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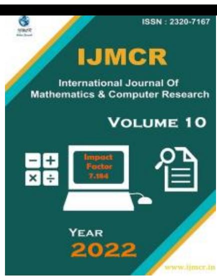

Conserved Quantities of a Nonlinear Coupled System of Korteweg-De Vries Equations

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ARTICLE INFO	ABSTRACT
Published Online: 07 May 2022 Corresponding Author: **Joseph Owuor**	The conserved vectors from a system of coupled Kortewegde Vries equations that have modelled the propagation of shallow water waves, ion-acoustic waves in plasmas, solitons, and nonlinear perturbations along internal surfaces between layers of different densities in stratified fluids, for example propagation of solitons of long internal waves in oceans. Notable applications have been to model shock wave formation, turbulence, boundary layer behavior, and mass transport. This paper illustrates the computation of conserved quantities using two approaches. First, by the multiplier method and by an application of new conservation theorem developed by Nail Ibragimov. 2010 Mathematics Subject Classification. 47B47; 47A30.

KEYWORDS: coupled KdV equations; soliton; multipliers; conservation laws.

INTRODUCTION

Conserved quantities have been used to qualitatively understand solutions to partial differential equations and even to construct exact solutions for the same. The dynamics of shallow-water waves, ionacoustic waves in plasmas, and long internal waves in oceans can be studied by understanding coupled KdV equations, which can be deduced the classical kdv equation .

$$J_t + \alpha JJ_x + \beta J_{xxx} = 0, \tag{1}$$

for α and β as constants, we let

$$J(t,x) = u(t,x) + iv(t,x), \tag{2}$$

where $i^2 = -1$. Then substituting (2) into (1) and separating the real and imaginary parts, we obtain

$$\Delta_1 \equiv u_t + \alpha uu_x - \alpha vv_x + \beta u_{xxx} = 0,\ \Delta_2 \equiv v_t + \alpha uv_x + \alpha vu_x + \beta v_{xxx} = 0, \tag{3}$$

which is a nonlinear system of coupled KdV equations. This paper uses symmetries of kdV equation to construct conservation laws for a nonlinear coupled kdV system (3).

Preliminaries

In this section, we outline preliminary concepts which are useful in the sequel.

Local Lie groups. [5] In Euclidean spaces R^n of $x = x^i$ independent variables and R^m of $u = u^\alpha$ dependent variables, we consider the transformations

T_ϵ : $\bar{x}^i = \varphi^i(x^i,u^\alpha,\epsilon)$, $\bar{u}^\alpha = \psi^\alpha(x^i,u^\alpha,\epsilon)$,

involving the continuous parameter ϵ which ranges from a neighbourhood $N' \subset N \subset R$ of $\epsilon = 0$ where the functions φ^i and ψ^α differentiable and analytic in the parameter ϵ.

Definition 0.1. The set G of transformations given by (4) is a local Lie group if it holds true that

(1) (Closure) Given $T_{\epsilon 1}, T_{\epsilon 2} \in G$, for $\epsilon_1,\epsilon_2 \in N' \subset N$, then $T_{\epsilon 1}T_{\epsilon 2} = T_{\epsilon 3} \in G$, $\epsilon_3 = \phi(\epsilon_1,\epsilon_2) \in N$.

(2) (Identity) There exists a unique $T_0 \in G$ if and only if $\epsilon = 0$ such that $T_\epsilon T_0 = T_0 T_\epsilon = T_\epsilon$.

(3) (Inverse) There exists a unique $T_{\epsilon^{-1}} \in G$ for every transformation $T_\epsilon \in G$,

where $\epsilon \in N' \subset N$ and $\epsilon^{-1} \in N$ such that $T_\epsilon T_{\epsilon^{-1}} =$
$T_{\epsilon^{-1}} T_\epsilon = T_0$.

Remark 0.2. Associativity of the group G in (4) follows from (1).

Prolongations. In the system,

(5) $$\Delta_\alpha \quad x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)}\big) = \Delta_\alpha = 0,$$

the variables u^α are dependent. The partial derivatives $u_{(1)} = \{u_i^\alpha\}, u_{(2)} = \{u_{ij}^\alpha\}, \ldots, u_{(\pi)} = \{u_{i_1\ldots i_\pi}^\alpha\}$, are of the first, second, ..., up to the πth-orders.

Denoting

(6) $$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots,$$

the total differentiation operator with respect to the variables x^i and δ_i^j, the Kronecker delta, we have

(7) $$D_i(x^j) = \delta_i^j, ', \ u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j(D_i(u^\alpha)), \ldots,$$

where $u^\alpha{}_i$ defined in (7) are differential variables [8].

(1) **Prolonged groups** Consider the local Lie group G given by the transformations

(8)
$$\bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \varphi^i\Big|_{\epsilon=0} = x^i, \ \ \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha,$$

where the symbol $|$ means evaluated on $\epsilon = 0.\big|_{\epsilon=0}$

Definition 0.3. The construction of the group G given by (8) is an equivalence of the computation of infinitesimal transformations

(9)
$$\bar{x}^i \approx x^i + \xi^i(x^i, u^\alpha)\epsilon, \quad \varphi^i\Big|_{\epsilon=0} = x^i, \quad \bar{u}^\alpha \approx u^\alpha + \eta^\alpha(x^i, u^\alpha)\epsilon, \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha,$$

obtained from (4) by a Taylor series expansion of $\varphi^i(x^i,u^\alpha,\epsilon)$ and $\psi^i(x^i,u^\alpha,\epsilon)$ in ϵ about $\epsilon = 0$ and keeping only the terms linear in ϵ, where

(10) $$\xi^i(x^i, u^\alpha) = \frac{\partial \varphi^i(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \eta^\alpha(x^i, u^\alpha) = \frac{\partial \psi^\alpha(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}.$$

Remark 0.4. The symbol of infinitesimal transformations, X, is used to write (9) as

(11) $$\bar{x}^i \approx (1 + X)x^i, \quad \bar{u}^\alpha \approx (1 + X)u^\alpha,$$

where

(12) $$X = \xi^i(x^i, u^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha},$$

is the generator of the group G given by (8).

Remark 0.5. To obtain transformed derivatives from (4), we use a change of variable formulae

(13) $$D_i = D_i(\varphi^j)\bar{D}_j,$$

where $\bar{D}_j$ is the total differentiation in the variables $\bar{x}^i$. This means that

(14) $$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha), \ \ \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha).$$

If we apply the change of variable formula given in (13) on G given by (8), we get

(15) $$D_i(\psi^\alpha) = D_i(\varphi^j), \ \ \bar{D}_j(\bar{u}^\alpha) = \bar{u}_j^\alpha D_i(\varphi^j).$$

Expansion of (15) yields

(16) $$\left(\frac{\partial \varphi^j}{\partial x^i} + \bar{u}_i^\beta \frac{\partial \varphi^j}{\partial \bar{u}}\right)\bar{u}_j^\beta = \frac{\partial \psi^\alpha}{\partial x^i} + \bar{u}_i^\beta \frac{\partial \psi^\alpha}{\partial \bar{u}}.$$

The variables $\bar{u}_i^\alpha$ can be written as functions of $x^i,u^\alpha,u_{(1)}$, that is

(17) $$u_i^\alpha = \Phi^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon), \quad \Phi^\alpha\Big|_{\epsilon=0} = u_i^{\bar{\alpha}} \quad .$$

Definition 0.6. The transformations in the space of the variables $x^i,u^\alpha,u_{(1)}$ given in (8) and (17) form the first prolongation group $G^{[1]}$.

Definition 0.7. Infinitesimal transformation of the first derivatives is

(18) $$\bar{u}^\alpha{}_i \approx u^\alpha{}_i + \zeta_i^\alpha \epsilon, \quad \text{where} \quad \zeta_i^\alpha = \zeta_i^\alpha(x^i,u^\alpha,u_{(1)},\epsilon).$$

Remark 0.8. In terms of infinitesimal transformations, the first prolongation group $G^{[1]}$ is given by (9) and (18).

(2) **Prolonged generators**

Definition 0.9. By using the relation given in (15) on the first prolongation group $G^{[1]}$ given by Definition 0.6, we obtain [5, **?**]

(19)

$D_i(x^j + \xi^j\epsilon)(u^\alpha_j + \zeta_j^\alpha\epsilon) = D_i(u^\alpha + \eta^\alpha\epsilon)$, which gives $u_i^\alpha + \zeta_j^\alpha\epsilon + u_j^\alpha\epsilon D_i\xi^j = u_i^\alpha + D_i\eta^\alpha\epsilon$,and thus

(20) $\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j)$, is the first prolongation formula.

Remark 0.10. Similarly, we get higher order prolongations [9] ,

(21) $\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u^\alpha_{i\kappa}D_j(\xi^\kappa)$, ..., $\zeta_i^\alpha 1,\ldots,i_\kappa = D_{i\kappa}(\zeta_i^\alpha 1,\ldots,i\kappa{-}1) - u_i^\alpha 1,i2,\ldots,i\kappa{-}1jD_{i\kappa}(\xi^j)$.

Remark 0.11. The prolonged generators of the prolongations $G^{[1]},\ldots,G^{[\kappa]}$ of the group G are

$$(22)\quad X^{[1]} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \quad \ldots, X^{[\kappa]} = X^{[\kappa-1]} + \zeta^\alpha_{i_1,\ldots,i_\kappa}\frac{\partial}{\partial \zeta^\alpha_{i_1,\ldots,i_\kappa}}, \quad \kappa \geq 1,$$

where X is the group generator given by (12).

Group invariants.

Definition 0.12. A function $\Gamma(x^i,u^\alpha)$ is called an invariant of the group G of transformations given by (4) if

$$(23)\qquad \Gamma(\bar{x}^i, \bar{u}^\alpha) = \Gamma(x^i, u^\alpha).$$

Theorem 0.13. *A function* $\Gamma(x^i,u^\alpha)$ *is an invariant of the group* G *given by (4) if and only if it solves the following first-order linear PDE:*

[5]

$$(24)\qquad X\Gamma = \xi^i(x^i,u^\alpha)\frac{\partial\Gamma}{\partial x^i} + \eta^\alpha(x^i,u^\alpha)\frac{\partial\Gamma}{\partial u^\alpha} = 0.$$

From Theorem (0.13), we have the following result.

Theorem 0.14. *The local Lie group* G *of transformations in* R^n *given by (4)* [8] *has precisely* $n-1$ *functionally independent invariants. One can take, as the basic invariants, the left-hand sides of the first integrals*

$$(25)\qquad \psi_1(x^i,u^\alpha) = c_1,\ldots,\psi_{n-1}(x^i,u^\alpha) = c_{n-1},$$

of the characteristic equations for (24):

$$(26)\qquad \frac{dx^i}{\xi^i(x^i,u^\alpha)} = \frac{du^\alpha}{\eta^\alpha(x^i,u^\alpha)}.$$

Symmetry groups.

Definition 0.15. The vector field X (12) is a Lie point symmetry of the PDE system (5) if the determining equations

$$(27)\qquad X^{[\pi]}\Delta_\alpha\Big|_{\Delta_\alpha=0} = 0, \quad \alpha = 1,\ldots,m, \quad \pi \geq 1,$$

are satisfied, where $\Big|_{\Delta\ \ =0}$ means evaluated on $\Delta_\alpha = 0$ and $X^{[\pi]}$ is the

α π-th prolongation of X.

Definition 0.16. The Lie group G is a symmetry group of the PDE system given in (5) if the PDE system (5) is form-invariant, that is

$$(28)\qquad \Delta_\alpha\ \ \bar{x}^i, \bar{u}^\alpha, \bar{u}_{(1)},\ldots,\bar{u}_{(\pi)}\big) = 0.$$

Theorem 0.17. *Given the infinitesimal transformations in (8), the Lie group* G *in (4) is found by integrating the Lie equations*

(29)

$$\frac{d\bar{x}^i}{d\epsilon} = \xi^i(\bar{x}^i,\bar{u}^\alpha), \quad \bar{x}^i\Big|_{\epsilon=0} = x^i, \quad \frac{d\bar{u}^\alpha}{d\epsilon} = \eta^\alpha(\bar{x}^i,\bar{u}^\alpha), \quad \bar{u}^\alpha\Big|_{\epsilon=0} = u^\alpha.$$

Lie algebras.

Definition 0.18. A vector space V_r of operators [5] X (12) is a Lie algebra if for any two operators, $X_i,X_j \in V_r$, their commutator

(30) $[X_i,X_j] = X_iX_j - X_jX_i$, is in V_r for all $i,j = 1,\ldots,r$.

Remark 0.19. The commutator satisfies the properties of bilinearity, skew symmetry and the Jacobi identity [5].

Theorem 0.20. *The set of solutions of the determining equation given by (27) forms a Lie algebra*[5].

Exact solutions. The methods of (G’/G)-expansion method [21], Extended Jacobi elliptic function expansion [22] and Kudryashov [23] are usually applied after symmetry reductions.

Conservation laws. [10]

Fundamental operators. Let a system of πth-order PDEs be given by (5).

Definition 0.21. The Euler-Lagrange operator $\delta/\delta u^\alpha$ is

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\kappa\geq 1}(-1)^\kappa D_{i_1},\ldots,D_{i_\kappa}\frac{\partial}{\partial u^\alpha_{i_1 i_2\ldots i_\kappa}}, \tag{31}$$

and the Lie- Ba¨cklund operator in abbreviated form [5] is

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \ldots. \tag{32}$$

Remark 0.22. The Lie- Ba¨cklund operator (32) in its prolonged form is

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \sum_{\kappa\geq 1}\zeta_{i_1\ldots i_\kappa}\frac{\partial}{\partial u^\alpha_{i_1 i_2\ldots i_\kappa}}, \tag{33}$$

where (34) $\zeta i\alpha = Di(W\alpha) + \xi ju\alpha ij, \ldots, \zeta i\alpha 1\ldots i\kappa = Di1\ldots i\kappa(W\alpha) + \xi juji\alpha 1\ldots i\kappa, j = 1,\ldots,n$. and the Lie characteristic function is

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{35}$$

Remark 0.23. The characteristic form of Lie- Ba¨cklund operator (33) is

$$X = \xi^i D_i + W^\alpha\frac{\partial}{\partial u^\alpha} + D_{i_1\ldots i_\kappa}(W^\alpha)\frac{\partial}{\partial u^\alpha_{i_1 i_2\ldots i_\kappa}}. \tag{36}$$

Remark 0.24. Noether's Theorem is applicable to systems from variational problems

The method of multipliers.

Definition 0.25. A function $\Lambda^\alpha\ \ x^i, u^\alpha, u_{(1)},\ldots) = \Lambda^\alpha$, is a multiplier of the PDE system given by (5) if it satisfies the condition that [17]

$$\Lambda^\alpha\Delta_\alpha = D_iT^i, \text{ where } D_iT^i \text{ is a divergence expression.} \tag{37}$$

Definition 0.26. To find the multipliers Λ^α, one solves the determining equations (38) [3] ,

$$\frac{\delta}{\delta u^\alpha}(\Lambda^\alpha\Delta_\alpha) = 0. \tag{38}$$

Ibragimov's conservation theorem . The technique [10] enables one to construct conserved vectors associated with each Lie point symmetry of the PDE system given by (5).

Definition 0.27. The adjoint equations of the system given by (5) are

$$\Delta^*_\alpha\ \ x^i, u^\alpha, v^\alpha,\ldots,u_{(\pi)}, v_{(\pi)}) \equiv \frac{\delta}{\delta u^\alpha}(v^\beta\Delta_\beta) = 0, \tag{39}$$

where v^α is the new dependent variable.

Definition 0.28. Formal Lagrangian L of the system (5) and its adjoint equations (39) is [10]

$$\mathrm{L} = v^\alpha\Delta_\alpha(x^i,u^\alpha,u_{(1)},\ldots,u_{(\pi)}). \tag{40}$$

Theorem 0.29. *Every infinitesimal symmetry Xof the system given by (5) leads to conservation laws* [10]

$$D_iT^i\Big|_{\Delta_\alpha=0} = 0, \tag{41}$$

where the conserved vector

(42)

$$T^i = \xi^i\mathcal{L} + W^\alpha\left[\frac{\partial\mathcal{L}}{\partial u^\alpha_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial u^\alpha_{ij}}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u^\alpha_{ijk}}\right) - \ldots\right] + D_j(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u^\alpha_{ij}} - D_k\left(\frac{\partial\mathcal{L}}{\partial u^\alpha_{ijk}}\right) + \ldots\right] + D_jD_k(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u^\alpha_{ijk}} - \ldots\right].$$

Main results

An illustrative example with a the classical kdV equation can be found in [6]. We now present our results in this section.

Nonlinear Coupled Korteweg-de Vries (KdV) Equations.

Lie point symmetries and solutions of the nonlinear coupled KdV Equations (3). The infinitesimal transformations of the Lie group with parameter ϵ are

(43) $\bar{t} = t + \xi^t(t,x,u,v)\epsilon,\ \bar{x} = x + \xi^x(t,x,u,v)\epsilon,\ \bar{u} = u + \eta^u(t,x,u,v)\epsilon,\ \bar{v} = v + \eta^v(t,x,u,v)\epsilon$. The vector field

(44)

$$X = \xi^t(t,x,u,v)\frac{\partial}{\partial t} + \xi^x(t,x,u,v)\frac{\partial}{\partial x} + \eta^u(t,x,u,v)\frac{\partial}{\partial u} + \eta^v(t,x,u,v)\frac{\partial}{\partial v},$$

is a Lie point symmetry of (3) if

$$\mathrm{X}^{[3]}\Delta_1\Big|_{\Delta_1=0,\ \Delta_2=0} = 0, \mathrm{X}^{[3]}\Delta_2\Big|_{\Delta_1=0,\ \Delta_2=0} = 0. \tag{45}$$

Expanding (45) and and splitting on derivatives of v and u, we have an overdetermined system of ten PDEs, namely,

(46)
$$\xi^t_u = 0,\ \xi^t_v = 0,\ \xi^t_x = 0,\ \xi^x_u = 0,\ \xi^x_v = 0,\ \xi^t_{tt} = 0,\ \xi^x_{tt} = 0,\ 3\xi^x_x - \xi^t_t = 0,$$
$$3\eta^v + 2\xi^t_t v = 0,\ 3\alpha\eta^u + 2\alpha\xi^t_t u - 3\xi^x_t = 0.$$

Solving the system (46) yields

(47)
$$\xi^t = A_1 + 3A_2 t,\quad \xi^x = A2x + \alpha A_3 t + A_4,\ \eta^u = -2A_2 u + A_3,\ \eta^v = -2A_2 v,$$

for arbitrary constants A_1,A_2,A_3,A_4. Hence from (47), the infinitesimal symmetries of the coupled KdV Equations (3) is a Lie algebra generated by the vector fields

(48)
$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \alpha t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\ X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}.$$

0.0.1. *Commutator table.* The set of all infinitesimal symmetries of coupled KdV equations forms a Lie algebra and yield the following commutation relations in Table 0.0.1.

$[X_i,X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	αX_2	$3X_1$
X_2	0	0	0	X_2
X_3	$-\alpha X_2$	0	0	$-2X_3$
X_4	$-3X_1$	$-X_2$	$2X_3$	0

A commutator table for the Lie algebra generated by the symmetries of coupled KdV equation.

0.0.2. *Local Lie groups.* The following Lie groups, for i = 1,2,3,4, are obtained

$$(49)\qquad T_{\epsilon_1} : \bar{t} = t + \epsilon_1,\ \bar{x} = x,\ \bar{u} = u,\ \bar{v} = v,$$
$$(50)\qquad T_{\epsilon_2} : \bar{t} = t,\ \bar{x} = x + \epsilon_2,\ \bar{u} = u,\ \bar{v} = v,$$
$$(51)\qquad T_{\epsilon_3} : \bar{t} = t,\ \bar{x} = x + \alpha\epsilon_3 t,\ \bar{u} = u + \epsilon_3,\ \bar{v} = v,$$
$$(52)\qquad T_{\epsilon_4} : \bar{t} = te^{3\epsilon_4},\ \bar{x} = xe^{\epsilon_4},\ \bar{u} = ue^{-2\epsilon_4},\ \bar{v} = ve^{-2\epsilon_4}.$$

Conservation laws of the coupled KdV Equations (3). Construction of conserved vectors for the coupled KdV Equations (3) is done using two methods; the method of multipliers and a theorem due to Ibragimov.

Conservation laws of (3) using the multipliers. We look for local conservation law multipliers for the system (3), whose determining equations are given by

$$(53)\quad \frac{\delta}{\delta u}\left[\Lambda^1\Delta_1 + \Lambda^2\Delta_2\right] = 0,\quad \frac{\delta}{\delta v}\left[\Lambda^1\Delta_1 + \Lambda^2\Delta_2\right] = 0,$$

where

$$(54)\quad \frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} - D_x^3\frac{\partial}{\partial u_{xxx}} + \ldots,$$
$$(55)\quad \frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} + D_x^2\frac{\partial}{\partial v_{xx}} - D_x^3\frac{\partial}{\partial v_{xxx}} + \cdots,$$

are the Euler-Lagrange operators and

(56)
$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + \cdots,$$

(57)
$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{tx}\frac{\partial}{\partial u_t} + v_{tx}\frac{\partial}{\partial v_t} + \cdots,$$

are total derivatives operators. If we consider second order multipliers

$$(58)\qquad \Lambda^n = \Lambda^n(t,x,u,u_x,u_{xx},v,v_x,v_{xx}),\qquad n = 1,2,$$

then the determining Equations (53) become

(59)

$$\frac{\delta}{\delta u}\left[\Lambda^1\{u_t+\alpha uu_x-\alpha vv_x+\beta\ u_{xxx}\}+\Lambda^2\{v_t+\alpha uv_x+\alpha vu_x+\beta\ v_{xxx}\}\right]=0,$$

(60)

$$\frac{\delta}{\delta v}\left[\Lambda^1\{u_t+\alpha uu_x-\alpha vv_x+\beta\ u_{xxx}\}+\Lambda^2\{v_t+\alpha uv_x+\alpha vu_x+\beta\ v_{xxx}\}\right]=0.$$

Expanding (59)-(60) and splitting on derivatives of u and v yields an over-determined system of 22 PDEs, namely

(61)

$$\Lambda^1_{xx}=0,\ \Lambda^2_{xx}=0\ \ \Lambda^1_{vx}=0,\ \ \Lambda^2_{vx}=0,\ \ \Lambda^1_{xv_{xx}}=0,\ \ \Lambda^2_{xv_{xx}}=0,\beta\ \ \Lambda^1_{vv}-\alpha\Lambda^2_{v_{xx}}=0,$$
$$\beta\ \ \Lambda^2_{vv}+\alpha\Lambda^1_{vv_{xx}}=0,\ \ \Lambda^1_{vv_{xx}}=0,\ \ \Lambda^2_{vv_{xx}}=0,\ \ \Lambda^1_{v_{xx}v_{xx}}=0,\ \ \Lambda^2_{v_{xx}v_{xx}}=0,\ \ \Lambda^1_u+\Lambda^2_v=0,$$
$$\Lambda^1_t+\alpha\ \ \Lambda^2_x v+\Lambda^1_x u)=0,\ \ \Lambda^2_t+\alpha\ \ \Lambda^2_x u-\Lambda^1_x v)=0,\ \ \Lambda^2_u-\Lambda^1_v=0,\ \ \Lambda^1_{u_x}=0,\ \ \Lambda^2_{u_x}=0,$$
$$\Lambda^1_{u_{xx}}+\Lambda^2_{v_{xx}}=0,\ \ \Lambda^2_{u_{xx}}-\Lambda^1_{v_{xx}}=0,\ \ \Lambda^2_{v_x}=0\ \ \Lambda^1_{v_x}=0.$$

Calculations reveal the solution of the system (61) as

(62)

$$\Lambda^1=\frac{\alpha}{\beta 2}\ \ c_3\{u^2-v^2\}+2c_4uv)+(c_2t+c_5)u+(c_1t+c_6)v+c_3u_{xx}+c_4v_{xx}+c_7-\frac{1}{\alpha}c_2x,$$
$$\Lambda^2=\frac{\alpha}{\beta 2}\ \ c_4\{u^2-v^2\}-2c_3uv+)+(c_1t+c_6)u-(c_2t+c_5)v+c_4u_{xx}-c_3v_{xx}+c_8-\frac{1}{\alpha}c_1x,$$

for arbitrary constants $c_1,\dots,c_8$.

Remark 0.30. Essentially, the nonlinear coupled system of KdV Equations (3) has eight sets of local conservation law multipliers.

Solving (53),, we obtain conserved quantitites corresponding to each set of multipliers as shown below. (i) The multiplier

(63) $$\Lambda^1_1,\Lambda^2_1)=\left(tv,tu-\frac{x}{\alpha}\right),$$

has the conserved vectors

(64)

$$T^t_1=tuv-\frac{xv}{\alpha},\ \ T^x_1=\beta\ \ \left[t\{vu_{xx}+uv_{xx}-v_xu_x\}+\frac{1}{\alpha}\{v_x-xv_{xx}\}\right]+\alpha\left[t\left(u^2v-\frac{v^3}{3}\right)\right]$$

(65) $$-xuv.$$

(ii) The multiplier

(66) $$\Lambda^1_2,\Lambda^2_2)=\left(tu-\frac{x}{\alpha},-tv\right),$$

has the conserved vectors

(67)

$$T^t_2=\frac{t}{2}\{u^2-v^2\}-\frac{xu}{\alpha},\ \ T^x_2=\beta\ \ \left[t\left(uu_{xx}-vv_{xx}+\frac{1}{2}\{v_x^2-u_x^2\}\right)+\frac{1}{\alpha}\{u_x-xu_{xx}\}\right]+$$
$$\alpha t\left[\frac{u^3}{3}-uv^2\right]+\frac{x}{2}\{v^2-u^2\}.$$

(iii) The multiplier

(68) $$\Lambda^1_3,\Lambda^2_3)=\left(\frac{\alpha}{\beta 2}\{u^2-v^2\}+u_{xx},-\{\frac{\alpha uv}{\beta}+v_{xx}\}\right),$$

has the conserved vectors

(69)

$$T^t_3=\frac{\alpha}{\beta 2}\left(\frac{u^3}{3}-uv^2\right),\ \ T^x_3=\frac{\alpha}{2}\left[(u^2-v^2)u_{xx}-v^2v_{xx}\right]-\alpha uvv_{xx}+$$

(70) $$\frac{\beta}{2}\left[u_{xx}^2-v_{xx}^2\right]+u_tu_x-v_tv_x+\frac{\alpha^2}{\beta 4}\left[\frac{1}{2}\{u^4+v^4\}-3u^2v^2\right].$$

(iv) The multiplier

(71) $$\Lambda^1_4,\Lambda^2_4)=\left(\{\frac{\alpha uv}{\beta}+v_{xx}\},\frac{\alpha[u^2-v^2]}{\beta 2}+u_{xx}\right),$$

has the conserved vectors

(72)

$$T_4^t = \frac{\alpha}{2\beta}\left(u^2 v - \frac{v^3}{3}\right),$$

(73)

$$T_4^x = \frac{\alpha^2}{2\beta}\left[(u^3 v - uv^3)\right] + v_t u_x + u_t v_x + \frac{\alpha}{2}(u^2 - v^2)v_{xx} + \{\alpha uv + \beta\ v_{xx}\}u_{xx}.$$

(v) The multiplier

(74) $\Lambda_5^1, \Lambda_5^2) = (u, -v)$, has the conserved vectors

(75)

$$T_5^t = \frac{1}{2}\{u^2 - v^2\}, \quad T_5^x = \beta\ \left(uu_{xx} - vv_{xx} + \frac{v_x^2 - u_x^2}{2}\right) + \alpha\left(\frac{u^3}{3} - uv^2\right).$$

(vi) The multiplier

(76) $\Lambda_6^1, \Lambda_6^2) = (v, u)$, has the conserved vectors

(77) $$T_6^t = uv, \quad T_6^x = \beta\ (vu_{xx} + uv_{xx} - u_x v_x) + \alpha\left(u^2 v - \frac{v^3}{3}\right).$$

(vii) The multiplier

(78) $\Lambda_7^1, \Lambda_7^2) = (1, 0)$, has the conserved vectors

(79) $$T_7^t = u, \quad T_7^x = \frac{\alpha}{2}\{u^2 - v^2\} + \beta\ u_{xx}.$$

(viii) The multiplier has

(80) $$\Lambda_8^1, \Lambda_8^2) = (0, 1),$$

the conserved vectors

(81) $$T_8^t = v, \quad T_8^x = \alpha uv + \beta\ v_{xx}.$$

Remark 0.31. It can be shown that

(82) $$D_t T_i^t + D_x T_i^x\Big|_{\Delta_1=0,\ \Delta_2=0} = 0,$$

for $i = 1,...,8$.

Remark 0.32. The expressions in (82) are eight conservation laws for the coupled KdV system (3).

Remark 0.33. The presence of multipliers

(83) $$\Lambda_7^1, \Lambda_7^2) = (1, 0), \quad \Lambda_8^1, \Lambda_8^2) = (0, 1)$$

manifest that the coupled KdV equations are themselves conservation laws.

Conservation laws of (3) using Ibragimov's theorem. In this section, we derive conserved vectors for coupled KdV equations (3) by a new theorem due to Ibragimov. The adjoint equations for the nonlinear system coupled KdV Equations (3) are

(84)

$$\Delta_1^* \equiv f_t + \alpha\ uf_x + \alpha v g_x + \beta f_{xxx} = 0, \quad \Delta_2^* \equiv g_t - \alpha v f_x + \alpha u g_x + \beta g_{xxx} = 0.$$

The formal Lagrangian L for the nonlinear coupled system of the KdV Equations (3) and its adjoint Equations (84) is given by

(85) $$\mathcal{L} = f\{u_t + \alpha u u_x - \alpha v v_x + \beta u_{xxx}\} + g\{v_t + \alpha u v_x + \alpha v u_x + \beta v_{xxx}\},$$

where f and g are new variables. We shall use the Lie point symmetries of the system (3) ,namely

(86)

$$X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = \alpha t\partial_x + \partial_u,\ X_4 = 3t\partial_t + x\partial_x - 2u\partial_u - 2v\partial_v,$$

to derive conserved vectors corresponding to each symmetry below.

Case (i) The symmetry $X_1 = \frac{\partial}{\partial t}$, yields Lie characteristic functions given by

(87) $$W_1^1 = -u_t, \quad W_1^2 = -v_t.$$

Hence by Ibragimov's theorem [10], the associated conserved vector is given by

$$\begin{aligned} T_1^t &= \alpha\left[f\{uu_x - vv_x\} + g\{vu_x + uv_x\}\right] + \beta\{fu_{xxx} + gv_{xxx}\}, \\ T_1^x &= \alpha\left[f\{-uu_t + vv_t\} - g\{vu_t + uv_t\}\right] \\ &\quad + \beta\{f_x u_{tx} + g_x v_{tx} - u_t f_{xx} - v_t g_{xx} - fu_{txx} - gv_{txx}\}. \end{aligned} \quad (88)$$

Case (ii) The symmetry $X_2 = \frac{\partial}{\partial x}$, yields Lie characteristic functions

$$(89) \qquad W_2^1 = -u_x, \quad W_2^2 = -v_x.$$

Therefore by Ibragimov's theorem [10], the associated conserved vector is

$$(90)$$
$$T_2^t = -u_x f - v_x g, \quad T_2^x = f u_t + g v_t + \beta\{-u_x f_{xx} - v_x g_{xx} + f_x u_{xx} + g_x v_{xx}\}.$$

Case (iii) The symmetry

$$(91) \qquad X_3 = \alpha t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}$$

yields Lie characteristic functions given by

$$(92) \qquad W_3^1 = 1 - \alpha t u_x, \quad W_3^2 = -\alpha t v_x.$$

Hence by Ibragimov's theorem [10], the associated conserved vector is given by

(93)

$$T_3^t = f - \alpha t\{u_x f + v_x g\} \quad ,$$

$$T_3^x = \alpha \left[fu + gv + t\{u_t f + v_t g\} + \beta \;\; t\{\frac{f_{xx}}{\alpha t} - u_x f_{xx} - v_x g_{xx} + f_x u_{xx} + g_x v_{xx}\} \right].$$

Case (iv) The symmetry

$$(94) \qquad X_4 = 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} - 2v \frac{\partial}{\partial v}$$

yields the Lie characteristic functions

$$(95) \qquad W_4^1 = -2u - 3t u_t - x u_x, \quad W_4^2 = -2v - 3t v_t - x v_x.$$

Consequently by Ibragimov's theorem [10], the corresponding conserved vector is given by

$$(96)$$
$$\begin{aligned}
T_4^t &= \alpha \left[3t\{f u u_x - f v v_x + g u v_x + g v u_x\}\right] + \beta \;\; \left[3t\{f u_{xxx} + g v_{xxx}\}\right] \\
&\quad - 2\{fu + gv\} - x\{f u_x + g v_x\}, \\
T_4^x &= x\{f u_t + g v_t\} + \beta \;\; \left[3 \;\; f_x u_x + g_x v_x + t\{f_x u_{tx} + g_x v_{tx}\}) \right] \\
&\quad - \alpha \left[2 \;\; f\{u^2 - v^2\} + 2guv) + 3t \;\; f\{u u_t - v v_t\} + g\{v u_t + u v_t\}) \right] \\
&\quad + \beta \;\; \left[x\{u_x f_{xx} + v_x g_{xx} - f_x u_{xx} - g_x v_{xx}\} + 2\{u f_{xx} + v g_{xx}\}\right] \\
&\quad + \beta \;\; \left[3t\{f_{xx} u_t + g_{xx} v_t + f u_{txx} + g v_{txx}\} + 4\{f u_{xx} + g v_{xx}\}\right].
\end{aligned}$$

Remark 0.34. The appearance of arbitrary functions $f(t,x)$ and $g(t,x)$ in the conserved quantities proves the existence of infinite conservation laws for coupled KdV system obtained by Ibagimov's method.

CONCLUSION

In this paper, Lie group analysis was employed in studying a nonlinear coupled kdV system. We used multiplier approach to compute conserved quantities for a nonlinear coupled kdv equations. A fourdimensional Lie algebra of symmetries was found for the nonlinear coupled system of KdV equations. This was spanned by space and time translations, Galilean boost and scaling symmetries where the scaling symmetry acts on four variables. Lastly, associated to each symmetry, we employed Ibragimov's theorem in the construction of infinitely many conserved quantities. From this work, one can see that mass, momentum and energy are conserved quantities in the evolution of a nonlinear coupled KdV system. In fact, only some of the first laws have a physical interpretation. Higher-order laws aid in understanding the qualitative properties of solutions. These conservation laws are very important in explaining the integrability of a system and the effectiveness of numerical methods used in approximating solutions. The above results show a very interesting property of the KdV equation. Most important to note is that the infinite number of conservation laws for the coupled system show that the KdV equation is completely integrable, meaning that the behavior of the system can be determined by initial conditions and can be integrated from the prescribed initial conditions. Indeed, the KdV equation gives rise to multiple-soliton solutions thus emphasizing the importance of the KdV equation in the theory of integrable systems. The beautiful KdV equation is ubiquitous, having applications in various settings. In future, the obtained conservation laws will be used to construct exact solutions to the nonlinear coupled system of Korteweg-de Vries equations.

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Author’s contribution

The author contributed wholly in writing this article and declares no conflict of interest.

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