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Numerical Solution of Fuzzy Differential Equation by Fifth Order Runge-Kutta-Fehlberg Method

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Abstract

In this paper numerical solutions for solving Fuzzy ordinary differential equations based on Seikala derivative of fuzzy process are considered. We propose a novel numerical method based on the Runge-Kutta-Fehlberg method of order five and is followed by a complete error analysis.

Keywords

Fuzzy Differential Equations-Fifth Order fuzzy Runge-Kutta-Fehlberg Method-Error Analysis

1. Introduction

Consider the numerical solution of the initial value problem for the system of ordinary differential equation.

$$y'(x) = f(x, y(x)), \quad x \in [x_0, b],$$
$$y(x_0) = y_0$$

One of the most common methods for solving numerically [3] is Runge-Kutta method. Most efforts to increase the order of accuracy of the Runge-Kutta method have been accomplished by increasing the number of terms used in the Taylor's series and thus the number of functional evaluations. In Runge-Kutta-Fehlberg method of order, six number of function evaluations is required per step. Many authors have attempted to increase the efficiency of Runge-Kutta methods with a lower number of function evaluations required.

2. Preliminaries

Consider the initial value problem

$$y'(t) = \begin{cases} f(t, y(t)), & t_0 \le t \le b \\ y(t_0) = y_0 \end{cases} \qquad (2.1)$$

We assume that

1. $f(t, y(t))$ is defined and continuous in the strip with $t_0 \le t \le b$, $-\infty < y < \infty$ with t_0 and b are finite.
2. There exists a constant L such that for any t in $[t_0, b]$ and any two numbers y and y^*
   $$\left| f(t, y) - f(t, y^*) \right| \le L \left| y - y^* \right|$$

These conditions are sufficient to prove that ∃ on $[t_0, b]$ a unique continuous differentiable solution y(t) satisfying (2.1) which is continuous and differentiable.

The basis of all Runge-Kutta methods is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=0}^{m} w_i k_i \qquad (2.2)$$

where w_i are constants, i = 1, 2, 3, . . . m

$$k_i = hf(t_n+a_ih,\ y_n + \sum_{j=1}^{i-1} c_{ij}k_j) \qquad (2.3)$$

Equation (2.2) is to be exact for powers of h through h^m, because it is to be coincident with Taylor series of order m. Therefore, the truncation error T_m, can be written as

$$T_m = \gamma_m h^{m+1} + O(h^{m+2})$$

The true magnitude of γ_m will generally be much less than the bound of theorem-2.1. Thus, if the $O(h^{m+2})$ term is small compared with $\gamma_m h^{m+1}$, as we expect to be so if h is small, then the bound on $\gamma_m h^{m+1}$, will usually be a bound on the error as a whole.

The method proposed in [7] introduces new terms involving higher order derivatives of 'f' in the Runge-Kutta k_i terms (i > 1) to obtain a higher order of accuracy without a corresponding increase in evaluations of f, but with the addition of evaluations of 'f' by Fifth order Runge-Kutta-Fehlberg method for autonomous system proposed in [15]. Consider,

$$y(t_{n+1}) = y(t_n) + w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4 + w_5k_5 + w_6k_6$$

where

$$\begin{aligned}
k_1 &= hf(t_n, y(t_n))\\
k_2 &= hf(t_n+c_2h, y(t_n)+a_{21}k_1)\\
k_3 &= hf(t_n+c_3h, y(t_n)+a_{31}k_1+a_{32}k_2)\\
k_4 &= hf(t_n+c_4h, y(t_n)+a_{41}k_1+a_{42}k_2+a_{43}k_3) \qquad (2.4)\\
k_5 &= hf(t_n+c_5h, y(t_n)+a_{51}k_1+a_{52}k_2+a_{53}k_3+a_{54}k_4)\\
k_6 &= hf(t_n+c_6h, y(t_n)+a_{61}k_1+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)
\end{aligned}$$

Utilizing the Taylor's series expansion techniques, Runge-Kutta-Fehlberg method of order fifth is given by,

$$y_{n+1} = y_n + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{1}{55}k_6$$

$$\begin{aligned}
k_1 &= hf(t_n, y(t_n))\\
k_2 &= hf(t_n+\frac{h}{3}, y(t_n)+\frac{1}{4}k_1)\\
k_3 &= hf(t_n+\frac{3h}{8}, y(t_n)+\frac{3}{32}k_1+\frac{9}{32}k_2)\\
k_4 &= hf(t_n+\frac{12h}{13}, y(t_n)+\frac{1932}{2197}k_1-\frac{7200}{2197}k_2+\frac{7296}{2197}k_3) \qquad (2.5)\\
k_5 &= hf(t_n+h, y(t_n)+\frac{439}{216}k_1-8k_2+\frac{3680}{513}k_3-\frac{845}{4104}k_4)\\
k_6 &= hf(t_n+\frac{h}{2}, y(t_n)-\frac{8}{27}k_1+2k_2-\frac{3544}{2565}k_3+\frac{1859}{4104}k_4-\frac{11}{40}k_5)
\end{aligned}$$

where

$a = t_0 \le t_1 \le t_2 \le . . . \le t_N = b$ and

$$h = \frac{b-a}{N} = t_{n+1} - t_n \qquad (2.6)$$

Theorem – 2.1

Let f(t,y) belong to $C^4[a,b]$ and let it's partial derivatives are bounded and assume there exists, P, Q positive numbers such that

$|f(t,y)| < P$, $\left|\frac{\partial^{i+j} f}{\partial t^i \partial y^j}\right| < \frac{P^{i+j}}{Q^{j-1}}$, $i + j \le m$, then in the Runge-Kutta method of order five,

$$y_{n+1} - y_n \approx \frac{11987}{12960} h^6 QP^5 + O(h^7)$$

Definition – 2.1

A fuzzy number u as a fuzzy subset of R ie) u : R → [0, 1] satisfying the following conditions.

i). u is normal, ie $\exists\, x_0 \in R \ni u(x_0) = 1$
ii). u is a convex fuzzy set ie) $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall\, t \in [0, 1]$ and $x, y \in R$
iii). u is upper semi continuous on R
iv). $\overline{\{x \in R, u(x) > 0\}}$ is compact

The set E is the family of fuzzy numbers and arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$ that satisfies the following requirements

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0, 1] w.r.to any 'r'.
2. $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0, 1] w.r.to any 'r'.
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$, r-level cut is $[u]_r = \{x/u(x) \geq r\}$, $0 \leq r \leq 1$ is a closed & bounded interval denoted by $[u]_r = [\underline{u}(r), \overline{u}(r)]$ and $[u]_0 = \{x/u(x) > 0\}$ is compact.

Definition – 2.2

A triangular fuzzy number u is a fuzzy set in E that is characterised by an ordered triple $(u_l, u_c, u_r) \in R^3$ with $u_l < u_c < u_r$ such that $[u]_0 = [u_l : u_r]$ and $[u]_1 = [u_c]$. The membership function of the triangular fuzzy number u is given by

$$u(x) = \begin{cases} \dfrac{x - u_l}{u_c - u_l}, & u_l \leq x \leq u_c \\ 1 & x = u_c \\ \dfrac{u_r - x}{u_r - u_c}, & u_c \leq x \leq u_r \end{cases}$$

and we will have

$u > 0$ if $u_l > 0$; $u \geq 0$ if $u_l \geq 0$; $u < 0$ if $u_r < 0$; $u \leq 0$ if $u_r \leq 0$

Let I be a real interval. A mapping y : I → E is called a fuzzy process and its α - level set is denoted by $[y(t)]_\alpha = [\underline{y}(t, y), \overline{y}(t, y)]$, $t \in I$, $0 < \alpha \leq I$.

The Seikkala derivative y(t) of a fuzzy process is defined by $[y^1(t)]_\alpha = [\underline{y}^1(t, y), \overline{y}^1(t, y)]$, $t \in I$, $0 < \alpha \leq I$ provided the equation defines fuzzy number as in [11].

For u, v ∈ E and $\lambda \in \Re$, the addition u + v and the product λu can be defined by

$$[u + v]_\alpha = [u]_\alpha + [v]_\alpha$$
$$[\lambda u]_\alpha = \lambda [u]_\alpha$$

where $\alpha \in [0, 1]$ and $[u]_\alpha + [v]_\alpha$ means the addition of two intervals of $\Re$ and $[u]_\alpha$ means the product between a scalar and a subset of $\Re$. Arithmetic operation of arbitrary fuzzy numbers

$u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$ and $\lambda \in \Re$ can be defined as

i). u = v if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$

ii). $u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$

iii). $u - v = (\underline{u}(r) - \overline{u}(r), \overline{u}(r) - \overline{v}(r))$

iv). $u \,.\, v = \begin{matrix} \min\{(\underline{u}(r).\underline{v}(r), \underline{u}(r).\overline{v}(r), \overline{u}(r).\underline{v}(r), \overline{u}(r).\overline{v}(r))\}, \\ \max\{(\underline{u}(r).\underline{v}(r), \underline{u}(r).\overline{v}(r), \overline{u}(r).\underline{v}(r), \overline{u}(r).\overline{v}(r))\} \end{matrix}$

v). $\lambda u = (\lambda \underline{u}(r), \lambda \overline{u}(r))$ if $\lambda \geq 0$

$= (\lambda \overline{u}(r), \lambda \underline{u}(r))$ if $\lambda < 0$

3. A Fuzzy Cauchy Problem

Consider the fuzzy initial value problem

$$y'(t) = \begin{cases} f(t, y(t)), & 0 \leq t \leq T \\ y(0) = y_0 \end{cases} \qquad (3.1)$$

where f is a continuous mapping from $R_+ \times R \rightarrow R$ and $y_0 \in E$ with r-level sets $[y_0]_r = \left[\underline{y}(0;r), \overline{y}(0;r)\right]$, $r \in [0, 1]$.

The extension principle of Zadeh leads to the following definition of f(t, y) when y = y(t) is a fuzzy number.

$f(t,y)(s)=\sup\{y(\tau)\backslash s=f(t, \tau)\}, s \in R$

It follows that

$$[f(t,y)]_r = \left[\underline{f}(t,y;r), \overline{f}(t,y;r)\right], \; r \in [0, 1]$$

where

$$\underline{f}(t,y;r) = \min\left\{f(t,u) \backslash u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\}$$

$$\overline{f}(t,y;r) = \max\left\{f(t,u) \backslash u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\} \qquad (3.2)$$

Theorem:

Let f satisfy $\left|f(t,v) - f(t,\bar{v})\right| \le g\left(t, \left|v - \bar{v}\right|\right)$, $t \ge 0$ and v, $\bar{v} \in R$, where $g : R_+ \times R_+ \rightarrow R_+$ is a continuous mapping such that $r \rightarrow g(t, r)$ is non-decreasing. An initial value problem

$$u'(t) = g(t, u(t)), \; u(0) = u_0, \qquad (3.3)$$

has a solution on R_+ for $u_0 > 0$ and that u(t) = 0 is the only solution of (3.3) for $u_0 = 0$. Then the fuzzy initial value problem (3.1) has a unique fuzzy solution.

$u^1(t) = g(t, u(t)), u(0) = u_0$

4. Fifth Order fuzzy Runge–Kutta-Fehlberg Method

Let the exact solution of the given differential equation $[Y(t)]_r = \left[\underline{Y}(t;r), \overline{Y}(t;r)\right]$ is approximated by some $[y(t)]_r = \left[\underline{y}(t;r), \overline{y}(t;r)\right]$. From (2.2) and (2.3) we define

$$\underline{y}(t_{n+1}:r) - \underline{y}(t_n:r) = \sum_{i=1}^{6} w_i \underline{k_i}$$

$$\overline{y}(t_{n+1}:r) - \overline{y}(t_n:r) = \sum_{i=1}^{6} w_i \overline{k_i} \qquad \ldots (4.1)$$

where w_i's are constants and

$[k_i(t, y(t,r))]_r = \left[\underline{k_i}(t, y(t,r)), \overline{k_i}(t, y(t,r))\right]$ where i = 1, 2, 3, 4, 5 and 6

$$\underline{k_1}(t, y(t:r)) = hf\left(t_n, \underline{y}(t_n:r)\right)$$

$$\overline{k_1}(t, y(t:r)) = hf\left(t_n, \overline{y}(t_n:r)\right)$$

$$\underline{k_2}(t, y(t:r)) = hf\left(t_n + \frac{h}{4}, \underline{y}(t_n:r) + \frac{1}{4}\underline{k_1}\right)$$

$$\overline{k_2}(t, y(t:r)) = hf\left(t_n + \frac{h}{4}, \overline{y}(t_n:r) + \frac{1}{4}\overline{k_2}\right)$$

$$\underline{k_3}(t, y(t:r)) = hf\left(t_n + \frac{3h}{8}, \underline{y}(t_n:r) + \frac{3}{32}\left(\underline{k_1} + 3\underline{k_2}\right)\right)$$

$$\overline{k_3}(t, y(t:r)) = hf\left(t_n + \frac{3h}{8}, \overline{y}(t_n:r) + \frac{3}{32}\left(\overline{k_1} + 3\overline{k_2}\right)\right)$$

$$\underline{k_4}(t, y(t:r)) = hf\left(t_n + \frac{12h}{13}, \underline{y}(t_n:r) + \frac{1932}{2197}\underline{k_1} - \frac{7200}{2197}\underline{k_2} + \frac{7296}{2197}\underline{k_3}\right)$$

$$\overline{k_4}(t, y(t:r)) = hf\left(t_n + \frac{12h}{13}, \overline{y}(t_n : r) + \frac{1932}{2197}\overline{k_1} - \frac{7200}{2197}\overline{k_2} + \frac{7296}{2197}\overline{k_3}\right)$$

$$\underline{k_5}(t, y(t:r)) = hf\left(t_n + h, \underline{y}(t_n : r) + \frac{439}{216}\underline{k_1} - 8\underline{k_2} + \frac{3680}{513}\underline{k_3} - \frac{845}{4104}\underline{k_4}\right)$$

$$\overline{k_5}(t, y(t:r)) = hf\left(t_n + h, \overline{y}(t_n : r) + \frac{439}{216}\overline{k_1} - 8\overline{k_2} + \frac{3680}{513}\overline{k_3} - \frac{845}{4104}\overline{k_4}\right)$$

$$\underline{k_6}(t, y(t:r)) = hf\left(t_n + \frac{h}{2}, \underline{y}(t_n : r) - \frac{8}{27}\underline{k_1} + 2\underline{k_2} - \frac{3544}{2565}\underline{k_3} + \frac{1859}{4104}\underline{k_4} - \frac{11}{40}\underline{k_5}\right)$$

$$\overline{k_6}(t, y(t:r)) = hf\left(t_n + \frac{h}{2}, \overline{y}(t_n : r) - \frac{8}{27}\overline{k_1} + 2\overline{k_2} - \frac{3544}{2565}\overline{k_3} + \frac{1859}{4104}\overline{k_4} - \frac{11}{40}\underline{k_5}\right) \quad \ldots (4.2)$$

$$F(t, y(t:r)) = \frac{16\underline{k_1}(t, y(t:r))}{135} + \frac{6656\underline{k_3}(t, y(t:r))}{12825} + \frac{28561\underline{k_4}(t, y(t:r))}{56430} - \frac{9\underline{k_5}(t, y(t:r))}{50} + \frac{2\underline{k_6}(t, y(t:r))}{55} \quad \ldots (4.3)$$

$$G(t, y(t:r)) = \frac{16\overline{k_1}(t, y(t:r))}{135} + \frac{6656\overline{k_3}(t, y(t:r))}{12825} + \frac{28561\overline{k_4}(t, y(t:r))}{56430} - \frac{9\overline{k_5}(t, y(t:r))}{50} + \frac{2\overline{k_6}(t, y(t:r))}{55}$$

The exact and approximate solution at t_n, $0 \le n \le N$ are denoted by $[Y(t_n)]_r = [\underline{Y}(t_n;r), \overline{Y}(t_n;r)]$ and $[y(t_n)]_r = [\underline{y}(t_n;r), \overline{y}(t_n;r)]$ respectively. The solution calculated by grid points at (2.6). By (4.1) and (4.3) we have

$$\underline{y}(t_{n+1} : r) = \underline{y}(t_n : r) + F[t_n, \underline{y}(t_n : r)]$$
$$\overline{y}(t_{n+1} : r) = \overline{y}(t_n : r) + G[t_n, \overline{y}(t_n : r)] \quad \ldots (4.4)$$

The following lemmas will be applied to show the convergence of these approximates

$$\lim_{h\to 0} \underline{y}(t:r) = \underline{Y}(t:r) \text{ and } \lim_{h\to 0} \overline{y}(t:r) = \overline{Y}(t:r)$$

Lemma-1:

Let the sequence of numbers $\{W\}_{n=0}^{N}$ satisfy $|W_{n+1}| \le A|W_n| + B$, $0 \le n \le N-1$ for some given positive constants A and B. Then $|W_n| \le A^n |W_0| + B\frac{A^n - 1}{A - 1}$, $0 \le n \le N$.

Lemma-2:

Let the sequences of numbers $\{W\}_{n=0}^{N}$ and $\{V\}_{n=0}^{N}$ satisfy the condition

$|W_{n+1}| \le |W_n| + A \max\{|W_n|, |V_n|\} + B$ and

$|V_{n+1}| \le |V_n| + A \max\{|W_n|, |V_n|\} + B$

for some given positive constants A and B and denote $U_n = |W_n| + |V_n|$, $0 \le n \le N$.
Then,

$|U_n| \le \overline{A}^n |U_0| + \overline{B}^n \frac{\overline{A}^n - 1}{\overline{A} - 1}$, $0 \le n \le N$, where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$

Theorem-4.1

Let F(t, u, v) and G(t, u, v) belongs to $C^4(K)$ and let the partial derivatives of F and G be bounded over K. Then, for arbitrary fixed r, $0 \le r \le 1$, the approximately solutions $\underline{y}(t_n;r)$, $\overline{y}(t_n;r)$ are converges to the exact solutions of $\underline{Y}(t_n;r)$ and $\overline{Y}(t_n;r)$ uniformly in t.[9]

5. Numerical Example

Consider the fuzzy initial value problem,

$y^1(t) = y(t)$, $t \in [0, 1]$

with $y(0) = (0.85 + 0.15r, 1.1 - 0.10r)$ where $0 \le r \le 1$

Solution:

The exact solution is given by

$$\underline{Y}(t;r)=\underline{y}(0;r)e^{t} \text{ and } \overline{Y}(t:r)=\overline{y}(0;r)e^{t}$$

which is at t = 1,

$$y(1;r)=[(0.85+0.15r)\ e,\ (1.1-0.10r)\ e],\ 0 \le r \le 1.$$

The exact and approximate solutions obtained by the fuzzy Runge-Kutta-Fehlberg fifth order method and fuzzy Runge-Kutta forth order method with h=0.1 are compared and plotted at t = 1 in Figure-5.1

Table – 5.1

r	Exact Solution		Fifth order RKF method		Fourth order RK method	
	Lower	Upper	Lower	Upper	Lower	Upper
0.0	2.3105395542	2.9901100113	2.3105392456	2.9901101589	2.3126564026	2.9928538799
0.1	2.3513137816	2.9629271930	2.3513138294	2.9629275799	2.3534719944	2.9656441212
0.2	2.3920880090	2.9357443747	2.3920881748	2.9357445240	2.3942780495	2.9384415150
0.3	2.4328622365	2.9085615565	2.4328622818	2.9085612297	2.4350955486	2.9112331867
0.4	2.4736364639	2.8813787382	2.4736363888	2.8813791275	2.4759082794	2.8840217590
0.5	2.5144106913	2.8541959199	2.5144107342	2.8541955948	2.5167136192	2.8568160534
0.6	2.5551849188	2.8270131016	2.5551846027	2.8270127773	2.5575318336	2.8296048641
0.7	2.5959591462	2.7998302833	2.5959587097	2.7998304367	2.5983390808	2.8024001122
0.8	2.6367333736	2.7726474650	2.6367337704	2.7726473808	2.6391501427	2.7751901150
0.9	2.6775076010	2.7454646467	2.6775078773	2.7454645634	2.6799674034	2.7479805946
1.0	2.7182818285	2.7182818285	2.7182817459	2.7182817459	2.7207753658	2.7207753658

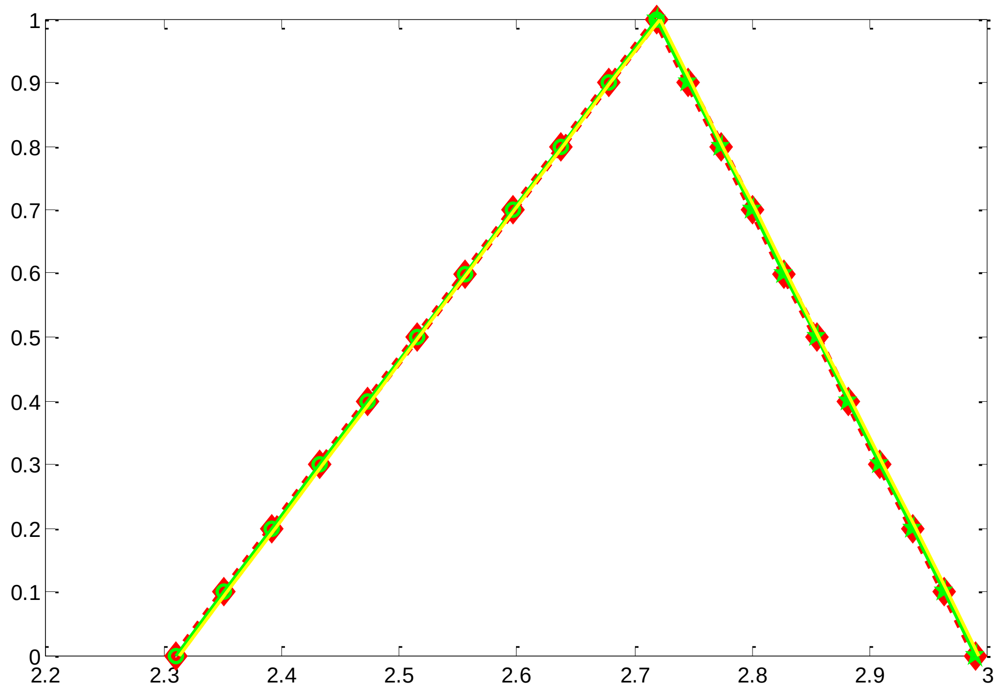
The error between the approximate solution by the method of fuzzy Runge-Kutta fourth order and the exact solution is computed. Also the error between the approximate solution by fuzzy Runge-Kutta-Fehlberg method and the exact solution of the problem are listed below.

Table – 5.2

r	Exact and Proposed Method		Exact and RK of order Four	
	Lower	Upper	Lower	Upper
0.0	0.0000003086	0.0000001476	0.0021168484	0.0027438686
0.1	0.0000000478	0.0000003869	0.0021582128	0.0027169282
0.2	0.0000001658	0.0000001493	0.0021900405	0.0026971403
0.3	0.0000000453	0.0000003268	0.0022333121	0.0026716302
0.4	0.0000000751	0.0000003893	0.0022718155	0.0026430208
0.5	0.0000000429	0.0000003251	0.0023029279	0.0026201335

0.6	0.0000003161	0.0000003243	0.0023469148	0.0025917625
0.7	0.0000004365	0.0000001534	0.0023799346	0.0025698289
0.8	0.0000003968	0.0000000842	0.0024167691	0.0025426500
0.9	0.0000002763	0.0000000833	0.0024598024	0.0025159479
1.0	0.0000000826	0.0000000826	0.0024935373	0.0024935373

Figure – 5.1



Green - Fuzzy Fifth order Runge-Kutta-Felhberg Method
Yellow - Fuzzy Fourth order Runge-Kutta Method
Red - Exact Solution

6. Conclusion

In this work, we have proposed fifth order fuzzy Runge-Kutta-Fehlberg method to find the numerical solutions of fuzzy differential equations. Taking into account that the convergence order of the Euler method is O(h), a higher order of convergence is obtained for the proposed method as $O(h^3)$. Comparison of the solutions of example shows that the proposed method gives a better solution than the fuzzy Runge-Kutta fourth order method.

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