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Mathematical Modelling of HIV/AIDS Transmission Dynamics with Optimal Control Strategy

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ABSTRACT

Human Immunodeficiency Virus is the causative agent of Acquired Immunodeficiency Syndrome. HIV can be transmitted to person through the exchange of a variety of body fluids from infected individuals, such as blood, breast milk, semen, and vaginal secretions. In this paper, mathematical model for HIV/AIDS transmission dynamics was formulated and analyze using the stability theory of differential equations.The basic reproduction number that represents the epidemic indicator is obtained by using next generation matrix.Both local and global stability of the disease free equilibrium and endemic equilibrium point of the model equation was established. The results show that, if the basic reproduction number is less than one then the solution converges to the disease free steady state and the disease free equilibrium is asymptotically stable. The endemic states are considered to exist when the basic reproduction number for each disease is greater than one. Sensitivity analysis of the model equation was performed on the key parameters in order to determine their impact on the disease transmission dynamics. The system was extended into an optimal control strategies by including time-dependent control variables: prevention of the recruitment to susceptible, reduction of spread of HIV, screen and treatment of infected individuals. Numerical simulations are performed and the pertinent results are presented graphically and discussed quantitatively.

KEYWORDS: Mathematical Model,Stability,Sensitivity,Optimal control, Numerical Simulation

1. INTRODUCTION

HIV/AIDS is one of the sexual transmitted diseases that have claimed and continue to claim the lives of millions of people worldwide. HIV is an RNA retrovirus which translates RNA to DNA with a viral enzyme called reverse transcriptase [1]. The target cell of HIV is CD4 T cells. A healthy human body has about $1000/mm^3$ of CD4 T cells. When the CD4 T cells of a patient decline to $200/mm^3$ or below, then that person is classified as having AIDS [2]. When the CD4 T cells decline, they cannot mount a strong response. This results in weak responses from CTL and antibodies which cannot clear the infection [3]. HIV is transmitted primarily through unprotected sexual intercourse with an infected individual, through exchange of infected blood or blood products, or to the newborn from an infected mother. However, antiretroviral (ART) treatment improves health, pro-longs life, and substantially reduces the risk of HIV transmission. More than 90% of adults in sub-Saharan Africa acquire HIV infection from unprotected sexual intercourse with infected partners [4].

According to the updated statistics on the state of AIDS epidemic by UNAIDS, 36.9 million people, globally,were living with HIV in 2017, of which 21.7 million individuals were accessing ART (antiretroviral therapy) treatment and 1.8 million became newly infected with HIV in 2017. A total of 77.3 million individuals have become infected with HIV since the start of the epidemic in 1981. Figures of death indicate that 940,000 people died of AIDS-related illnesses in 2017, with a total of 35.4 million people that have died from AIDS-related illnesses since the start of the epidemic[5].

Mathematical models have played a major role in increasing our understanding of of the dynamics of sexually transmitted diseases. Several models have been proposed to study the effects of some factors on the transmission dynamics of these sexually transmitted diseases including HIV/AIDS and to provide guidelines as to how the spread can be controlled. Among these models Anderson et al [6]

presented a simple mathematical HIV transmission model to investigate the effects of various factors on the overall pattern of the AIDS epidemic. Stilianakis et al [7] who proposed and gave a detailed analysis of a dynamical model that describes the pathogenesis of HIV, and Tripathi et al [8] who proposed a model to study the effects of screening of unaware infective on the transmission dynamics of HIV/AIDS. K.O. Okosun [9] presented the impact of optimal control on the treatment of HIV/AIDS and screening of unaware infective on the transmission dynamics of the disease in a homogeneous population with constant immigration of susceptible incorporating use of condom, screening of unaware infective and treatment of the infected. In [10] a mathematical model for HIV/AIDS transmission has been proposed, along with a control problem in which the objective was to determine the pre-exposure prophylaxis (PrEP) strategy that minimizes the number of individuals with pre-AIDS HIV infection, balanced against the costs associated with PrEP. The paper by Mukandavire et al [11] compares the impact of increasing condom use or HIV PrEP use among sex workers. The authors found that condom promotion interventions should remain the mainstay HIV prevention strategy for female sex workers (FSWs), with PrEP only being implemented once condom interventions have been maximized or to fill prevention gaps where condoms cannot be used. In [12], the authors develop a model of HIV risk and compare HIV-risk estimates before and after the introduction of PrEP to determine the maximum tolerated reductions in condom use with regular partners and clients for HIV risk not to change. With a case study of FSWs in South Africa, in [12] it is found that PrEP is likely to be of benefit in reducing HIV risk, even if reductions in condom use do occur.

So far, few mathematical studies have been undertaken to model Human Immunodeficiency Virus mathematically, but they did not considered protected compartment in their studies.

2. MODEL DESCRIPTION AND FORMULATION

The model divides the total population, denoted by $N(t)$ into six subclasses with respect to their disease status in the system. $P(t)$ is the class of individuals which are protected against the disease over a period of time. $S(t)$ is the class of individuals who are healthy but can contract the disease. $E(t)$ is the class of individuals which are infected but not yet infectious. $A_s(t)$ is the class of an infectious without symptoms of disease. $I(t)$ is the class of an infectious with symptoms of disease and $D(t)$ is the class of individuals with AIDS.

The model assumes that a fraction of the population has been protected before the disease out break at rate of $\theta\Pi$ and $(1-\theta)\Pi$ fraction of population susceptible. The susceptible class is increased from protected class by losing protection with φ rate. Susceptible individuals are exposed to HIV infection with force of infection $\lambda = \frac{\beta(I+qA)}{N}$ where β is contact rate and q is transmission coefficient for the asymptomatic. If $q > 1$ then, the asymptomatic infect susceptible more likely than infective. If $q = 1$, then both asymptomatic and infective have equal chance to infect the susceptible, but if $q < 1$ then, the infective have good chance to infect susceptible than asymptomatic. Exposed individuals progress to the symptomatic infectious class with probability $p\eta$ and to the asymptomatic infectious class with probability $(1-p)\eta$, where η is the per capita rate of becoming infectious. The asymptomatic individuals can develop disease symptom or can screen them selves and join the symptomatic class with a rate ϕ and others join the AIDS class with rate γ. Individuals in symptomatic class join the AIDS class with rate α. All infectious individuals ξ is the disease induced mortality rate due to infectious. Also, in all class μ is the natural mortality rate of individuals and all parameters in the model are positive.

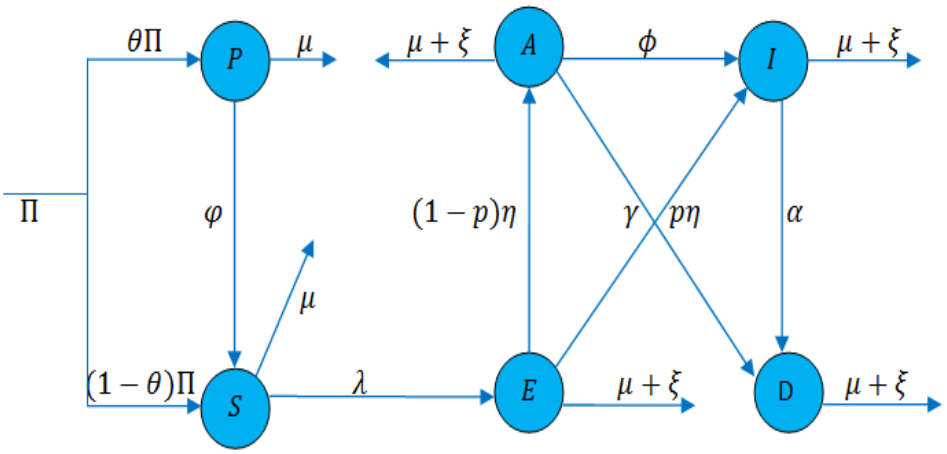

Figure 1: Schematic diagram of the model.

Based on the model assumptions and the schematic diagram,the model equations are formulated and given as follows:

$$\begin{cases} \frac{dP(t)}{dt} = \theta\Pi - (\varphi + \mu)P \\ \frac{dS(t)}{dt} = (1-\theta)\Pi + \varphi P - (\lambda + \mu)S \\ \frac{dE(t)}{dt} = \lambda S - (\eta + \mu + \xi)E \\ \frac{dA(t)}{dt} = (1-p)\eta E - (\phi + \gamma + \mu + \xi)A \\ \frac{dI(t)}{dt} = p\eta E + \phi A - (\alpha + \mu + \xi)I \\ \frac{dD(t)}{dt} = \gamma A + \alpha I - (\mu + \xi)D \end{cases} \quad (1)$$

With initial condition $P(0) = P_0, S(0) = S_0, E(0) = E_0, A(0) = A_0, I(0) = I_0, D(0) = D_0$.

3. MATHEMATICAL ANALYSIS OF THE MODEL

3.1 Invariant Region

In the model equation1 that governs human population, all the variables and parameters used in the model equation are non-negative. We consider a biologically-feasible region $\{\Omega = \{(P,S,E,A,I,D) \in \mathbb{R}_+^6 : N \leq \frac{\Pi}{\mu}\}$.

We adhere to the following steps to show the positive invariance of Ω, that is all the solution of model equation 1 that initiate in Ω remains in the region Ω and is bounded in Ω.

We have the total population

$N(t) = P(t) + S(t) + E(t) + A(t) + I(t) + D(t)$

The rate of change of the total population by adding all the equations considered in model equation1 is given by

$\frac{dN}{dt} = \Pi - \mu N - \xi(E(t) + A(t) + I(t) + D(t))$

In the absence of mortality due to disease it becomes

$\frac{dN}{dt} \leq \Pi - \mu N$

Thus, the particular solution can be expressed as

$$0 \leq N(t) \leq \frac{\Pi}{\mu} + (N_0 - \frac{\Pi}{\mu})e^{-\mu t} \quad (2)$$

As $t \longrightarrow \infty$ in equation 2 , the population size $N \longrightarrow \frac{\Pi}{\mu}$ which implies that $0 \leq N \leq \frac{\Pi}{\mu}$.

Thus the feasible solution set of the model equation remain in the the region

$\Omega = \{(P,S,E,A,I,D) \in \mathbb{R}_+^6 : N \leq \frac{\Pi}{\mu}\}$

Therefore, the basic model is wellposed epidemologically and mathematically. Hence , it is sufficient to study the dynamics of the basic model in region Ω.

3.2 Existence and Uniqueness of the Solutions of the Model

The validity and authenticity of any mathematical model depends on whether the given system of equations has a solution, and if the solution exists then it is unique. We shall use the Lipchitz condition to verify the existence and uniqueness of solution for the system of equation 1.

Theorem 1 Let Ω denote th region $1 \leq \alpha \leq \mathbb{R}$. Then the model equations (1) together with the initial condition $P(0) > 0, S(0) > 0, E(0) \geq 0, A(t)(0) \geq 0, I(t)(0) \geq 0, D(t)(0) \geq 0$ exist in $\mathbb{R}_+^6$ and has a unique solution. i.e., the model variables $P(t), S(t), E(t), A(t), I(t)$ and $D(t)$ exist for all t and will remain in $\mathbb{R}_+^6$.

Proof. We have to show that $\frac{\partial f_i}{\partial x_j}, i,j = 1,2,3,4,5,6$ are continues and bounded in Ω.

Let the right hand side of the system of equation (1) can be expressed as follows:

$$\begin{cases} f_1(P,S,E,A,I,D) = \theta\Pi - (\varphi + \mu)P \\ f_2(P,S,E,A,I,D) = (1-\theta)\Pi + \varphi P - (\lambda + \mu)S \\ f_3(P,S,E,A,I,D) = \lambda S - (\eta + \mu + \xi)E \\ f_4(P,S,E,A,I,D) = (1-p)\eta E - (\phi + \gamma + \mu + \xi)A \\ f_5(P,S,E,A,I,D) = p\eta E + \phi A - (\alpha + \mu + \xi)I \\ f_6(P,S,E,A,I,D) = \gamma A + \alpha I - (\mu + \xi)D \end{cases} \quad (3)$$

According to Derrick and Groosman theorem, let Ω denote the region $\Omega = (P,S,E,A,I,D) \in \mathbb{R}_+^6; N \leq (\Pi/\mu)$. Then equations (1) have a unique solution if $(\partial f_i)/(\partial x_j), i,j = 1,2,3,4,5,6$ are continuous and bounded in Ω. Here, $x_1 = P, x_2 = S, x_3 = E, x_4 = A, x_5 = I, x_6 = D$ and $\lambda = \frac{\beta}{N}(I + qA)$. The continuity and the boundedness are verified here under table 1.

Table 1: Continuity and boundedness of the model solution

For f_1	For f_2	For f_3
$\lvert\frac{\partial f_1}{\partial P}\rvert = \lvert-(\varphi+\mu)\rvert < \infty$	$\lvert\frac{\partial f_2}{\partial P}\rvert = \lvert\varphi\rvert < \infty$	$\lvert\frac{\partial f_3}{\partial P}\rvert = 0 < \infty$
$\lvert\frac{\partial f_1}{\partial S}\rvert = 0 < \infty$	$\lvert\frac{\partial f_2}{\partial S}\rvert = \lvert-(\lambda+\mu)\rvert < \infty$	$\lvert\frac{\partial f_3}{\partial S}\rvert = \lvert\lambda\rvert < \infty$
$\lvert\frac{\partial f_1}{\partial E}\rvert = 0 < \infty$	$\lvert\frac{\partial f_2}{\partial E}\rvert = 0 < \infty$	$\lvert\frac{\partial f_3}{\partial E}\rvert = \lvert-(\eta+\mu+\xi)\rvert < \infty$

$\lvert\frac{\partial f_1}{\partial A}\rvert = 0 < \infty$	$\lvert\frac{\partial f_2}{\partial A}\rvert = \lvert-\frac{\beta qS}{N}\rvert < \infty$	$\lvert\frac{\partial f_3}{\partial A}\rvert = \lvert\frac{\beta\ qS}{N}\rvert < \infty$
$\lvert\frac{\partial f_1}{\partial I}\rvert = 0 < \infty$	$\lvert\frac{\partial f_2}{\partial I}\rvert = \lvert\frac{-\beta S}{N}\rvert < \infty$	$\lvert\frac{\partial f_3}{\partial I}\rvert = \lvert\frac{\beta S}{N}\rvert < \infty$
$\lvert\frac{\partial f_1}{\partial D}\rvert = 0 < \infty.$	$\lvert\frac{\partial f_2}{\partial D}\rvert = 0 < \infty.$	$\lvert\frac{\partial f_3}{\partial D}\rvert = 0 < \infty.$
Forf_4	**Forf_5**	**Forf_6**
$\lvert\frac{\partial f_4}{\partial P}\rvert = 0 < \infty$	$\lvert\frac{\partial f_5}{\partial P}\rvert = 0 < \infty$	$\lvert\frac{\partial f_6}{\partial P}\rvert = 0 < \infty$
$\lvert\frac{\partial f_4}{\partial S}\rvert = 0 < \infty$	$\lvert\frac{\partial f_5}{\partial S}\rvert = 0 < \infty$	$\lvert\frac{\partial f_6}{\partial S}\rvert = 0 < \infty$
$\lvert\frac{\partial f_4}{\partial E}\rvert = \lvert(1-p)\eta\rvert < \infty$	$\lvert\frac{\partial f_5}{\partial E}\rvert = \lvert p\eta\rvert < \infty$	$\lvert\frac{\partial f_6}{\partial E}\rvert = 0 < \infty$
$\lvert\frac{\partial f_4}{\partial A}\rvert = \lvert-(\phi+\gamma+\mu+\xi)\rvert < \infty$	$\lvert\frac{\partial f_5}{\partial A}\rvert = \lvert\phi\rvert < \infty$	$\lvert\frac{\partial f_6}{\partial A}\rvert = \lvert\gamma\rvert < \infty$
$\lvert\frac{\partial f_4}{\partial I}\rvert = 0 < \infty$	$\lvert\frac{\partial f_5}{\partial I}\rvert = \lvert-(\alpha+\mu+\xi)\rvert < \infty$	$\lvert\frac{\partial f_6}{\partial I}\rvert = \lvert\alpha\rvert < \infty$
$\lvert\frac{\partial f_4}{\partial D}\rvert = 0 < \infty.$	$\lvert\frac{\partial f_5}{\partial D}\rvert = 0 < \infty.$	$\lvert\frac{\partial f_6}{\partial D}\rvert = \lvert-(\mu+\xi)\rvert < \infty.$

Thus, all the partial derivatives $\frac{(\partial f_i)}{(\partial x_j)}, i,j = 1,2,3,4,5,6$ exist, continuous and bounded in Ω. Hence, by Derrick and Groosman theorem, a solution for the model (1) exists and it is unique.

3.3.Positivity of the solution of the model

In this section we aim to obtain the non negative solution when dealing with human populations. Therefore, the next discussion below targets on the conditions under which the model being studied has a non negative solution.

Theorem 2. Let $\Omega = \{(P,S,E,A,I,D) \in \mathbb{R}^6_+ : P_0 > 0, S_0 > 0, E_0 \geq 0, A_0 \geq 0, I_0 \geq 0, D_0 \geq 0.\}$ then the solution of $\{P,S,E,A,I,D\}$ are positive for all $t \geq 0$.

Proof: From the system of differential equation(1), let us take the first equation such that.

$\frac{dP}{dt} = \theta\Pi - (\varphi+\mu)P$, eliminating the positive terms $\theta\Pi$ we get

$\frac{dP}{dt} \geq -(\varphi+\mu)P$, using variables separable method we get,

$\frac{dP}{P} \geq -(\varphi+\mu)dt$ integrating both side we can get,

$\int \frac{dP}{P} \geq -\int (\varphi+\mu)dt$ we obtain:

$\ln(S) \geq -(\varphi+\mu)t + \ln(C)$ where $ln(C)$ is any arbitrary constant.

Then after solving for P we obtain:

$P(t) \geq Pe^{-(\varphi+\mu)t}$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, i.e., the exponential function $e^{-(\varphi+\mu)t}$ is a non-negative quantity. Hence,it can be concluded that $S(t) > Ce^{-(\varphi+\mu)t} \geq 0$.

Therefore $P(t) > 0$ for all $t \geq 0$

From the system of differential equation(1), let us take the second equation such that.

$\frac{dS}{dt} = (1-\theta)\Pi + \varphi P - (\lambda+\mu)S$, eliminating the positive terms $(1-\theta)\Pi + \varphi P$ we get

$\frac{dS}{dt} \geq -(\lambda+\mu)S$, using variables separable method we get,

$\frac{dS}{S} \geq -(\lambda+\mu)dt$ integrating both side we can get,

$\int \frac{dS}{S} \geq -\int (\lambda+\mu)dt$ we obtain:

$\ln(S) \geq -(\lambda+\mu)t + \ln(C)$ where $ln(C)$ is any arbitrary constant.

Then after solving for S we obtain:

$S(t) \geq Ce^{-(\lambda+\mu)t}$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, i.e., the exponential function $e^{-(\lambda+\mu)t}$ is a non-negative quantity. Hence,it can be concluded that $S(t) > Ce^{-(\lambda+\mu)t} \geq 0$.

Therefore $S(t) > 0$ for all $t \geq 0$

From the system of differential equation1,let us take the third equation such that:

$\frac{dE}{dt} = \lambda S - (\eta+\mu+\xi)E$, eliminating the positive terms λS we get

, $\frac{dE}{dt} \geq -(\eta+\mu+\xi)E$ using variables separable method we get,

$\frac{dE}{E} \geq -(\eta+\mu+\xi)dt$ integrating both side we can get,

$\int \frac{dE}{E} \geq -\int (\eta+\mu+\xi)dt$ we obtain:

$\ln(E) \geq -(\eta+\mu+\xi)t + \ln(C)$ where $ln(C)$ is any arbitrary constant.

Then after solving for E we obtain:

$E(t) \geq Ce^{-(\eta+\mu+\xi)t}$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, i.e., the exponential function $e^{-(\eta+\mu+\xi)t}$ is a non-negative quantity.

Hence,it can be concluded that $E(t) > Ce^{-(\eta+\mu+\xi)t} \geq 0$.

Therefore $E(t) \geq 0$for all $t \geq 0$

From the system of differential equation(1),let us take the fourth equation such that:

$\frac{dA}{dt} = (1-p)\eta E - (\phi+\gamma+\mu+\xi)A$, eliminating the positive terms $(1-p)\eta E$ we get

, $\frac{dA}{dt} \geq -(\phi+\gamma+\mu+\xi)A$ using variables separable method we get,

$\frac{dA}{A} \geq -(\phi+\gamma+\mu+\xi)dt$ integrating both side we can get,

$\int \frac{dA}{A} \geq -\int (\phi+\gamma+\mu+\xi)dt$ we obtain:

$\ln(A) \geq -(\phi+\gamma+\mu+\xi)t + \ln(C)$ where $ln(C)$ is any arbitrary constant.

Then after solving for A we obtain:

$A(t) \geq Ce^{-(\phi+\gamma+\mu+\xi)t}$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, i.e., the exponential function $e^{-(\phi+\gamma+\mu+\xi)t}$ is a non-negative quantity. Hence,it can be concluded that $A(t) \geq Ce^{-(\phi+\gamma+\mu+\xi)t} \geq 0$.

Therefore $A(t) \geq 0$ for all $t \geq 0$

From the system of differential equation(1),let us take the fifth equation such that:

$\frac{dI}{dt} = p\eta E + \phi A - (\alpha+\mu+\xi)I$,eliminating the positive terms $p\eta E + \phi A$ we get

, $\frac{dI}{dt} \geq -(\alpha+\mu+\xi)I$ using variables separable method we get,

$\frac{dI}{I} \geq -(\alpha+\mu+\xi)dt$ integrating both side we can get,

$\int \frac{dI}{I} \geq -\int (\alpha+\mu+\xi)dt$ we obtain:

$\ln(I) \geq -(\alpha+\mu+\xi)t + \ln(C)$ where $ln(C)$ is any arbitrary constant.

Then after solving for I we obtain:

$I(t) \geq Ce^{-(\alpha+\mu+\xi)t}$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, i.e., the exponential function $e^{-(\alpha+\mu+\xi)t}$ is a non-negative quantity. Hence,it can be concluded that $I(t) \geq Ce^{-(\alpha+\mu+\xi)t} \geq 0$.

Therefore $I(t) \geq 0$ for all $t \geq 0$

From the system of differential equation1,let us take the sixth equation such that:

$\frac{dD}{dt} = \gamma A + \alpha I - (\mu+\xi)D$, eliminating the positive terms $\gamma A + \alpha I$ we get

, $\frac{dD}{dt} \geq -(\mu+\xi)D$ using variables separable method we get,

$\frac{dD}{D} \geq -(\mu+\xi)dt$ integrating both side we can get,

$\int \frac{dD}{D} \geq -\int (\mu+\xi)dt$ we obtain:

$\ln(D) \geq -(\mu+\xi)t + \ln(C)$ where $ln(C)$ is any arbitrary constant.

Then after solving for D we obtain:

$D(t) \geq Ce^{-(\mu+\xi)t}$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, i.e., the exponential function $e^{-(\mu+\xi)t}$ is a non-negative quantity. Hence,it can be concluded that $D(t) \geq Ce^{-(\mu+\xi)t} \geq 0$.

Therefore $D(t) \geq 0$ for all $t \geq 0$

3.4. Disease Free Equilibrium Points (DFE)

Disease free equilibrium points are steady state solutions where there is no disease in the population. In the absence of disease in the population, implies that $E(t) = 0, A(t) = 0, I(t) = 0$ and $D(t) = 0$ and the equilibrium points require that the right hand side of the model equation set equal to zero. We denote disease-free equilibrium point by E_1.

These requirements reflect in reducing the model equations(1) as

$$\begin{cases} \theta\Pi - (\varphi+\mu)P = 0 \\ (1-\theta)\Pi + \varphi P - (\lambda+\mu)S = 0 \\ \lambda S - (\eta+\mu+\xi)E = 0 \\ (1-p)\eta E - (\phi+\gamma+\mu+\xi)A = 0 \\ p\eta E + \phi A - (\alpha+\mu+\xi)I = 0 \\ \gamma A + \alpha I - (\mu+\xi)D = 0 \end{cases} \quad (4)$$

Then solving the system of differential equation 4 simultaneously,we obtain

$$E_0 = \{P^0, S^0, E^0, A^0, I^0, D^0\} = \{\frac{\theta\Pi}{(\varphi+\mu)}, \frac{\Pi(\varphi+\mu-\theta)}{(\varphi+\mu)(\lambda+\mu)}, 0,0,0,0\}$$

3.5. The Basic Reproduction Number(R_0)

The basic reproduction number denoted by R_0 is the average number of secondary infections caused by an infectious individual during his or her entire period of infectiousness (Diekmann et. al) [13] The basic reproduction number is an important non-dimensional quantity in epidemiology as it sets the threshold in the study of a disease both for predicting its outbreak and for evaluating its control strategies [13]. Thus, whether a disease becomes persistent or dies out in a community depends on the value of the reproduction number, R_0. Furthermore, the stability of the equilibrium point can be analyzed using R_0 . If $R_0 < 1$ it means that every infectious individual will cause less than one secondary infection which is impossible and hence the disease will die out and when $R_0 > 1$ every infectious individual will cause more than one secondary infection and hence the disease will invade the population. It is Obtained by taking the largest (dominant) eigenvalue (spectral radius)

$$R_0 = [\frac{\partial F_i(E_0)}{\partial x_j}][\frac{\partial V_i(x_0)}{\partial x_j}]^{-1}$$

where f_i be the rate of appearance of new criminal in compartments, v_i is the transfer of individuals out of the compartment by another means, E_0 is the disease free equilibrium point. We compute the basic reproduction number using the next generation matrix approch.

Thus the associated matrices F and V for the new infectious terms and the remaining transition terms are

respectively given by:

$$F_i = \begin{bmatrix} \frac{\beta(\ I + qA)S}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix}, V_i = \begin{bmatrix} (\eta + \mu + \xi)E \\ -(1-p)\eta E + (\phi + \gamma + \mu + \xi)A \\ -p\eta E - \phi A + (\alpha + \mu + \xi)I - \gamma A - \alpha I + (\mu + \xi)D \end{bmatrix}$$

Thus the jacobian matrix of F and V at the disease free equilibrium point E_0 take the form respectively as: $F(E_0) =$

$$\begin{bmatrix} 0 & \beta q & \beta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } V(E_0) = \begin{bmatrix} a & 0 & 0 & 0 \\ -(1-p)\eta & b & 0 & 0 \\ -p\eta & -\phi & c & 0 \\ 0 & -\gamma & -\alpha & d \end{bmatrix}$$

where $a = \eta + \mu + \xi, b = \phi + \gamma + \mu + \xi, c = \alpha + \mu + \xi, d = \mu + \xi.$

It can be verified that that the matrix $V(E_0)$ is non-singular as its determinant $\det(V(E_0)) = abcd \neq 0$ is non-zero. That is $V(E_0) \neq 0$ then it is invertable and the inverse is given by .

$$(V(E_0))^{-1} = \frac{Adj(V)}{det(V)} \tag{5}$$

Then after some algebraic computations the inverse matrix is constructed as follows:

$$[V(E_0)]^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 & 0 \\ \frac{(1-p)\eta}{ab} & \frac{1}{b} & 0 & 0 \\ \frac{(1-p)\eta + bp\eta}{abc} & \frac{-\phi}{bc} & \frac{1}{c} & 0 \\ \frac{(1-p)\eta(\phi\alpha + \gamma c) - b\alpha p\eta}{abcd} & \frac{\phi\alpha + c\gamma}{bcd} & \frac{\alpha}{cd} & \frac{1}{d} \end{bmatrix} \tag{6}$$

Now,

$$[F(E_0)][V(E_0)]^{-1} = \begin{bmatrix} \frac{\beta\eta q(1-p)(cq+1) + \beta bq\eta}{abc} & \frac{\beta q}{b} - \frac{\beta\phi}{bc} & \frac{\beta}{c} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

Thus the eigenvalues of the matrix 7 are: $\lambda_1 = \frac{\beta\eta q(1-p)(cq+1) + \beta bq\eta}{abc}, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$ Then from $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ the dominant eigenvalue is $\lambda_1 = \frac{\beta\eta q(1-p)(cq+\phi) + \beta bq\eta}{abc}$. Therefore the basic reproduction number is given by

$R_0 = \frac{\beta\eta q(1-p)(cq+\phi) + \beta bq\eta}{abc}$

3.6. Local Stability of Disease Free Equilibrium Points (DFE)

Theorem 3: The DFE E_0 of the system (1) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof. Consider the right hand side expressions of the equations (1) as functions so as to find the Jacobian matrix as follows:

Thus, the Jacobian matrix J of model at the disease free equilibrium E_0 is given by

$$J(E_0) = \begin{bmatrix} -(\varphi + \mu) & 0 & 0 & 0 & 0 & 0 \\ \varphi & -\mu & 0 & -\beta q & -\beta & 0 \\ 0 & 0 & -a & \beta q & \beta & 0 \\ 0 & 0 & (1-p)\eta & -b & 0 & 0 \\ 0 & 0 & p\eta & \phi & -c & 0 \\ 0 & 0 & 0 & \gamma & \alpha & -d \end{bmatrix} \tag{8}$$

The eigenvalues of the jacobian matrix $J(E_0)$ are required to be found as follows.

$$\begin{vmatrix} -(\varphi + \mu) - \lambda & 0 & 0 & 0 & 0 & 0 \\ \varphi & -\mu - \lambda & 0 & -\beta q & -\beta & 0 \\ 0 & 0 & -a - \lambda & \beta q & \beta & 0 \\ 0 & 0 & (1-p)\eta & -b - \lambda & 0 & 0 \\ 0 & 0 & p\eta & \phi & -c - \lambda & 0 \\ 0 & 0 & 0 & \gamma & \alpha & -d - \lambda \end{vmatrix} = 0$$

The characteristic equation of the Jacobian matrix at the disease free equilibrium point is

$(-(\varphi+\mu)-\lambda)(-\mu-\lambda)(-d-\lambda)(\lambda^3+e_1\lambda^2+e_2\lambda+e_3)=0$

Where $e_1=(a+b+c), e_2=(ac+bc+ab-\beta q(1-p)\eta-\beta p\eta), e_3=abc(1-R_0)$

Then, $\lambda_1=-(\varphi+\mu), \lambda_2=-\mu, \lambda_3=-d$. From this the first three eigenvalues $\lambda_1,\lambda_2,\lambda_3$ are real, distinct and negative, which is stable. To determine the sign of the eigenvalues we use the Routh-Hurwitz criterian for the cubic equation; $\lambda^3+e_1\lambda^2+e_2\lambda+e_3=0$.

According to the Routh-Hurwitz criteria the three roots of a polynomial of order three of type $p(\lambda)=\lambda^3+e_1\lambda^2+e_2\lambda+e_3$, are real distinct and negative if the coefficients satisfy the conditions $e_1>0, e_2>0, e_3>0$ and $e_1e_2>e_3$

It is straight forward to verify that this conditions are satisfied and hence the last three eigenvalues are real distinct and negative.i.e

$e_1>0$ if $a+b+c>0$

$e_2>0$ if $ac+cb+ab>(\beta q(1-p)\eta-\beta p\eta)$

$e_3>0$ if $R_0<1$

Clearly it can be observed that the first three conditions of the Routh-Hurwitz criteria are satisfied and the fourth condition is satisfied provided that : $a_1a_2>a_3$ if $(a+b+c)(ac+bc+ab-\beta q(1-p)\eta+\beta p\eta)>abc(1-R_0)$.

Therefore the disease free equilibrium point of the system of ordinary differential equation (1) is locally asymptotically stable if $R_0<1$.

3.7. Global stability of the disease free equilibrium point (DFE)

Theorem 4: The disease free equilibrium point E_0 of the model equation (1 is globally asymptotically stable if $R_0<1$.

proof To establish the global stability of the disease-free equilibrium point, we construct a Lyapunov function.

let $\Omega\subseteq\mathbb{R}_+^6$ be an open neighborhood of the disease free equilibrium point E_0.

Then the function $L:\Omega\longrightarrow\mathbb{R}_+^6$ defined by:

$$L(P,S,E,A,I,D)=\frac{B_1}{2}(E(t))^2+\frac{B_2}{2}(A(t))^2+\frac{B_3}{2}(I(t))^2+\frac{B_4}{2}(D(t))^2 \quad (9)$$

where B_i , for $i=1,2,3,4$ are some positive constants to be chosen later.

Then $L(P,S,E,A,I,D)$ should satisfies the following properties:

$i)L$ is continuously differentiable.

$ii)L>0, \forall x\in\Omega: E_0$ and $L(E_0)=0$,as $(E(t))^2\geq 0,(A(t))^2\geq 0,(I(t))^2\geq 0,\ (D(t))^2\geq 0.$

$iii)\frac{dL}{dt}\leq 0$ in Ω, then E_0 is stable.

The first two condition holds, as L it is continuously differentiable and $L>0,\ \forall x\in\Omega: E_0$ and $L(E_0)=0$. Now let we check the third condition $\frac{dL}{dt}\leq 0$ in Ω.

$$\frac{dL}{dt}=B_1\frac{dE}{dt}+B_2\frac{dA}{dt}+B_3\frac{dI}{dt}+B_4\frac{dD}{dt}$$

$$=B_1(\lambda S-aE)+B_2((1-p)\eta E-bA)+B_3(p\eta E+\phi A-cI)+B_3(\gamma A+\alpha I-dD)$$

$$=B_1(\frac{\beta(I+qA)}{N}S-aE)+B_2((1-p)\eta E-bA)+B_3(p\eta E+\phi A-cI)+B_4(\alpha I-dD), S\leq N \text{ at } E_0.$$

$$=B_1(\beta(I+qA)-aE)+B_2((1-p)\eta E-bA)+B_3(p\eta E+\phi A-cI)+B_4(\alpha I-dD)$$

$$=B_1\beta I+B_1\beta qA-B_1aE+B_2(1-p)\eta E-B_2bA+B_3p\eta E+B_3\phi A-B_3cI+B_4\gamma A+B_4\alpha I-B_4dD$$

$$=(B_2(1-p)\eta+B_3p\eta-B_1a)E+(B_1\beta q-B_2b+B_3\phi+B_4\gamma)A+(B_1\beta-B_3c+B_4\alpha)I-B_4dD$$

$$=B_1a(\frac{(B_2(1-p)\eta+B_3p\eta}{B_1a}-1)E+B_3\phi A-B_3cI+B_4\gamma A+B_4\alpha I-B_4dD$$

Now choosing $B_1=bc, B_2=\beta(cq+\phi), B_3=\beta b, B_4=0$. Then,

$$\frac{dL}{dt}=abc(\frac{\beta(cq+\phi)(1-p)\eta+\beta bp\eta}{abc}-1)E+(bcq\beta-bcq\beta-\beta b\phi+\beta b\phi)A+(bc\beta-\beta bc)I$$

$$=abc(\frac{\beta(cq+\phi)(1-p)\eta+\beta bp\eta}{abc}-1)E$$

$$=abc(R_0-1)E$$

Therefore $\frac{dL}{dt}\leq bcd(R_0-1)E<0$ if $R_0<1$ which implies that $\frac{dL}{dt}\leq 0$. Therefore the largest compact invariant set in Ω is singleton set E_0. Hence LaSalle's invariant principle implies that E_0 is globally asymptotically stable.

3.8. Endemic Equilibrium Points

The endemic equilibrium point denoted by $E_1=\{P^*,S^*,E^*,A^*,I^*,D^*\}$ is a steady state solution where the disease persists in the population. The endemic equilibrium point is obtained by setting rates of changes of variables with respect to time in model equations (1) equal to zero. That is, setting

$$\begin{cases}\theta\Pi-(\varphi+\mu)P=0\\(1-\theta)\Pi+\varphi P-(\lambda+\mu)S=0\\\lambda S-(\eta+\mu+\xi)E=0\\(1-p)\eta E-(\phi+\gamma+\mu+\xi)A=0\\p\eta E+\phi A-(\alpha+\mu+\xi)I=0\\\gamma A+\alpha I-(\mu+\xi)D=0\end{cases} \quad (10)$$

Then solving they system of differential equation 10 by substitution and after some algebraic simplificaton we obtain $E_1=\{P^*,S^*,E^*,A^*,I^*,D^*\}$ where:

$P^*=\frac{\theta\Pi}{(\varphi+\mu)}$

$$S^*=\frac{\Pi(\varphi+\mu-\theta\mu)}{(\lambda^*+\mu)}$$

$$E^*=\frac{\lambda^*S^*}{(\eta+\mu+\xi)}$$

$$A^*=\frac{(1-p)\eta E^*}{(\phi+\gamma+\mu+\xi)}$$

$$I^* = \frac{p\eta E^* + \phi A^*}{(\alpha + \mu + \xi)}$$

$D^* = \frac{\gamma A^* + \alpha I^*}{\mu + \xi}$.

On substituting the expression for A^* and I^* into the force of infection, that is, $\lambda^* = \frac{\beta(I^* + qA^*)}{N}$ obtained as

$\lambda^* = \mu[R_0(\varphi + \mu - \theta\mu) - 1]$

$\lambda^* \leq 0$, if $R_0(\varphi + \mu - \theta\mu) < 1$, i.e $R_0 < \frac{1}{\varphi + \mu - \theta\mu}$

From this, we see that, there is no endemic equilibrium for this model. Therefore, this condition shows that it is not possible for backward bifurcation in the model if $R_0 < 1$.

Lemma: A unique endemic equilibrium point E_1 exists and positive if $R_0 > 1$.

3.9. Global Stability of Endemic Equilibrium

Theorem: 5 The endemic equilibrium point of the model equation(1) is globally asymptotically stable whenever $R_0 > 1$.

Proof: To prove the global stability of the endemic equilibrium we use the method of Lyapunov functions. Define Define:

$$L(P^*, S^*, E^*, A^*, I^*, D^*) = [P - P^* - P^*\ln(\tfrac{P}{P^*})][S - S^* - S^*\ln(\tfrac{S}{S^*})] + [E - E^* - E^*\ln(\tfrac{E}{E^*})]$$
$$+[A - A^* - A^*\ln(\tfrac{A}{A^*})] + [I - I^* - I^*\ln(\tfrac{I}{I^*})] + [D - D^* - D^*\ln(\tfrac{D}{D^*})] \quad (11)$$

Then by taking the time derivative of $L(P^*, S^*, E^*, A^*, I^*, D^*)$, we obtain:

$$\frac{dL}{dt} = (P' - \frac{P^*}{P}P') + (S' - \frac{S^*}{S}S') + (E' - \frac{E^*}{E}E') + (A' - \frac{A^*}{A}A') + (I' - \frac{I^*}{I}I') + (D' - \frac{D^*}{D}D')$$
$$= (1 - \frac{P^*}{P})\frac{dP}{dt} + (1 - \frac{S^*}{S})\frac{dS}{dt} + (1 - \frac{E^*}{E})\frac{dE}{dt} + (1 - \frac{A^*}{A})\frac{dA}{dt} + (1 - \frac{I^*}{I})\frac{dI}{dt} + (1 - \frac{D^*}{D})\frac{dD}{dt}$$

By substituting the value $\frac{dD}{dt}, \frac{dS}{dt}, \frac{dE}{dt}, \frac{dA}{dt}, \frac{dI}{dt}, \frac{dD}{dt}$ from model equation1 we obtain

$$\frac{dL}{dt} = (1 - \frac{P^*}{P})[\theta\Pi - (\varphi + \mu)P] + [1 - \frac{S^*}{S}][(1 - \theta)\Pi + \varphi P - (\lambda + \mu)S] + [1 - \frac{E^*}{E}][\lambda S - (\eta + \mu + \xi)E] + [1 - \frac{A^*}{A}][(1 - p)\eta E$$
$$- (\phi + \gamma + \mu + \xi)A] + [1 - \frac{I^*}{I}][p\eta E + \phi A - (\alpha + \mu + \xi)I] + [1 - \frac{D^*}{D}][\gamma A + \alpha I - (\mu + \xi)D]$$
$$= (\pi + \theta R + \lambda S^* + \mu S^* - \lambda S - \mu S - \pi\frac{S^*}{S} - \theta R\frac{S^*}{S}) + (\lambda S + \mu E^* + \eta E^* + \varphi E^* - \mu E - \eta E - \varphi E - \lambda S\frac{E^*}{E}) + (\eta E + \mu A^* + \gamma A^*$$
$$+ \frac{A^*}{A}p\eta E - p\eta E - \mu A - \gamma A - \frac{A^*}{A}\eta E) + (p\eta E\mu I^* + \alpha I^* - \mu I - \alpha I - \frac{I^*}{I}P\eta E) + (\alpha I + \gamma I + \varphi E + \theta R^* + \mu R^*$$
$$- \theta R - \mu R - \alpha I\frac{R^*}{R} - \gamma A\frac{R^*}{R} - \varphi E\frac{R^*}{R})$$

Now after some simplifications i.e cancelling like terms which is opposite in sign we obtain:

$$= \underbrace{[\Pi + P^*\varphi + \lambda^* S^* + \eta E^* + (\phi + \gamma)A^* + \alpha I^* + \mu(N^* - N) + \xi[(E^* + A^* + I^* + D^*) - (E + A + I + D)]]}_{\hat{Q}}$$
$$\underbrace{- [\theta\Pi\frac{P^*}{P} + ((1 - \theta)\Pi + \varphi P)\frac{S^*}{S} + \lambda^* S\frac{E^*}{E} + ((1 - p)\eta E)\frac{A^*}{A} + (p\eta E + \phi A)\frac{I^*}{I} + (\gamma A + \alpha I)\frac{D^*}{D}]}_{\hat{K}}$$
$$\frac{dL}{dt} = Q - K$$

Thus if $Q < K$, then $\frac{dL}{dt} \leq 0$. Noting that $\frac{dL}{dt} = 0$ if and only if $P = P^*, S = S^*, E = E^*, A = A^*, I = I^*, D = D^*$. Therefore, the largest compact invariant set in $\{(P^*, S^*, E^*, A^*, I^*, D^*) \in \Omega; \frac{dL}{dt} = 0\}$ is the singleton E_1 is the endemic equilibrium of the system 1. By LaSalle's invariant principle (LaSalle's,1976),it implies that $E_1$is globally asymptotically stable in Ω if $Q < K$.

4. SENSITIVITY ANALYSIS OF MODEL PARAMETERS

One of the most important concerns about any infectious disease is its ability to invade a population. The basic reproduction number, R_0 is a measure of the potential for disease spread in a population, and is inarguably 'one of the foremost and most valuable ideas that mathematical thinking has brought to epidemic theory' [14]. A large value of R_0 may indicate the possibility of a major epidemic. We thus, carried out sensitivity analysis of the basic reproduction number, R_0 with respect to the model parameters in order to determine the relative importance of the different factors responsible for the transmission and prevalence of the disease. This will assist in curtailing the transmission of the disease by using appropriate intervention strategies. There are more than a dozen ways of conducting sensitivity analysis, all resulting in a slightly different sensitivity ranking [15]. Following [14], we used the normalized forward sensitivity index also called elasticity as it is the backbone of nearly all other sensitivity analysis techniques [15] and are computationally efficient[16]. The normalized forward sensitivity index of the basic reproduction number, R_0 with respect to a parameter value, P is given by:

$$S_p^{R_0} = \frac{\partial R_0}{\partial P} \times \frac{P}{R_0} \quad (12)$$

The sensitivity indices of the basic reproductive number with respect to main parameters are arranged orderly in Table 2. Those parameters that have positive indices show that they have great impact on expanding the disease in the community if their values are increasing. Due to the reason that the basic reproduction number increases as their values increase, it means that the average number of secondary cases of infection increases in the community. Furthermore, those parameters in which their sensitivity indices are negative have an influence of minimizing the burden of the disease in the community as their values increase while the others are left constant. And also as their values increase, the basic reproduction number decreases, which leads to minimizing the endemicity of the disease in the community.

Table 2: Sensitivity indices Table.

Parameter symbol	Sensitivity indices
$\boldsymbol{\beta}$	+ve
$\boldsymbol{\eta}$	+ve
$\boldsymbol{q}$	+ve
$\boldsymbol{\mu}$	-ve
$\boldsymbol{p}$	-ve
$\boldsymbol{\alpha}$	-ve
$\boldsymbol{\phi}$	-ve
$\boldsymbol{\gamma}$	-ve
$\boldsymbol{\xi}$	-ve

5. FORMULATION OF AN OPTIMAL CONTROL PROBLEM

The purpose of this section is to extend model equation (1) into an optimal control problem. The controls are defined as follows:

1. u_1 is the control variable for prevention of the recruitment to susceptible individuals.
2. u_2 is the control variable for reduction of the spread/contact of HIV infection.
3. u_3 is the control variable for screen of the exposed individuals.
4. u_4 is the control variable for treatment of the asymptomatic and symptomatic individuals.

After incorporating the controls, the corresponding state system for model equation1 is given as:

$$\begin{cases} \frac{dP(t)}{dt} = \theta\Pi - (1-u_1)\varphi P - \mu P \\ \frac{dS(t)}{dt} = (1-\theta)\Pi + (1-u_1)\varphi P - (1-u_2)\lambda S - \mu S \\ \frac{dE(t)}{dt} = (1-u_2)\lambda S - (1-u_3)\eta E - (\mu+\xi)E \\ \frac{dA(t)}{dt} = (1-u_3)(1-p)\eta E - (1-u_4)(\phi+\gamma)A - (\mu+\xi)A \\ \frac{dI(t)}{dt} = (1-u_3)p\eta E + (1-u_4)\phi A - (1-u_4)\alpha I - (\mu+\xi)I \\ \frac{dD(t)}{dt} = (1-u_4)\gamma A + (1-u_4)\alpha I - (\mu+\xi)D \end{cases} \quad (13)$$

With initial condition $P(0) \geq 0, S(0) \geq 0, E(0) \geq 0, A(0) \geq 0, I(0) \geq 0, D(0) \geq 0$ with a bounded Lebesgue measurable control set is represented as

$U = \{u = (u_1, u_2, u_3, u_4), 0 \leq u_i \leq u_{i_{max}}, i = 1,2,3,4,\}$ and $t \in [0, T]$

The aim objective is to minimize the number of infected population while minimizing the rate of interventions u_1, u_2, u_3 and u_4 on a fixed time period T. Therefore, the optimal control problem for the model 13 is to minimize the objective functional:

$$J(u) = \int_0^T [g(\phi, u)]dt = \int_0^T [M_1 S + M_2 E + M_3 A + M_4 I + + \frac{1}{2}\sum_{i=1}^4 k_i u_i^2(t)]dt \longrightarrow min \quad (14)$$

where, $i = 1,2,3,4$ and $\phi = (P, S, E, A, I, D)$ solves equation 13 for the specified control u.

In the intervention of controls the solution $\phi = (P, S, E, A, I, D)$ depends on the controls. The constants w_1, w_2, w_3 and w_4 measures the cost or effort required for the implementation of each of the four control measures adopted while M_1, M_2, M_3 and M_4 measures the relative importance of reducing the associated classes on the spread of the disease. Thus, we need to find the optimal controls $u^* = (u_1^*, u_2^*, u_3^*, u_4^*)$ such that

$$J(u^*) = \min_U J(u_1, u_2, u_3, u_4) \quad (15)$$

Hence, the basic setup of the optimal control problem is to

check the existence and uniqueness of the optimal controls and to characterize them.

5.1. Existence of an optimal controls

Theorem: 6 Given $J(u)$ subject to system 13 with $P(0) \geq 0, S(0) \geq 0, E(0) \geq 0, A(0) \geq 0, I(0) \geq 0, D(0) \geq 0$, then there exists an optimal control u^* and corresponding $(P^*, S^*, E^*, A^*, I^*, D^*)$, that minimizes $J(u)$ over U. The proof is based on the following assumption and by Fleming and Rishel's [17] theorem.

1. The set of controls and corresponding state variable is nonempty.
2. The measurable control set is convex and closed.
3. All the right hand sides of equations of the state system is continuous, bounded above by a sum of bounded control and state, and can be written as a linear function of u with coefficients depending on time and state.
4. The integrand $g(\phi, u)$ of the objective functional is convex.
5. There exist constants $c_1, c_2, c_3, c_4, c_5 \geq 0$ and $\tau^* \geq 1$ such that the integrand of the objective functional satisfies $g(\phi, u) \geq c_1 + c_2|u_1|^\tau + c_3|u_2|^\tau + c_4|u_3|^\tau + c_5|u_4|^\tau$.

Proof:

1. U is a nonempty set of measurable functions on $0 \leq T$ with values in real numbers $\mathbb{R}$. The system 13 has bounded coefficients and hence any solutions are bounded on [0, T]. The corresponding solutions for the system 13 exists.
2. Assume that $u_1, u_2, u_3, u_4 \in U$ such that $\| u_i \| \leq 1, i = 1,2,3,4$. Now, let us take any controls $u_1, u_2 \in U$ and $\lambda \in [0,1]$, then $0 \leq \lambda u_1 + (1-\lambda)u_2$. Additionally, we observe that

$\| \lambda u_1 \| \leq \lambda \| u_1 \| \leq \lambda$ and $\| (1-\lambda)u_2 \| \leq (1-\lambda) \| u_2 \| \leq (1-\lambda)$

Then for any $\lambda \in [0,1]$,

$$\| \lambda u_1 + (1-\lambda)u_2 \| \leq \| \lambda u_1 \| + \| (1-\lambda)u_2 \| \leq \lambda \| u_1 \| + (1-\lambda) \| u_2 \| \leq \lambda + (1-\lambda) = 1$$

Hence, $0 \leq \lambda u_1 + (1-\lambda)u_2 \leq$, for all $u_1, u_2 \in U$ and $\lambda \in [0,1]$.

Therefore, the control space $U = \{u = (u_1, u_2, u_3, u_4), 0 \leq u_i \leq u_{i_{max}}, i = 1,2,3,4,\}$ and $t \in [0, T]$ is convex and closed by definition.

3. By definition, each right hand side of system 13 is continuous. All variables P, S, E, A, I, D and u are bounded on $[0, T]$. To prove the boundedness we use the method in []. To do so we use the fact the super-solutions of system 13 is written as:

$$\begin{cases} \frac{dP(t)}{dt} = \theta\Pi \\ \frac{dS(t)}{dt} = (1-\theta)\Pi + (1-u_1)\varphi P \\ \frac{dE(t)}{dt} = (1-u_2)\lambda S \\ \frac{dA(t)}{dt} = (1-u_3)(1-p)\eta E \\ \frac{dI(t)}{dt} = (1-u_3)p\eta E + (1-u_4)\phi A \\ \frac{dD(t)}{dt} = (1-u_4)\gamma A + (1-u_4)\alpha I \end{cases} \tag{16}$$

are bounded on a finite time interval. System 17 can be written as;

$$\phi = \begin{bmatrix} P' \\ S' \\ E' \\ A' \\ I' \\ D' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ (1-u_1)\varphi & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-u_2)\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-u_3)\eta & 0 & 0 & 0 \\ 0 & 0 & (1-u_3)\eta p & (1-u_4)\phi & 0 & 0 \\ 0 & 0 & 0 & (1-u_4)\gamma & (1-u_4)\alpha & 0 \end{bmatrix} \begin{bmatrix} \overline{P} \\ \overline{S} \\ \overline{E} \\ \overline{A} \\ \overline{I} \\ \overline{D} \end{bmatrix} + \begin{bmatrix} \theta\Pi \\ (1-\theta)\Pi \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

The system is linear in finite time with bounded coefficients, then the super-solutions $\overline{P}, \overline{S}, \overline{E}, \overline{A}, \overline{I}$ and $\overline{D}$ are uniformly bounded. Since the solution to each state equation is bounded, we observe that,

$$|f(t, \phi, u)| \leq \left\| \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ (1-u_1)\varphi & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-u_2)\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-u_3)\eta & 0 & 0 & 0 \\ 0 & 0 & (1-u_3)\eta p & (1-u_4)\phi & 0 & 0 \\ 0 & 0 & 0 & (1-u_4)\gamma & (1-u_4)\alpha & 0 \end{bmatrix} \begin{bmatrix} \overline{P} \\ \overline{S} \\ \overline{E} \\ \overline{A} \\ \overline{I} \\ \overline{D} \end{bmatrix} \right\| + \tag{18}$$

$$\left\|\begin{bmatrix}\theta\Pi\\(1-\theta)\Pi\\0\\0\\0\\0\end{bmatrix}\right\| + \left\|\begin{bmatrix}S\\E\\A\\I\end{bmatrix}\begin{bmatrix}0\\u_1\\u_2\\u_3\\u_4\\0\end{bmatrix}\right\|$$

$$\leq K|\phi| + M|u| + N$$

Where K depends on the coefficients of the system. Thus, the assumption holds.

4. The integrand in the objective functional, which is a cost function $g(\phi,u)$ is an affine function. Recall that any affine function is a convex and the sum of a convex function is a convex. Therefore, $g(\phi,u)$ is convex on U.

5. Assume that there exists constants $c_1,c_2,c_3,c_4,c_5 \geq 0$ and $\tau^* \geq 1$ such that $g(\phi,u)$ satisfies $g(\phi,u) \geq c_1 + c_2|u_1|^\tau + c_3|u_2|^\tau + c_4|u_3|^\tau + c_5|u_4|^\tau$. Thus, the state variables are being bounded.

Let $c_1 = \inf_{t\in[0,T]}[M_1S + M_2E + M_3A + M_4I], c_2 = \frac{w_1}{2}, c_3 = \frac{w_2}{2}, c_4 = \frac{w_3}{2}, c_5 = \frac{w_4}{2}$ and $\tau = 2$ then it follows that $g(\phi,u) \geq c_1 + c_2|u_1|^\tau + c_3|u_2|^\tau + c_4|u_3|^\tau + c_5|u_4|^\tau$

Thus, this assumption is justified.

Therefore, the optimal control u exists.

5.2. Characterization of an optimal control

In order to determine the necessary conditions for the optimal control the Pontryagin's maximum principle [18] is used. To apply this we need to convert the optimal control problem into a problem of minimizing point wise a Hamiltonian, H, with respect to u. The Hamiltonian associated to our problem is:

$$H(\phi,u,\lambda) = M_1S + M_2E + M_3A + M_4I + \frac{w_1u_1^2}{2} + \frac{w_2u_2^2}{2} + \frac{w_3u_3^2}{2} + \frac{w_4u_4^2}{2} \quad (19)$$

$$+\lambda_1[\theta\Pi - (1-u_1)\varphi P - \mu P] + \lambda_2[(1-\theta)\Pi + (1-u_1)\varphi P - (1-u_2)\lambda S - \mu S]$$

$$+\lambda_3[(1-u_2)\lambda S - (1-u_3)\eta E - (\mu+\xi)E] + \lambda_4[(1-u_3)(1-p)\eta E - (1-u_4)(\phi+\gamma)A - (\mu+\xi)A]$$

$$+\lambda_5[(1-u_3)p\eta E + (1-u_4)\phi A - (1-u_4)\alpha I - (\mu+\xi)I] + \lambda_6[(1-u_4)\gamma A + (1-u_4)\alpha I - (\mu+\xi)D]$$

Based on [], if the controlu^* and the corresponding state ϕ^* are an optimal couple, necessarily there exists a non trivial adjoint vector $\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)$ satisfying the following equality

$$\begin{cases}\frac{d\phi}{dt} = \frac{\partial H(\phi,u,\lambda)}{\partial\lambda}\\ \frac{d\lambda}{dt} = -\frac{\partial H(\phi,u,\lambda)}{\partial\phi}\\ \frac{\partial H(\phi,u,\lambda)}{\partial u} = 0\end{cases} \quad (20)$$

Which gives after derivation

$$\begin{cases}u_i^* = 0, if\ \frac{\partial H}{\partial u_i} < 0\\ 0 \leq u_i^* \leq u_{imax}, if\ \frac{\partial H}{\partial u_i} = 0\\ u_i^* = u_{imax}, if\ \frac{\partial H}{\partial u_i} > 0\end{cases} \quad (21)$$

Now we apply the necessary conditions to the Hamilton function,H.

Theorem: 7 Given an optimal control u^* and a solution to the corresponding state13, ϕ, then there exist an adjoint vector λ and this satisfies the following adjoint equation:

$$\begin{cases}\frac{d\lambda_1}{dt} = \lambda_1[(1-u_1)\varphi+\mu] - \lambda_2(1-u_1)\varphi \\ \frac{d\lambda_2}{dt} = -M_1 + \lambda_2[(1-u_2)\lambda^* + \mu] - \lambda_3(1-u_2)\lambda^* \\ \frac{d\lambda_3}{dt} = -M_2 + \lambda_3[(1-u_3)\eta + (\mu+\xi)] - \lambda_4[(1-u_3)(1-p)\eta] - \lambda_5[(1-u_3)p\eta] \\ \frac{d\lambda_4}{dt} = -M_3 + \lambda_2[(1-u_2)\frac{\beta qS}{N}] - \lambda_3[(1-u_2)\frac{\beta qS}{N}] + \lambda_4[(1-u_4)(\gamma+\phi) + (\mu+\xi)] - \lambda_5[(1-u_4)\phi] \\ -\lambda_6[(1-u_4)\gamma] \\ \frac{d\lambda_5}{dt} = -M_4 + \lambda_2[(1-u_2)\frac{\beta S}{N}] - \lambda_3[(1-u_2)\frac{\beta S}{N}] + \lambda_5[(1-u_4)\alpha + (\mu+\xi)] - \lambda_6[(1-u_4)\alpha] \\ \frac{d\lambda_6}{dt} = \lambda_6(\mu+\xi) \\ \lambda_i(T) = 0, i = 1,2,3,4,5,6.\end{cases}$$

$\lambda_i(T) = 0$ is the transversality condition. Moreover, the optimal control u^* given by

$$\begin{cases}u_1^* = \min\{\max\{\frac{(\lambda_2-\lambda_1)\varphi P}{w_1}, 0\}, u_{1max}\} \\ u_2^* = \min\{\max\{\frac{(\lambda_3-\lambda_2)\lambda^* S}{w_2}, 0\}, u_{2max}\} \\ u_3^* = \min\{\max\{\frac{(\lambda_4(1-p)-\lambda_3)\eta E+\lambda_5 p\eta E}{w_3}, 0\}, u_{3max}\} \\ u_4^* = \min\{\max\{\frac{(\lambda_6-\lambda_4)\gamma A+(\lambda_5-\lambda_4)\phi A+(\lambda_6-\lambda_5)\alpha I}{w_4}, 0\}, u_{4max}\}\end{cases} \quad (22)$$

Proof: The adjoint equation is obtained by differentiating the Hamiltonian equation 19 with respect to $\phi = (P, S, E, A, I, D)$. That is $\frac{d\lambda}{dt} = -\frac{\partial H(\phi,u,\lambda)}{\partial \phi}$. Assuming that the final states $P(T), S(T), E(T), A(T), I(T), D(T)$ are free we get the transversality conditions $\lambda(T) = 0$. The optimal controls u are found from the optimality conditions and using the property of the control space U. The optimality condition of the Hamiltonian gives $\frac{\partial H}{\partial u} = 0$. That is

$$\begin{cases}\frac{\partial H}{\partial u_1} = 0 \Rightarrow u_1^* = \frac{(\lambda_2-\lambda_1)\varphi P}{w_1} \\ \frac{\partial H}{\partial u_2} = 0 \Rightarrow u_2^* = \frac{(\lambda_3-\lambda_2)\lambda^* S}{w_2} \\ \frac{\partial H}{\partial u_3} = 0 \Rightarrow u_3^* = \frac{(\lambda_4(1-p)-\lambda_3)\eta E+\lambda_5 p\eta E}{w_3} \\ \frac{\partial H}{\partial u_4} = 0 \Rightarrow u_4^* = \frac{(\lambda_6-\lambda_4)\gamma A+(\lambda_5-\lambda_4)\phi A+(\lambda_6-\lambda_5)\alpha I}{w_4}\end{cases} \quad (23)$$

And using the property of the control space U, the controls are given as

$$\begin{cases}u_1^* = 0, if(\lambda_2-\lambda_1)\varphi P < 0, \\ u_1^*, if 0 \le (\lambda_2-\lambda_1)\varphi P \le w_1 u_{1max} \\ u_{1max}, if(\lambda_2-\lambda_1)\varphi P > w_1 u_{1max}\end{cases} \quad (24)$$

$$\begin{cases}u_2^* = 0, if(\lambda_3-\lambda_1)\lambda^* S < 0, \\ u_2^*, if 0 \le (\lambda_3-\lambda_1)\lambda^* S \le w_2 u_{2max} \\ u_{2max}, if(\lambda_3-\lambda_1)\lambda^* S > w_2 u_{2max}\end{cases} \quad (25)$$

$$\begin{cases}u_3^* = 0, if(\lambda_4(1-p)-\lambda_3)\eta E + \lambda_5 p\eta E < 0, \\ u_3^*, if 0 \le (\lambda_4(1-p)-\lambda_3)\eta E + \lambda_5 p\eta E \le w_3 u_{3max} \\ u_{3max}, if(\lambda_4(1-p)-\lambda_3)\eta E + \lambda_5 p\eta E > w_3 u_{3max}\end{cases} \quad (26)$$

$$\begin{cases}u_4^* = 0, if(\lambda_6-\lambda_4)\gamma A + (\lambda_5-\lambda_4)\phi A + (\lambda_6-\lambda_5)\alpha I < 0, \\ u_4^*, if 0 \le (\lambda_6-\lambda_4)\gamma A + (\lambda_5-\lambda_4)\phi A + (\lambda_6-\lambda_5)\alpha I \le w_4 u_{4max} \\ u_{4max}, if(\lambda_6-\lambda_4)\gamma A + (\lambda_5-\lambda_4)\phi A + (\lambda_6-\lambda_5)\alpha I > w_4 u_{4max}\end{cases} \quad (27)$$

5.3. The Optimality System

The optimality system consists of the state system 13 with its initial conditions coupled with the adjoint system ?? with its transversality conditions together with the characterization of the optimal controls. It is written as follows:

$$\begin{cases}
\frac{dP(t)}{dt} = \theta\Pi - (1-u_1)\varphi P - \mu P \\
\frac{dS(t)}{dt} = (1-\theta)\Pi + (1-u_1)\varphi P - (1-u_2)\lambda S - \mu S \\
\frac{dE(t)}{dt} = (1-u_2)\lambda S - (1-u_3)\eta E - (\mu+\xi)E \\
\frac{dA(t)}{dt} = (1-u_3)(1-p)\eta E - (1-u_4)(\phi+\gamma)A - (\mu+\xi)A \\
\frac{dI(t)}{dt} = (1-u_3)p\eta E + (1-u_4)\phi A - (1-u_4)\alpha I - (\mu+\xi)I \\
\frac{dD(t)}{dt} = (1-u_4)\gamma A + (1-u_4)\alpha I - (\mu+\xi)D \\
\frac{d\lambda_1}{dt} = \lambda_1[(1-u_1)\varphi+\mu] - \lambda_2(1-u_1)\varphi \\
\frac{d\lambda_2}{dt} = -M_1 + \lambda_2[(1-u_2)\lambda^* + \mu] - \lambda_3(1-u_2)\lambda^* \\
\frac{d\lambda_3}{dt} = -M_2 + \lambda_3[(1-u_3)\eta + (\mu+\xi)] - \lambda_4[(1-u_3)(1-p)\eta] - \lambda_5[(1-u_3)p\eta] \\
\frac{d\lambda_4}{dt} = -M_3 + \lambda_2[(1-u_2)\frac{\beta qS}{N}] - \lambda_3[(1-u_2)\frac{\beta qS}{N}] + \lambda_4[(1-u_4)(\gamma+\phi) + (\mu+\xi)] - \lambda_5[(1-u_4)\phi] \\
-\lambda_6[(1-u_4)\gamma] \\
\frac{d\lambda_5}{dt} = -M_4 + \lambda_2[(1-u_2)\frac{\beta S}{N}] - \lambda_3[(1-u_2)\frac{\beta S}{N}] + \lambda_5[(1-u_4)\alpha + (\mu+\xi)] - \lambda_6[(1-u_4)\alpha] \\
\frac{d\lambda_6}{dt} = \lambda_6(\mu+\xi)
\end{cases}$$

Where $\lambda^* = \frac{\beta(I+qA)}{N}, \lambda_i(T) = 0, i = 1,2,3,4,5,6.$

5.4. Uniqueness of the optimality system

In order to successively discuss uniqueness of the optimality system we notice that the adjoint system is also linear in λ_i for $i = 1,2,3,4,5,6$ with bounded coefficients. Thus, there exists a $M > 0$ such that $|\lambda_i(t)| < M$ for $i = 1,2,3,4,5,6$ on $[0,T]$.

Theorem 8. [?] For T sufficiently small the solution to the optimality system is unique.

6. NUMERICAL SIMULATION

6.1. Numerical Simulation of the autonomous system

In this subsection, numerical simulation study of the autonomous system 1 are carried out using the software MATLAB R2015b with ODE45 solver. To conduct the study, a set of physically meaningful values are assigned to the model parameters. These values are either taken from literature or assumed on the basis of reality. Using the parameter values given in Table 3 and the initial conditions $P(0) = 100, S(0) = 80, E(0) = 50, A(0) = 25, I(0) = 15$ and $D(0) = 30$ a simulation study is conducted and the results are given in the following Figures.

Table 3: Parameter values

Parameter	Value	Source
$\boldsymbol{\theta}$	0.007	Assumed
$\boldsymbol{\Pi}$	0.004	[19]
$\boldsymbol{\beta}$	0.067	Assumed
$\boldsymbol{\mu}$	0.054	[19]
$\boldsymbol{\gamma}$	0.015	Assumed
$\boldsymbol{\alpha}$	0.16	[19]
$\boldsymbol{\eta}$	0.012	Assumed
$\boldsymbol{p}$	0.06	Assumed
$\boldsymbol{\xi}$	0.0001	[19]
$\boldsymbol{\phi}$	0.04	[19]
$\boldsymbol{\varphi}$	0.03	Assumed
$\boldsymbol{q}$	0.01	Assumed

Figure 2 shows that the protected individuals decreases due to more number of protected join susceptible class and converges to disease free equilibrium. Similarly susceptible individual decreases due to more number of infectious individuals. Figure 3 illustrate that the Exposed individuals decreases due to more number of exposed join asymptomatic and symptomatic class and converges to disease free equilibrium. However, in this figure asymptomatic individual increases firstly as the consequence of some number of exposed individuals joined the asymptomatic class but decline because of some asymptomatic individuals joined symptomatic class after developing the symptom of the disease and others join AIDS class. Similarly, in Figure 4 individuals with AIDS increases firstly as the consequence of some number of asymptomatic individuals and symptomatic individuals joined the AIDS class but decline because of natural death and disease death(induced death). Moreover, Figure 5 shows that the number of all infectious increases in the beginning as a result of infectious from susceptible enters it and decreases due to death rate. Also, Figure 6 and Figure 7 indicating that contact rate has an effect on reducing the disease from community. An increases in level of contact rate among individuals has an effect on reducing the prevalence of infectious and AIDS the disease.

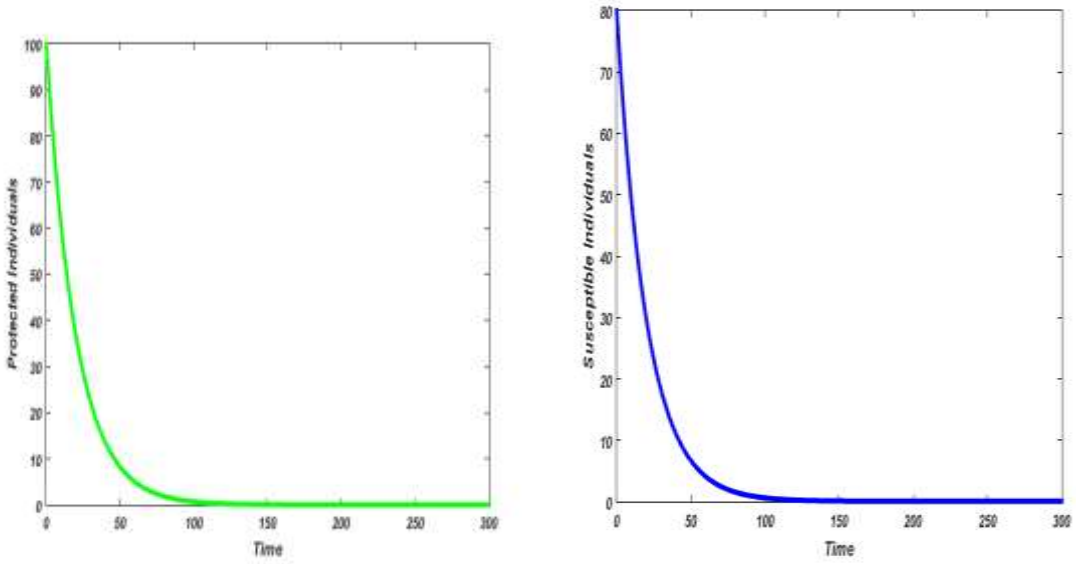

Figure 2: Dynamics of Protected individuals and Susceptible Individuals

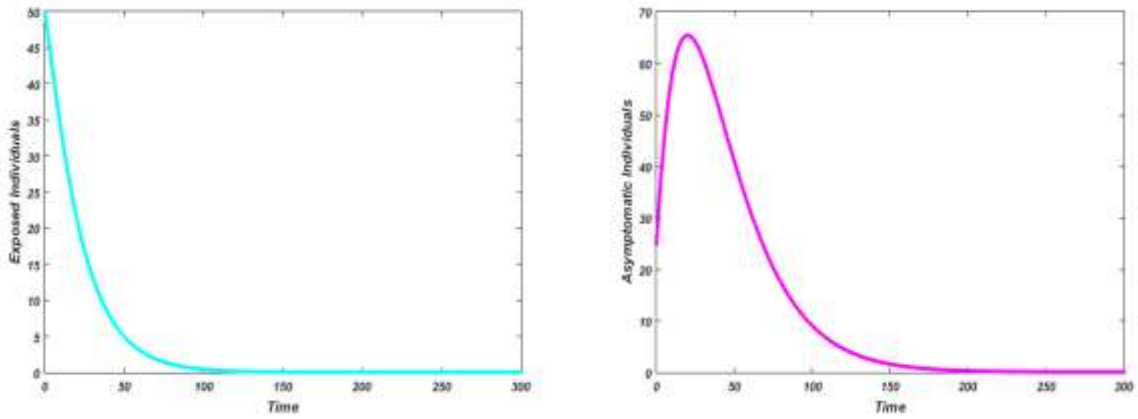

Figure 3: Dynamics of Exposed individuals and Asymptomatic Individuals

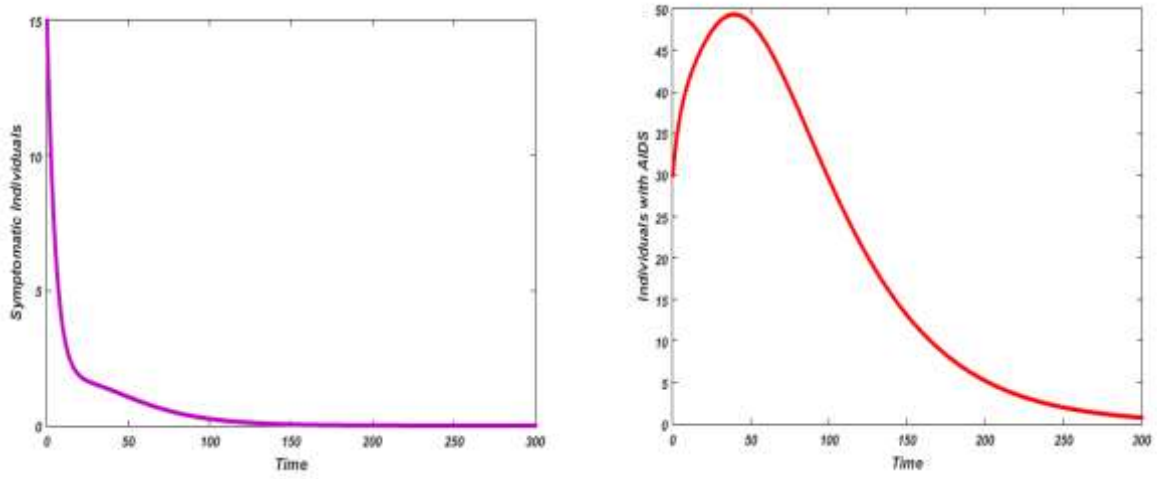

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Figure 4: Dynamics of Symptomatic individuals and AIDS Individuals

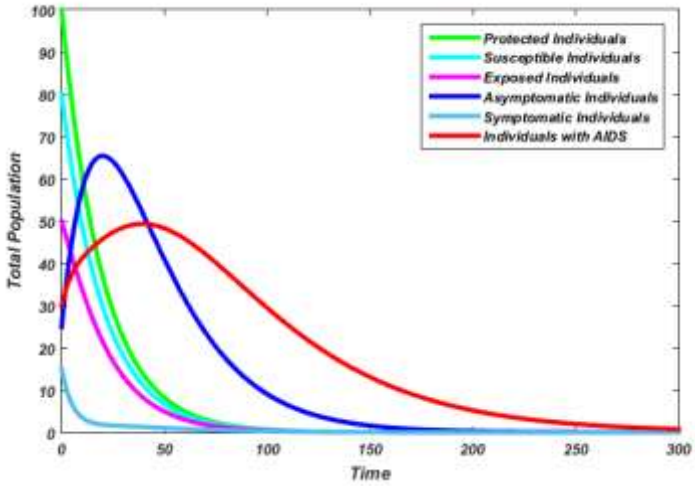


Figure 5: Dynamics of total population

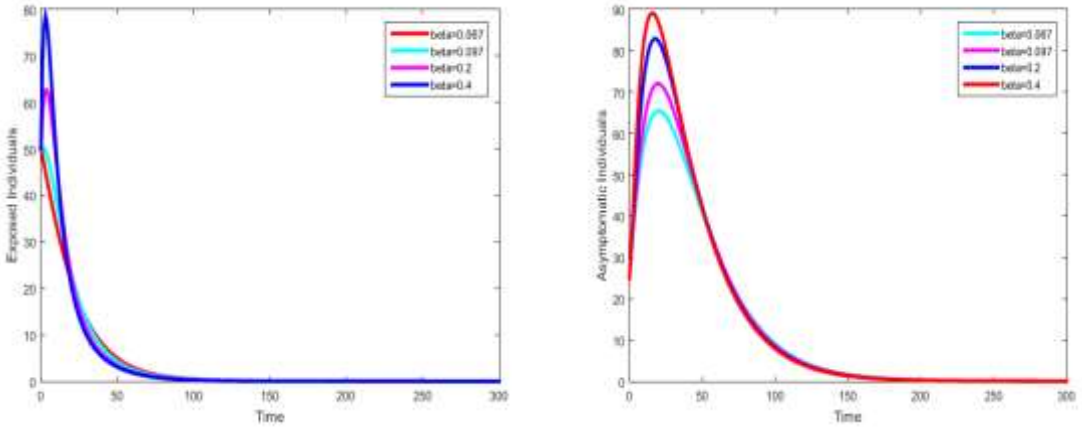


Figure 6: Effect of increasing contact rate on Exposed and Asymptomatic Individuals

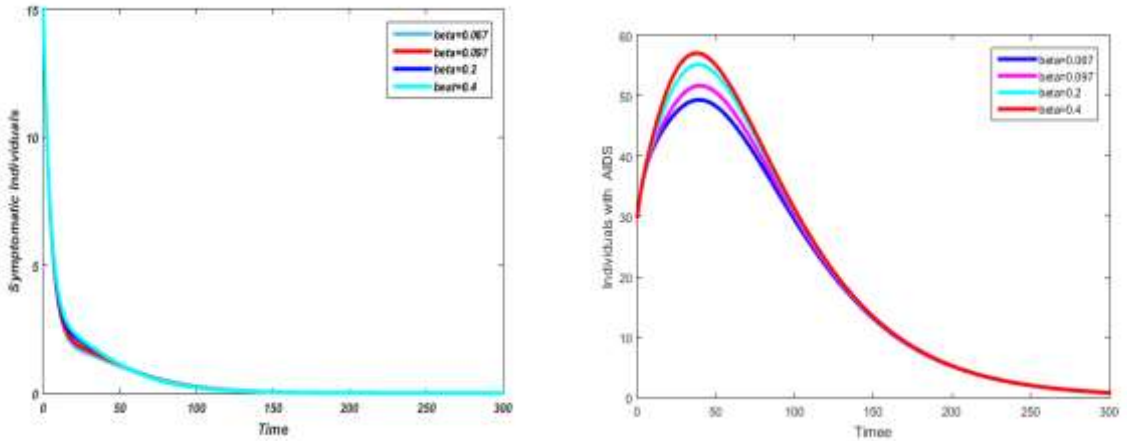
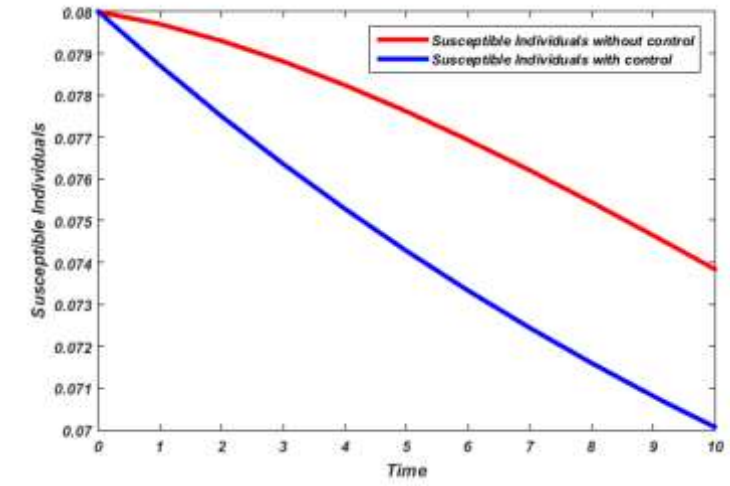


Figure 7: Effect of increasing contact rate on Symptomatic and AIDS Individuals

6.2. Numerical Simulation of the optimality system

In this section, the result obtained by numerically solving the optimality system was presented. In our control problem, we have initial conditions for the state variables and terminal conditions for the adjoints. That is, the optimality system is a two-point boundary value problem with separated boundary conditions at times step $i = 0$ and $i = T$. The simulations are consistent for all the scenarios under consideration,varying only in the margins of growth and reduction. We, consequently, only present and discuss results for the most effective combination. In Figure 8,9 and 10, the number of individuals are lower under control as contrasted to without control. In fact, the number individuals reduces under control while there is increase in time without control.



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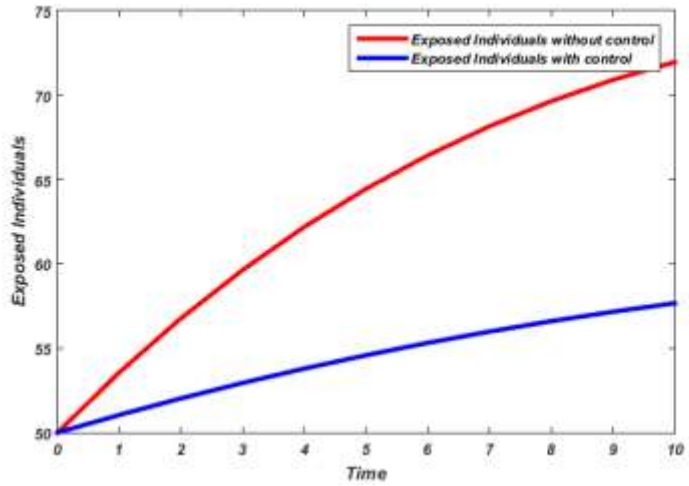

Figure 8: Simulations of Susceptible individuals and Exposed individuals with control strategy

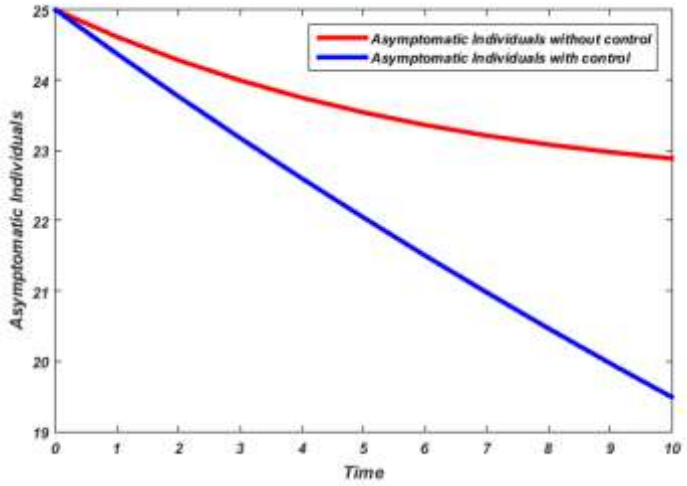

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Figure 9: Simulations of Asymptomatic individuals with control strategy

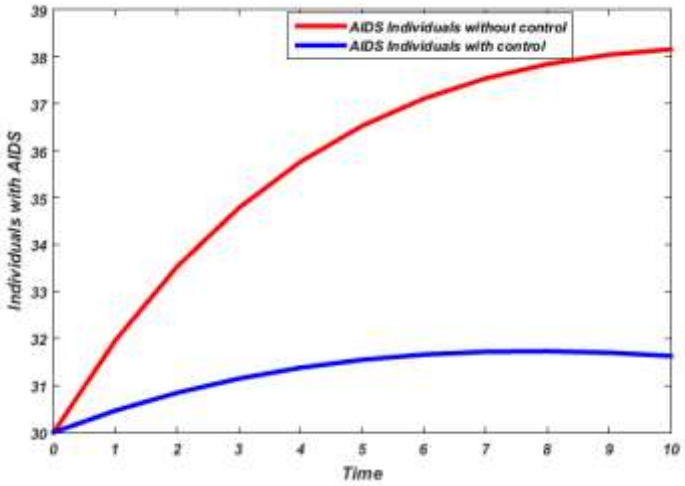

Figure 10: Simulations of AIDS individuals with control strategy

7. CONCLUSION

In this paper, a non linear deterministic model of HIV/AIDS was formulated and analyzed. The well possedness of the modified model are performed. The study also obtained the basic reproduction number that governs the disease transmission from the largest eigenvalue of the next-generation matrix. The equilibria points of the model are obtained and their local as well as global stability condition was established. The model exhibits a backward bifurcation and the sensitivity analysis was performed. The optimal control problem is designed by incorporating continuous controls: prevention, reduction, screening and treatment. The results from the optimal control problem suggest that the disease may be reduces by implementing controls. Control policies implementing either of the strategies presented in this paper could reduce the number of infection in a community. It could be concluded that the qualitative analysis targeted to the autonomous system and the optimal control approach together provide a complete picture of the possible outcomes of the system behavior.

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