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Complementary Degree Equitable Dominating Sets in Graphs

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Abstract

Let G= (V,E) be a simple graph. A subset D of V(G) is said to be a dominating set of G if for every vertex v∈V-D there exists a vertex u∈D such that u and v are adjacent in G. A subset D of V(G) is called a complementary degree equitable dominating set (cdged-set) of G if D is a dominating set of G and V-D is a degree equitable set in G. The minimum cardinality of a minimal cdged-set of G is called the complementary degree equitable domination number of G and is denoted by γ^{cdged} (G). The maximum cardinality of a minimal cdged-set of G is called the upper complementary degree equitable domination number of G and is denoted by Γ^{cdged}(G). Complementary degree equitable domination is super hereditary. Therefore, complementary degree equitable domination is minimal if and only if it is 1-minimal. Interesting results are proved with respect to the new parameters.

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Introduction 1.0 Let u,v∈V(G). u and v are said to be degree equitable if |deg(u)-deg(v)|≤1. A subset S of V(G) is said to be degree equitable if any two vertices of S are degree equitable. Such sets are studied in detail by Arumugam et al [1,2]. Instead of requiring that the set is degree equitable, equitability in degree is imposed in the complement of a set. Complementary degree equitable dominating sets and complementary equitable independent sets are studied.

Definition 1.1 Let G=(V,E) be a simple graph. A subset D of V(G) is called a complementary degree equitable dominating set (cdged-set) of G if D is a dominating set of G and V-D is a degree equitable set in G.

V is always a complementary degree equitable dominating set of G and hence the existence of a complementary degree equitable dominating set is guaranteed in any graph.

Definition 1.2 The minimum cardinality of a minimal cdged-set of G is called the complementary degree equitable domination number of G and is denoted by γ^{cdged} (G). The maximum cardinality of a minimal cdged-set of G is called the upper complementary degree equitable domination number of G and is denoted by Γ^{cdged} (G).

Complementary degree equitable domination is super hereditary. Therefore, complementary degree equitable dominating set is minimal if and only if it is 1-minimal.

Remark 1.3

Let G be a simple graph. Then $\gamma(G) \le \gamma^{cdged}(G)$.

γ^{cdged}(G) for standard graphs

1. $\gamma^{cdged}(K_n) = 1$

2. $\gamma^{cdged}(K_{1,n}) = 1$

3. $\gamma^{cdged}(K_{m,n}) = \begin{cases} 2, if\ |m-n| \leq 1 \\ m+1, if\ |m-n| \geq 2\ and\ m \leq n \end{cases}$

4. $\gamma^{cdged}(D_{r,s}) = 2$

5. $\gamma^{cdged}(P_n) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$

6. $\gamma^{cdged}(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil, n \geq 3$

7. $\gamma^{cdged}(W_n) = 1$

8. $\gamma^{cdged}(P) = \gamma(P) = 3$ where P is the Petersen graph.

Proposition 1.4 Let G be a simple graph. Then γ^{cdged}(G)=n if and only if $G = \overline{K_n}$.

Observation 1.5

$\gamma^{cdged}(K_n + K_{1,m}) = \min\{n+1, m\}$.

Observation 1.6

$$\gamma^{cdged}(K_n + P_m) = \begin{cases} 1, if\ m = 1,2,3 \\ 2, if\ m = 4 \\ \min\{m,n\}, if\ m \geq 5 \end{cases}$$

Observation 1.7

$$\gamma^{cdged}(K_n + C_m) = \begin{cases} 1, if\ m = 3,4 \\ \min\{m,n\}, if\ m \geq 5 \end{cases}$$

Observation 1.8

$$\gamma^{cdged}(K_n + W_m) = \begin{cases} 1, if\ m \leq 5 \\ \mathrm{m}-1, if\ m \geq 6 \end{cases}$$

Definition 1.9 A vertex u is a complementary equitable full degree vertex if u is a full degree vertex and V(G)-{u} is degree equitable.

Remark 1.10 γ^{cdged}(G)=1 if and only if G has a complementary equitable full degree vertex.

Theorem 1.11 A cdged-set D is minimal if and only if for any vertex u∈D one of the following holds:

(i) u is an isolate of D
(ii) u has a private neighbour in V-D with respect to D
(iii) there exists a vertex v∈V-D such that |deg(u)-deg(v)|≥2

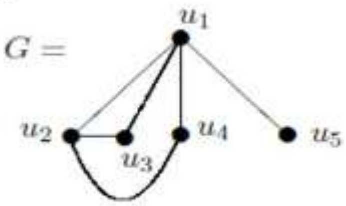
Proof: Routine.

Observation 1.12 Suppose G has an independent cdged-set. The minimum cardinality of a maximal independent cdged-set is defined as the independent cdged-number of G and is denoted by $i^{cdged}(G)$.

Remark 1.13 Let G be a simple graph. Then $\gamma^{cdged}(G) \leq i^{cdged}(G) \leq \beta_0(G)$.

Remark 1.14 There are graphs which admit independent cdged-sets.

For:



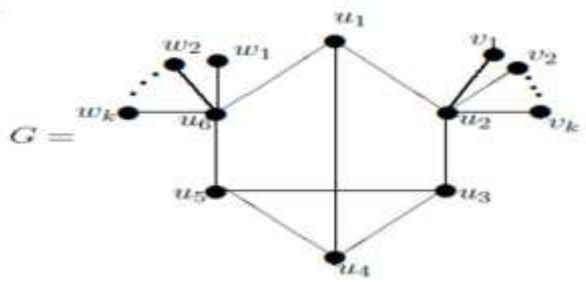
G has an independent cdged-set namely $\{u_3,u_4,u_5\}$.

Remark 1.15 There exist graphs in which no independent cdged-set exists.

Remark 1.16 A maximal independent set of a graph G is a minimal dominating set of G but need not be a cdged-set of G.

Remark 1.17 There are graphs in which the complement of a minimal cdged-set is not even a dominating set.

For:



$D=\{u_2,u_4,u_6,v_1,v_2,\ldots,v_k,w_1,w_2,\ldots,w_k\}$ is a minimal cdged-set. $V\text{-}D=\{u_1,u_3,u_5\}$. V-D is not even a dominating set.

Definition 1.18 Let $u\in V(G)$. $N^e_G(u)$ stands for the set of all neighbours of u which form an equitable set in G.

$\deg^e_G(u)=|N^e_G(u)|$. $\delta^e(G)=\min\{\deg^e_G(u) : u\in V(G)\}$. $\Delta^e(G)= \max\{\deg^e_G(u) : u\in V(G)\}$.

Theorem 1.19 Let G be a simple graph. Then $\gamma^{cdged}(G)\leq n-\Delta^e(G)$.

Proof: Let u be a vertex of G such that $\deg^e_G(u)= \Delta^e(G)$. Let $S=V-N^e_G(u)$. Therefore, S is a cdged-set of G. Therefore, $\gamma^{cdged}(G)\leq |S| \leq n-\Delta^e(G)$.

Theorem 1.20 Let G be a simple graph. Then $\frac{n}{1+\Delta^e(G)} \leq \gamma^{cdged}(G)$.

Proof: $\Delta^e(G)$ vertices can be equitably dominated by a vertex u of $\deg^e_G(u)= \Delta^e(G)$. Thus, $1+\Delta^e(G)$ vertices are covered by u. Therefore, to cover n vertices we require at least $\frac{n}{1+\Delta^e(G)}$ vertices. Therefore, $\frac{n}{1+\Delta^e(G)} \leq \gamma^{cdged}(G)$.

Remark 1.21 $\frac{n}{1+\Delta^e(G)} \leq \gamma^{cdged}(G) \leq n - \Delta^e(G)$.

Theorem 1.22 For any tree T, $\gamma^{cdged}(T)=n+1-\Delta^e(T)$ if and only if T is a wounded spider.

Proof: Let G be a wounded spider. Except for the centre of the spider, all other vertices are equitable. Therefore, $\gamma^{cdged}(T)=1+t$ where t is the number of vertices with degree two. $\Delta^e(T)$=degree of the central vertex=n+1-1-t. Therefore, $\gamma^{cdged}(T)+\Delta^e(T)=1+t+n+1-1-t=n+1$. Conversely, Let $\gamma^{cdged}(T)+\Delta^e(T)=n+1$. Let v be a vertex of equitable degree Δ^e. If $T-N^e[v]=\varphi$, then T is a star. Therefore, $N^e[v]=n+1$. Therefore, $N^e(v)=n$. Since T is a tree, the equitable neighbours of v are independent. Therefore, T is a star ($K_{1,n}$). Suppose $T-N^e[v]\neq\varphi$.

$n+1-|N^e[v]|=n+1-(\Delta^e(T)+1)$

$=n+1-\Delta^e(T)-1$

$=\gamma^{cdged}(T)-1$

Let D′ be a minimum cdged-set of G. Then

$|D'|\leq n+1-|N^e[v]|+1$

$=\gamma^{cdged}(T)-1+1$

$=\gamma^{cdged}(T)$

Therefore, $|D'|=\gamma^{cdged}(T)$. Therefore, |D′| contains v and $V-N^e[v]$. Let $u\in D'$. Then u and v are connected. That is, there exists a path $u,u_1,u_2,\ldots,u_k,v$ in T. If k=0, then u is adjacent with v. If k=1, then u is adjacent with u_1 and u_1 is adjacent with v. If $k\geq 2$, then $\gamma^{cdged}(T)<n+1-\Delta^e(T)$, a contradiction. Therefore, T is a wounded spider.

Theorem 1.23 Let G be a simple graph with $\gamma^{cdged}(G)\geq 2$. Then

$m \leq \frac{1}{2}\left[\left(n-\gamma^{cdged}(G)\right)\left(n-\gamma^{cdged}(G)+2\right)\right]$ where m is the size of G.

Proof: Let G be a simple graph with $\gamma^{cdged}(G)\geq 2$. The proof is by induction on the order n. If n=2, then G is K_2 or $\overline{K_2}$. m=1 or 0. $\gamma^{cdged}(G)=1$ or 2.

$$m \leq \frac{1}{2}\left[\left(n-\gamma^{cdged}(G)\right)\left(n-\gamma^{cdged}(G)+2\right)\right]$$

$$=\frac{1}{2}[(2-1)(2-1+2)]$$

$$=\frac{3}{2}$$

which is true since K_2 has exactly one edge. If $G=\overline{K_2}$,

$$m \leq \frac{1}{2}\left[\left(n-\gamma^{cdged}(G)\right)\left(n-\gamma^{cdged}(G)+2\right)\right]$$

$$=\frac{1}{2}[(2-2)(2-2+2)]$$

$$=0$$

Therefore, m=0 which is true since $\overline{K_2}$ has no edge. Assume the result for all graphs with order less than n and $\gamma^{cdged}(G)\geq 3$. Let v be a vertex of degree $\Delta^e(G)$. $|N^e(v)|=\Delta^e(G)\leq n-\gamma^{cdged}(G)$. That is, $|N^e(v)|=n-\gamma^{cdged}(G)-r$ where $0\leq r\leq n-\gamma^{cdged}(G)$. Let $S=V-N^e[v]$. Therefore, $|S|=\gamma^{cdged}(G)+r-1$. Then $S=N^e[v]$. Therefore, V-S is equitable. Let $u\in N^e(v)$. Let $T=(S-N^e(u))\cup\{u,v\}$. Therefore, T is cdged-set.

$\gamma^{cdged}(G) \leq T$

$= |S-N^e(u)|+2$

$\leq \gamma^{cdged}(G)+r-1-|S\cap N^e(u)|+2$

Therefore, $|S\cap N^e(u)| \leq r+1$. This is true for every $u \in N^e(v)$. Therefore, number of edges between $N^e(v)$ and S denoted by m_1 is at most $\Delta^e(G)(r+1)$. That is, $m_1 \leq \Delta^e(G)(r+1)$. If D is a γ^{cdged}-set of <S>, then $D\cup\{v\}$ is a cdged-set of G. Therefore, $\gamma^{cdged}(G) \leq |D|+1$. Therefore, $\gamma^{cdged}(<S>) \geq \gamma^{cdged}(G)-1 \geq 2$. Therefore, by induction hypothesis the number of edges in <S> say m_2 is

$$m_2 \leq \frac{1}{2}\Big[\big(|S|-\gamma^{cdged}(<S>)\big)(|S|-\gamma^{cdged}(<S>)+2)\Big]$$

$$\leq \frac{1}{2}[(\gamma^{cdged}(G)+r-1-(\gamma^{cdged}(G)-1)(\gamma^{cdged}(G)+r-1-(\gamma^{cdged}(G)-1)+2))]$$

$$=\frac{1}{2}[r(r+2)]$$

Let $m_3=|E(N^e[v])|$. Therefore, number of edges in

$G = m_1+m_2+m_3$

$$\leq \frac{1}{2}\Big[\big(n-\gamma^{cdged}(G)\big)(n-\gamma^{cdged}(G)+2)\Big]$$

By induction, the proof is complete when $\gamma^{cdged}(G) \geq 3$. If $\gamma^{cdged}(G) \geq 2$, by adding an isolated vertex to the graph, we get that $\gamma^{cdged}(G) \geq 3$ and order of the graph is n+1. The number of edges is not increased. Therefore,

$$|E(G)| \leq \frac{1}{2}\Big[\big(n-\gamma^{cdged}(G)\big)(n-\gamma^{cdged}(G)+2)\Big].$$

Theorem 1.24 Let G be a graph of order n and size m. Let $\gamma^{cdged}(G)=t$. Then every subset of V(G) of cardinality t is a cdged-set of G if and only if G is either K_n or $\overline{K_n}$ or $\overline{(\frac{n}{2})K_2}$.

Proof: Let G be a graph of order n, size m and $\gamma^{cdged}(G)=t$. Suppose every subset of V(G) of cardinality t is a cdged-set of G. If t=n, then $G=\overline{K_n}$. Let t<n. Let $u\in V(G)$. If $deg_{\bar{G}}(u) \geq t$, then u is not adjacent in G to at least t vertices say $v_1,v_2,\ldots,v_t$. Therefore, the set $\{v_1,v_2,\ldots,v_t\}$ is not a dominating set of G, a contradiction since any t element set is a cdged-set of G. Therefore, $deg_{\bar{G}}(u) < t$ for every $u\in V(G)$. Therefore, $deg_G(u) \geq n-t$. Therefore,

$$2m = \sum deg_G(u) \geq n(n-t).$$

Therefore, $m \geq \frac{n(n-t)}{2} = \frac{n\left(n-\gamma^{cdged}(G)\right)}{2} \ldots\ldots\ldots\ldots (1)$

When t>2, $m \leq \frac{1}{2}\Big[\big(n-\gamma^{cdged}(G)\big)(n-\gamma^{cdged}(G)+2)\Big]$, a contradiction to (1). Therefore, $t \leq 2$. Suppose t=1. Then $G=K_n$(since every vertex of G is a dominating vertex and hence $G=K_n$). Suppose t=2. Then $\gamma^{cdged}(G)=2$. $\Delta^e(G) \leq n-\gamma^{cdged}(G)=n-2$. $deg_G(u) \geq n-2$ for every $u\in V(G)$. Therefore, $\delta(G) \geq n-2$. If $\Delta^e(G)=n-1$, then $\Delta^e(u)=n-1$. Therefore, $\gamma^{cdged}(G)=1$, a contradiction. Therefore, $\Delta^e(G) \leq n-2$. Suppose $\delta(G)=n-1$ and $\Delta^e(G)=n-2$. Suppose there exists a vertex $u\in V(G)$ such that $\delta(u)=n-1$ and $\Delta^e(u)=n-2$. Then {u} is not a cdged-set. Let v be not degree equitable with u. If n=3, then $deg_G(u)=2$ and $deg_G(v)=2$ for every $v\in V(G)$. Therefore, $\Delta^e(G)=2$, a contradiction. Therefore, $n \geq 4$. Let $S=\{u,u_1\}$ where $u_1\neq v$. Then S is a γ^{cdged}-set of G but V-S is not degree equitable since v∈V-S. Therefore, Δ(G)=n-2 and δ(G)=n-2. Therefore, G is regular and every vertex has degree n-2. Therefore, $2m = \sum deg_G(u) = n(n-2)$. Therefore, $m = \frac{n(n-2)}{2}$. Therefore, n is even. Therefore, $G = \overline{(\frac{n}{2})K_2}$.

Conversely, If $G = K_n\ or\ \overline{K_n}\ or\ \overline{(\frac{n}{2})K_2}$, then any subset of V(G) of cardinality γ^{cdged}(G) is a minimum cdged-set of G. Hence the theorem.

Theorem 1.25 Let G be a graph of order n and γ^{cdged}(G)=t. Then the following are equivalent:

(i) every t-subset of V(G) is a cdged set

(ii) $G = K_n\ or\ \overline{K_n}\ or\ \overline{(\frac{n}{2})K_2}$

(iii) t=n-κ(G)

Proof: From the above theorem, (i) and (ii) are equivalent. It has been proved that (ii) and (iii) are equivalent in [5]. Hence the theorem.

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