepsilon\cdot r|<s$ for any positive numbers, r and s [5], then we have $|\varepsilon\cdot\omega|<\omega$.

Hence, from Eq.(6), we have $\left|\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^{s'}}\right| < +\infty$. From Lemma.3, it can be seen that $\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^{s}}$ is convergent for $1/2 < \sigma \le 1$ and hence $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{s}}$ is absolutely convergent for $1/2 < \sigma \le 1$.

(QED)

Corollary:

The Riemann hypothesis is true.

Proof

$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{s}}$ is absolutely convergent for $1/2 < \sigma \le 1$, which is identical to the Riemann hypothesis [4].

(QED)

5. CONCLUSION

By the Wigner-Ville distribution analysis, which is a tool developed for analyzing instantaneous spectrum of a signal. From which, absolutely convergence of the Dirichlet series of the Mobius function for $1/2 < \sigma \le 1$ can be derived, which is identical to the Riemann hypothesis.

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