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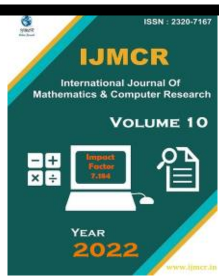

Comparison of Synchronization Speed of Networks Consisting of Two Ordinary Differential Systems of Fitzhugh – Nagumo Type with Bidirectionally and Unidirectionally Linear Coupling

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ARTICLE INFO	ABSTRACT
Published Online: 27 July 2022 Corresponding Author: **Phan Van Long Em**	This work considers networks of two nodes with bidirectionally and unidirectionally linear coupling. Each node is represented by a system of ordinary differential equations of FitzHugh-Nagumo type which is obtained by simplifying the famous Hodgkin-Huxley model. From two network topologies, the existence of global attractors, and the sufficient condition under the coupling strength are sought such that the synchronization phenomenon occurs. The result shows that the network with bidirectionally linear coupling synchronizes more easily than the other. The paper also shows this theoretical result numerically and see that there is a compromise.

KEYWORDS: coupling strength, FitzHugh-Nagumo model, global attractor, linear coupling, synchronization

I. INTRODUCTION

Synchronization is a ubiquitous feature in many natural systems and nonlinear science. The word *synchronization* means having the same behavior at the same time. Therefore, the synchronization of two dynamical systems usually means that one system copies the movement of the other. When the behaviors of many systems are synchronized, these systems are called *synchronous*. Aziz-Alaoui [1] and Corson [2] suggested that a phenomenon of synchronization may appear in a network of many weakly coupled oscillators. A broad variety of applications have emerged to increase the power of lasers, synchronize the output of electric circuits, control oscillations in chemical reactions or encode electronic messages for secure communications [1, 3].

In recent years, the synchronization has been extensively studied in many fields, many natural phenomena also reflect the synchronization such as the movement of birds forming the cloud, the movement of fishes in the lake, the movement of the parade, the reception and transmission of a group of cells, ...etc [1, 4-7]. Therefore, the study of the synchronization in the network of cells is very necessary. In order to make the study easier, the network of two neurons interconnected together with linear coupling is investigated and the sufficient condition on the coupling strength is sought to achieve the synchronization. Each neuron is represented by a dynamical system named FitzHugh-Nagumo model. It was introduced as a dimensional reduction of the well-known Hodgkin-Huxley model [4, 5, 7-10]. It is more analytically tractable and maintains some biophysical meaning. The model is constituted a common form of two equations in the two variables u and v. The first variable is the fast one called excitatory which represents the transmembrane voltage. The second is the slow recovery variable which describes the time dependence of several physical quantities, such as electrical conductivity of ion currents across the membrane. The FitzHugh-Nagumo equations (FHN) are given by:

$$\begin{cases} \varepsilon \dfrac{du}{dt} = u_t = f(u) - v \\ \dfrac{dv}{dt} = v_t = au - bv + c \end{cases} \qquad (1)$$

where a, b and c are constants (a and b are strictly positive), $0 < \varepsilon \ll 1$ and $f(u) = -u^3 + 3u$.

System (1) is considered as the model of a neuron, then we would like to form the model of a neural network of two neurons coupled bidirectionally and unidirectionally. Notice that a neural network describes a population of physically interconnected nerve cells. Communication between cells is mainly due to electrochemical processes. This work focuses on analyzing the behavior of a set of neurons connected together with a given topology by electrical. Thus, the complex system based on a network of interactions between neurons is considered in which each network node is modeled by FHN type. Specifically, the network of two neurons with bidirectionally linear coupling is given by the following system:

$$\begin{cases} \varepsilon \dfrac{du_1}{dt} = \varepsilon u_{1t} = f(u_1) - v_1 - g_{syn}(u_1 - u_2) \\ \dfrac{dv_1}{dt} = v_{1t} = au_1 - bv_1 + c \\ \varepsilon \dfrac{du_2}{dt} = \varepsilon u_{2t} = f(u_2) - v_2 - g_{syn}(u_2 - u_1) \\ \dfrac{dv_2}{dt} = v_{2t} = au_2 - bv_2 + c \end{cases} \quad (2)$$

And the network of two neurons with unidirectionally linear coupling is given by the following system:

$$\begin{cases} \varepsilon \dfrac{du_1}{dt} = \varepsilon u_{1t} = f(u_1) - v_1 - g_{syn}(u_1 - u_2) \\ \dfrac{dv_1}{dt} = v_{1t} = au_1 - bv_1 + c \\ \varepsilon \dfrac{du_2}{dt} = \varepsilon u_{2t} = f(u_2) - v_2 \\ \dfrac{dv_2}{dt} = v_{2t} = au_2 - bv_2 + c \end{cases} \quad (3)$$

In this work, we would like to prove the existence of global attractors of the system (2) and (3). The sufficient conditions for synchronization of those networks are also investigated. From those results, we would like to see the synchronization speed of those networks. The simulations in C++ are also shown in this paper to check out if there is a compromise between the theoretical results and the numerical ones.

II. EXISTENCE OF THE GLOBAL ATTRACTORS OF NETWORKS CONSISTING OF TWO NEURONS FHN LINEARLY COUPLED

In this section, the existence of global attractors of the system (2) and (3) in $\mathbb{R}^4$ is shown. The global attractor is a compact invariant set for the flow that attracts all trajectories (see for example [11]). Practically, it is very important since it is the set where all the solutions asymptotically evolve. In particular, all the patterns and solutions relevant for applications belong, asymptotically, to the global attractor (see [12, 13]).

Theorem 1: *There exists a positive constant K and $T > 0$ such that for all $(u_i(0), v_i(0)) \in \mathbb{R}^2$*

$$|u_i(t)| \le K, |v_i(t)| \le K, i = 1,2, \qquad \text{for all } t > T,$$

Where (u_i, v_i) is defined by system (2).

Proof. Let $\Phi_1(t) = \frac{1}{2}\left(\varepsilon a u_1^2 + v_1^2\right) + \frac{1}{2}\left(\varepsilon a u_2^2 + v_2^2\right)$.

By deriving the function $\Phi_1(t)$ with respect to t, there is the following:

$$\begin{aligned} \frac{d\Phi_1(t)}{dt} &= \varepsilon a u_1 u_{1t} + v_1 v_{1t} + \varepsilon a u_2 u_{2t} + v_2 v_{2t} \\ &= au_1\left[f(u_1) - v_1 - g_{syn}(u_1 - u_2)\right] + v_1(au_1 - bv_1 + c) \\ &\quad + au_2\left[f(u_2) - v_2 - g_{syn}(u_2 - u_1)\right] + v_2(au_2 - bv_2 + c) \\ &= au_1 f(u_1) - au_1v_1 - au_1 g_{syn}(u_1 - u_2) + au_1v_1 - bv_1^2 + cv_1 \\ &\quad + au_2 f(u_2) - au_2v_2 - au_2 g_{syn}(u_2 - u_1) + au_2v_2 - bv_2^2 + cv_2 \\ &= a\left[u_1 f(u_1) + u_2 f(u_2)\right] - a\left[u_1 g_{syn}(u_1 - u_2) + u_2 g_{syn}(u_2 - u_1)\right] \\ &\quad - b\left(v_1^2 + v_2^2\right) + c(v_1 + v_2) \end{aligned}$$

We can find a positive constant α such that:

$$\left|g_{syn}(u_1 - u_2)\right| \le \alpha\left(1 + |u_1| + |u_2|\right).$$

Hence,

$$\begin{aligned} \frac{d\Phi_1(t)}{dt} &\le a\left[u_1 f(u_1) + u_2 f(u_2)\right] + a\alpha\left(|u_1| + |u_2|\right) \\ &\quad + a\alpha\left(|u_1|.|u_1| + |u_1|.|u_2| + |u_2|.|u_1| + |u_2|.|u_2|\right) \\ &\quad - b\left(v_1^2 + v_2^2\right) + c(v_1 + v_2) \\ &\le a\left[u_1.\left(-u_1^3 + 3u_1\right) + u_2.\left(-u_2^3 + 3u_2\right)\right] + a\alpha\left(|u_1| + |u_2|\right) \\ &\quad + a\alpha\left(u_1^2 + 2|u_1|.|u_2| + u_2^2\right) - b\left(v_1^2 + v_2^2\right) + c(v_1 + v_2) \\ &\le -a\left(u_1^4 + u_2^4\right) + 3a\left(u_1^2 + u_2^2\right) + a\alpha\left(\frac{u_1^2}{2} + \frac{1}{2} + \frac{u_2^2}{2} + \frac{1}{2}\right) \\ &\quad + a\alpha\left[u_1^2 + 2\left(\frac{u_1^2}{2} + \frac{u_2^2}{2}\right) + u_2^2\right] - b\left(v_1^2 + v_2^2\right) + c(v_1 + v_2) \\ &\le -a\left(u_1^4 + u_2^4\right) + 3a\left(u_1^2 + u_2^2\right) + a\alpha\left(\frac{u_1^2}{2} + \frac{u_2^2}{2}\right) + 2a\alpha\left(u_1^2 + u_2^2\right) \\ &\quad - b\left(v_1^2 + v_2^2\right) + c(v_1 + v_2). \end{aligned}$$

We can find the constants $\beta > 0$, $K > 0$, and for all $h > 0$ such that:

$$\begin{aligned} \frac{d\Phi_1(t)}{dt} &\le -\beta\left(u_1^4 + u_2^4\right) + K - b\left(v_1^2 + v_2^2\right) + c(v_1 + v_2) \\ &\le -2\beta\left(u_1^2 + u_2^2\right) + \beta + K - b\left(v_1^2 + v_2^2\right) + \frac{hv_1^2}{2} + \frac{c^2}{2h} + \frac{hv_2^2}{2} + \frac{c^2}{2h} \\ &\le -2\beta\left(u_1^2 + u_2^2\right) + \beta + K - b\left(v_1^2 + v_2^2\right) + \frac{hv_1^2}{2} + \frac{hv_2^2}{2} + \frac{c^2}{h}. \end{aligned}$$

Finally, we can find the other constants $\beta > 0$ and $K > 0$ such that:

$$\frac{d\Phi_1(t)}{dt} \le -\beta\Phi_1(t) + K\beta.$$

This implies that:

$$\Phi_1(t) \le \exp(-\beta t)\Phi_1(0) + K(1 - \exp(-\beta t)).$$

Let t reach infinity, Theorem 1 will then be proved.

Theorem 2: *There exists a positive constant K and $T > 0$ such that for all $(u_i(0), v_i(0)) \in \mathbb{R}^2$*

$$|u_i(t)|\le K, |v_i(t)|\le K, i=1,2, \quad \textit{for all } t>T,$$

where (u_i, v_i) *is defined by system (3).*

Proof. Let $\Phi_2(t)=\frac{1}{2}\left(\varepsilon a u_1^2+v_1^2\right)+\frac{1}{2}\left(\varepsilon a u_2^2+v_2^2\right)$.

By deriving the function $\Phi_2(t)$ with respect to t, there is the following:

$$\begin{aligned}\frac{d\Phi_2(t)}{dt} &= \varepsilon a u_1 u_{1t} + v_1 v_{1t} + \varepsilon a u_2 u_{2t} + v_2 v_{2t}\\
&= a u_1\left[f(u_1)-v_1-g_{syn}(u_1-u_2)\right]+v_1(a u_1 - b v_1 + c)\\
&\quad + a u_2\left[f(u_2)-v_2\right]+v_2(a u_2 - b v_2 + c)\\
&= a u_1 f(u_1) - a u_1 v_1 - a u_1 g_{syn}(u_1-u_2) + a u_1 v_1 - b v_1^2 + c v_1\\
&\quad + a u_2 f(u_2) - a u_2 v_2 + a u_2 v_2 - b v_2^2 + c v_2\\
&= a\left[u_1 f(u_1) + u_2 f(u_2)\right] - a u_1 g_{syn}(u_1-u_2) - b\left(v_1^2+v_2^2\right) + c(v_1+v_2).\end{aligned}$$

We can find a positive constant α such that:

$$\left|g_{syn}(u_1-u_2)\right|\le \alpha\left(1+|u_1|+|u_2|\right).$$

Hence,

$$\begin{aligned}\frac{d\Phi_2(t)}{dt} &\le a\left[u_1 f(u_1)+u_2 f(u_2)\right]+a\alpha\left(|u_1|+|u_2|\right)\\
&\quad + a\alpha\left(|u_1|.|u_1|+|u_1|.|u_2|+|u_2|.|u_1|+|u_2|.|u_2|\right)-b\left(v_1^2+v_2^2\right)\\
&\quad + c(v_1+v_2)\\
&\le a\left[u_1.\left(-u_1^3+3u_1\right)+u_2.\left(-u_2^3+3u_2\right)\right]+a\alpha\left(|u_1|+|u_2|\right)\\
&\quad + a\alpha\left(u_1^2+2|u_1|.|u_2|+u_2^2\right)-b\left(v_1^2+v_2^2\right)+c(v_1+v_2)\\
&\le -a\left(u_1^4+u_2^4\right)+3a\left(u_1^2+u_2^2\right)+a\alpha\left(\frac{u_1^2}{2}+\frac{1}{2}+\frac{u_2^2}{2}+\frac{1}{2}\right)\\
&\quad + a\alpha\left[u_1^2+2\left(\frac{u_1^2}{2}+\frac{u_2^2}{2}\right)+u_2^2\right]-b\left(v_1^2+v_2^2\right)+c(v_1+v_2)\\
&\le -a\left(u_1^4+u_2^4\right)+3a\left(u_1^2+u_2^2\right)+a\alpha\left(\frac{u_1^2}{2}+\frac{u_2^2}{2}\right)+2a\alpha\left(u_1^2+u_2^2\right)\\
&\quad - b\left(v_1^2+v_2^2\right)+c(v_1+v_2).\end{aligned}$$

We can find the constants $\beta>0$, $K>0$, and for all $h>0$ such that:

$$\begin{aligned}\frac{d\Phi_2(t)}{dt} &\le -\beta\left(u_1^4+u_2^4\right)+K-b\left(v_1^2+v_2^2\right)+c(v_1+v_2)\\
&\le -2\beta\left(u_1^2+u_2^2\right)+\beta+K-b\left(v_1^2+v_2^2\right)+\frac{hv_1^2}{2}+\frac{c^2}{2h}+\frac{hv_2^2}{2}+\frac{c^2}{2h}\\
&\le -2\beta\left(u_1^2+u_2^2\right)+\beta+K-b\left(v_1^2+v_2^2\right)+\frac{hv_1^2}{2}+\frac{hv_2^2}{2}+\frac{c^2}{h}.\end{aligned}$$

Finally, we can find the other constants $\beta>0$ and $K>0$ such that:

$$\frac{d\Phi_2(t)}{dt}\le -\beta\Phi_2(t)+K\beta.$$

This implies that:

$\Phi_2(t)\le \exp(-\beta t)\Phi_2(0)+K(1-\exp(-\beta t))$.

Let t reach infinity, Theorem 2 will then be proved.

III. SYNCHRONIZATION SPEED OF NETWORKS CONSISTING OF TWO NEURONS FHN LINEARLY COUPLED

In this section, the sufficient conditions to obtain the synchronization in network of two neurons are found, and the minimal value of coupling strength to get the synchronization is investigated by numerical experiments.

Definition 1 (see [1]). Let $S_i=(u_i,v_i)$, $i=1,2,\ldots,n$ and $S=(S_1,S_2,\ldots,S_n)$ be a network. We say that S synchronizes identically if

$$\lim_{t\to+\infty}|u_j-u_i|=0 \;\; and \;\; \lim_{t\to+\infty}|v_j-v_i|=0, \;\; for\, all\;\, i,j=1,2,\ldots,n.$$

Let $M = \sup_{u\in B, x\in ℝ} \sum_{k=1}^{3}\frac{f^{(k)}(u)}{k!}x^{k-1}$, B is a compact interval including u and $f^{(k)}(u)$ is the kth derivative of f with respect to u. The existence of B is due to Theorem 1 and 2. We have then the following results.

Theorem 3. *If* $g_{syn}>\frac{M}{2}$, *the network (2) synchronizes in the sense of Definition 1.*

Proof. Let consider the Lyapunov function

$$W_1(t)=\frac{a\varepsilon}{2}\left(u_2-u_1\right)^2+\frac{1}{2}\left(v_2-v_1\right)^2.$$

By deriving the function $W_1(t)$ with respect to t, there is the following:

$$\begin{aligned}\frac{dW_1(t)}{dt} &= a\varepsilon\left(u_2-u_1\right)\left(u_{2t}-u_{1t}\right)+\left(v_2-v_1\right)\left(v_{2t}-v_{1t}\right)\\
&= a\left(u_2-u_1\right).\left[f(u_2)-f(u_1)-2g_{syn}\left(u_2-u_1\right)\right]-b\left(v_2-v_1\right)^2.\end{aligned}$$

By applying the Taylor formula for function f, we have then:

$$f(u_2)=f(u_1)+\sum_{k=1}^{3}\frac{f^{(k)}(u_1)}{k!}\left(u_2-u_1\right)^k.$$

Hence,

$$\begin{aligned}\frac{dW_1(t)}{dt} &= a\left(u_2-u_1\right)\left[\sum_{k=1}^{3}\frac{f^{(k)}(u_1)}{k!}\left(u_2-u_1\right)^k-2g_{syn}\left(u_2-u_1\right)\right]\\
&\quad - b\left(v_2-v_1\right)^2\\
&= a\left(u_2-u_1\right)^2\left[\sum_{k=1}^{3}\frac{f^{(k)}(u_1)}{k!}\left(u_2-u_1\right)^{k-1}-2g_{syn}\right]\\
&\quad - b\left(v_2-v_1\right)^2\\
&\le a\left(u_2-u_1\right)^2\left[M-2g_{syn}\right]-b\left(v_2-v_1\right)^2\end{aligned}$$

If $g_{syn}>\frac{M}{2}$, then

$$\frac{dW_1(t)}{dt}\le -\beta W_1(t)\Rightarrow W_1(t)\le W_1(0)e^{-\beta t},$$

where $\beta = \min\left(2\dfrac{2g_{syn} - M}{\varepsilon}, 2b\right)$. Thus, the synchronization occurs if the coupling strength verifies

$$g_{syn} > \frac{M}{2}.$$

Theorem 4. *If $g_{syn} > M$, the network (3) synchronizes in the sense of Definition 1.*

Proof. Let consider the Lyapunov function

$$W_2(t) = \frac{a\varepsilon}{2}(u_2 - u_1)^2 + \frac{1}{2}(v_2 - v_1)^2.$$

By deriving the function $W_2(t)$ with respect to t, there is the following:

$$\begin{aligned}\frac{dW_2(t)}{dt} &= a\varepsilon(u_2 - u_1)(u_{2t} - u_{1t}) + (v_2 - v_1)(v_{2t} - v_{1t})\\ &= a(u_2 - u_1).\left[f(u_2) - f(u_1) - g_{syn}(u_2 - u_1)\right]\\ &\quad - b(v_2 - v_1)^2.\end{aligned}$$

By using the same technic of the proof of Theorem 3. We have then:

$$\begin{aligned}\frac{dW_2(t)}{dt} &= a(u_2 - u_1)\left[\sum_{k=1}^{3}\frac{f^{(k)}(u_1)}{k!}(u_2 - u_1)^k - g_{syn}(u_2 - u_1)\right]\\ &\quad - b(v_2 - v_1)^2\\ &= a(u_2 - u_1)^2\left[\sum_{k=1}^{3}\frac{f^{(k)}(u_1)}{k!}(u_2 - u_1)^{k-1} - g_{syn}\right]\\ &\quad - b(v_2 - v_1)^2\\ &\le a(u_2 - u_1)^2\left[M - g_{syn}\right] - b(v_2 - v_1)^2\end{aligned}$$

If $g_{syn} > M$, then

$$\frac{dW_2(t)}{dt} \le -\beta W_2(t) \Rightarrow W_2(t) \le W_2(0)e^{-\beta t},$$

where $\beta = \min\left(2\dfrac{g_{syn} - M}{\varepsilon}, 2b\right)$. Thus, the synchronization occurs if the coupling strength verifies

$$g_{syn} > M.$$

As the results of Theorem 3 and 4 are given, we can easily see that to synchronize the network of two neurons with bidirectionally linear coupling is easier than to synchronize the one with unidirectionally linear coupling. Because the coupling strength g_{syn} to synchronize the system (2) is smaller than the one to synchronize the system (3).

IV. NUMERICAL SIMULATIONS

In this section, we make the simulations to check if the numerical results will meet the theoretical results above. The simulation results are obtained by integrating the system (2) and (3), with the following parameter values: $a = 1, b = 0.001, c = 0, \quad \varepsilon = 0.1.$ The integrations of those systems were realized by using C++ and the patterns are presented by Gnuplot.

Figure 1 below illustrates the phenomenon of synchronization for the network of two neurons with bidirectionally linear coupling. The simulations show that the system synchronizes from the value $g_{syn} = 1.4.$ Figures 1(a), 1(b), 1(c), 1(d) represent the phase portraits (u_1, u_2) corresponding to the different values of coupling strength. Before synchronization, for $g_{syn} = 0.0001$, Figure 1(a) represents the temporal dynamic of u_2 with respect to u_1; Figure 1(b) represents the temporal dynamic of u_2 with respect to u_1 for $g_{syn} = 0.01$; Figure 1(c) represents the temporal dynamic of u_2 with respect to u_1 for $g_{syn} = 0.5.$ The synchronization occurs for $g_{syn} = 1.4.$ It is easy to see that the synchronization occurs in Figure 1(d) for $g_{syn} = 1.4$, since $u_1 \approx u_2$.

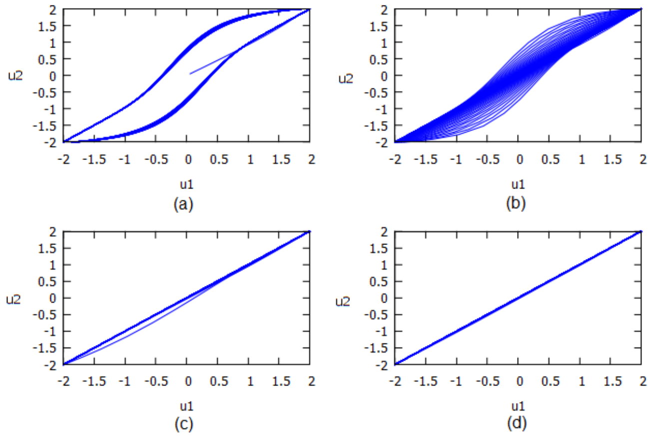

Figure 1. - Synchronization in network of two neurons with bidirectionally linear coupling. The synchronization occurs for $g_{syn} = 1.4$. Before synchronization, for $g_{syn} = 0.0001$, figure (a) represents the temporal dynamic of u_2 with respect to u_1 ; figure (b) represents the temporal dynamic of u_2 with respect to u_1 for $g_{syn} = 0.01$; figure (c) represents the temporal dynamic of u_2 with respect to u_1 for $g_{syn} = 0.5$. For the value $g_{syn} = 1.4$ in figure (d), the synchronization of two neurons occurs: $u_1 \approx u_2$

Figure 2 below illustrates the phenomenon of synchronization for the network of two neurons with unidirectionally linear coupling. The simulations show that the system synchronizes from the value $g_{syn} = 2.5$. Figures 1(a), 1(b), 1(c), 1(d) represent the phase portraits (u_1, u_2) corresponding to the different values of coupling strength. Before synchronization, for $g_{syn} = 0.5$, Figure 2(a) represents the temporal dynamic of u_2 with respect to u_1; Figure 2(b) represents the temporal dynamic of u_2 with respect to u_1 for $g_{syn} = 1.0$; Figure 2(c) represents the temporal dynamic of u_2 with respect to u_1 for $g_{syn} = 1.4$. The synchronization occurs for $g_{syn} = 2.5$. It is easy to see that the synchronization occurs in Figure 1(d) for $g_{syn} = 2.5$, since $u_1 \approx u_2$.

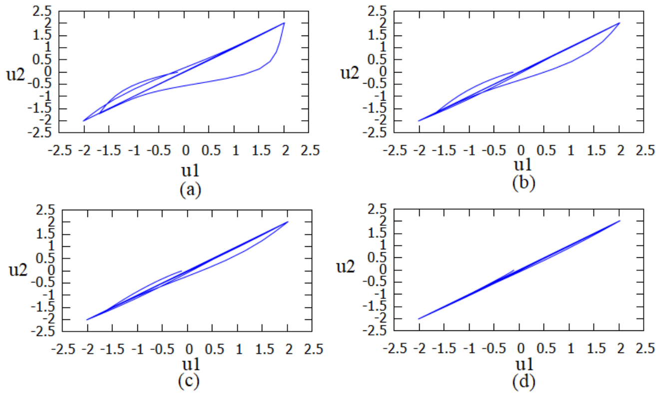

Figure 2. - Synchronization in network of two neurons with unidirectionally linear coupling. The synchronization occurs for $g_{syn} = 2.5$. Before synchronization, for $g_{syn} = 0.5$, figure (a) represents the temporal dynamic of u_2 with respect to u_1; figure (b) represents the temporal dynamic of u_2 with respect to u_1 for $g_{syn} = 1.0$; figure (c) represents the temporal dynamic of u_2 with respect to u_1 for $g_{syn} = 1.4$. For the value $g_{syn} = 2.5$ in figure (d), the synchronization of two neurons occurs: $u_1 \approx u_2$

As the numerical results are shown in Figures 1 and 2, we need $g_{syn} = 2.5$ to get the synchronization in the network of two neurons with unidirectionally linear coupling. Meanwhile, $g_{syn} = 1.4$ can occur synchronization in the network of two neurons with bidirectionally linear coupling. It means to synchronize the network of two neurons with bidirectionally linear coupling is easier than the other. This result completely meets the theoretical results above.

V. CONCLUSION

The paper shows that there exist the global attractors of the networks of two neurons linearly coupled, and synchronizing the network of two neurons with bidirectionally linear coupling is easier than synchronizing the one with unidirectionally linear coupling. Because the coupling strength to synchronize the system (2) is smaller than the one to synchronize the system (3). It also presents the numerical results, and there is a compromise between the theoretical results and the numerical ones.

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