International Journal of Mathematics and Computer Research
ISSN: 2320-7167
Volume 09 Issue 03 March 2021, Page no. - 2193-2196
Index Copernicus ICV: 57.55, Impact Factor: 7.184
DOI: 10.47191/ijmcr/v9i3.01

On Fuzzy gp*- neighborhood, interior and Closure in Fuzzy Topological Spaces

Firdose Habib
Department of mathematics Maulana Azad National Urdu University Hyderabad

ARTICLE INFO
Published Online:
06 March 2021
Corresponding Author:
Firdose Habib

ABSTRACT
Based on fuzzy gp*-closed sets and fuzzy gp*-open sets, in this paper we have introduced fuzzy gp*-neighborhoods, fuzzy gp*-Interior and fuzzy gp*-Closure. Also we investigated some of their elementary properties and discuss some important theorems of fuzzy gp*- neighborhoods, fuzzy gp*-Interior and fuzzy gp*-Closure.

KEYWORDS: Fuzzy topological spaces; fuzzy gp*-closed sets; fuzzy gp*-open sets; fuzzy gp*- neighborhoods; fuzzy gp*-Interior and fuzzy gp*-Closure.

1. INTRODUCTION

Introduced by Lotfi A. Zadeh in 1965[1] fuzzy set theory is the generalization of the classical set theory and thus fuzzy set extended the basic mathematical concept of a crisp set. So fuzzy mathematics is just a kind of mathematics developed in this framework and fuzzy topology is the generalization of topology in classical mathematics introduced by C.L Chang in 1968 [2]. Since then work started taking place in fuzzy topology at a rapid rate and various types of fuzzy sets were introduced and studied by various researchers, Like S.S Benchalli and G.P.Siddapur introduced fuzzy g* pre continuous maps[4], Hamid Reza Moradi and Anahid Kamali introduced fuzzy strongly g* -closed sets and g**-closed sets in 2015 [5], In 2020 Firdose Habib and Khaja Moinuddin introduced fuzzy gp*-closed sets and fuzzy gp*-open sets [7]. And almost all the mathematical, engineering, medicinal etc. concepts have been redefined using fuzzy theory and it has further deepened the understanding of basic set theory.

Based on these newly introduced fuzzy gp*-closed and fuzzy gp*-open sets, in this paper we have introduced and studied fuzzy gp*- neighborhoods, fuzzy gp*-Interior and fuzzy gp*-Closure. Also we investigated some of their elementary properties and discuss some important theorems.

2. PRELIMINARIES

Definition 2.1[1] Let X be a space of objects, with a generic element of X denoted by x. Then a *fuzzy set* A in X is a set of ordered pairs $\{(x, f_A(x))\}$ where $f_A(x)$ is called the membership function which associates each point in X a real number in the interval [0,1].

Definition 2.2 [2] A family τ of fuzzy sets of X is called *fuzzy topology* on X if 0 and 1 belong to τ and τ is closed with respect to arbitrary union and finite intersection. The elements of τ are called *fuzzy open* sets and there complements are called *fuzzy closed* sets. The space X with topology τ is called fuzzy topological space denoted by (X, τ).

Definition 2.3 [2] For a fuzzy set α of X, the closure $Cl\ \alpha$ and the interior $Int\ \alpha$ of α are defined respectively, as $Cl\ \alpha = \wedge\{\mu : \mu \geq \alpha, 1-\mu \in \tau\}$ and $\text{Int}\ \alpha = \vee\{\mu : \mu \leq \alpha, \mu \in \tau\}$

Definition 2.4 [2] A function f from a fts (X, τ) to a fts (Y, δ) is *fuzzy-continuous* iff the inverse of each δ-open fuzzy set in Y is τ-open fuzzy set in X.

Definition 2.5 [2] let (X, τ) be a fts. A fuzzy set h in X is a *neighborhood* of fuzzy set α in X iff there is $g \in \tau$ such that $\alpha \leq g \leq h$.

Definition 2.6 [3] A fuzzy set n in a fts (X, τ) is a neighborhood of a point $x \in X$ iff there is $g \in \tau$ such that $g \leq n$ and n(x) = g(x) >0. A neighborhood of a point x is frequently denoted by n_x. A neighborhood n_x is called an open neighborhood of x iff $n_x \in \tau$.

Definition 2.7 [2] Let A and B be fuzzy sets in fts (X, τ), and let $A \geq B$. Then B is called an interior fuzzy set of A iff A is nbhd of *B*. The union of all interior fuzzy sets of *A* is called the interior of *A* and is denoted by A^O.

Definition 2.8 [2] Let A and B be two fuzzy sets in a space X = {x} with the grades of membership of x in *A* and *B* denoted by μ_A (x) and μ_B (x), respectively. Then

A = B <=> μ_A (x) = μ_B (x) for all x ε X

A ⊂ B <=> μ_A (x) ≤ μ_B (x) for all x ε X

C = A B <=> μ_C (x) = Max [μ_A (x),μ_B (x)]for all x ε X

D = A B <=> μ_D (x) = Min [μ_A (x),μ_B (x)]for all x ε X

Definition 2.9 [6] A fuzzy set A of a fuzzy topological space (X , τ) is called fuzzy generalized pre- *closed or gp-closed* set if $pcl(A) \leq U$ whenever $A \leq U$ and U is a fuzzy open set in (X,τ). And complement of a Fuzzy gp-closed set is called *fuzzy generalized pre-open or gp-open* set.

Definition 2.10 [7] A fuzzy set λ of a fuzzy topological space (Y, τ) is called fuzzy generalized pre star closed (briefly fuzzy gp*-closed) if $cl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy generalized pre-open in *Y*.

Definition 2.11 [7] suppose a fuzzy set λ is fuzzy generalized pre star closed set in fuzzy topological space (Y,τ), Then its complement I.e. $1-\lambda$ is called fuzzy generalized pre star open (briefly fuzzy gp*- open) in (Y,τ).

Remark 2.12 [7] *All fuzzy closed sets are fuzzy gp* closed sets.*

Remark 2.13 [7] *All fuzzy open sets are fuzzy gp*-open.*

3. FUZZY GENERALIZED PRE STAR (FUZZY gp*) NEIGHBORHOOD

Definition 4.1 Suppose (Y,τ) is a fuzzy topological space & suppose that $y \in Y$. Then a subset *N* of *Y* is called *fuzzy gp*-neighborhood* of *y* if and only if there exists a fuzzy gp*-open subset *M* of *Y* such that $y \in M \leq N$.

Definition 4.2 Suppose (Y,τ) is a fuzzy topological space then a subset *N* of *Y* is said to be *fuzzy gp*- neighborhood* of a fuzzy set *L* in *Y* if and only if there exists a fuzzy gp*-open set *M* in *Y* such that $L \leq M \leq N$.

Theorem 4.3 *Every fuzzy neighborhood α of $y \in Y$ is fuzzy gp*-neighborhood of y.*

Proof: Suppose α is a fuzzy neighborhood of point $y \in Y$. So by definition of fuzzy neighborhood, we have a fuzzy open set μ in *Y* such that $y \in \mu \leq \alpha$. Now as every fuzzy open set is fuzzy gp*-open in Y, implies μ is fuzzy gp*-open set. So $y \in \mu \leq \alpha$, where μ is fuzzy gp*-open set implies α is fuzzy gp*-neighborhood of *y*.

Theorem 4.4 *α is fuzzy gp*-neighborhood of each of its points in fuzzy topological space (Y,τ), if α is fuzzy gp*-open in Y.*

Proof: Let α is a fuzzy gp*-open set in *Y* and let $y \in \alpha$. Now, the proof that α is fuzzy gp*-neighborhood of *y* is clear as $y \in \alpha \leq \alpha$, and α is fuzzy gp*-open set in *Y*. So α follows the definition of fuzzy gp*- neighborhood. Now as *y* is any arbitrary point in α implies that α is fuzzy gp*-neighborhood of each of its points.

Theorem 4.5 *Suppose (Y,τ) is a fuzzy topological space & A is a fuzzy gp*-closed subset of Y, and let $y \in 1-A$. Then ∃ fuzzy gp*-neighborhood B of y such that $A \wedge B = \phi$.*

Proof: Given that *A* is a fuzzy gp*-closed subset of *Y*, implies *1-A* is a fuzzy gp*-open subset in *Y*, such that $y \in 1-A$. So by Theorem 4.4, *(1-A)* is a fuzzy gp*-neighborhood of *y*. Now by Definition 4.2 ∃ a fuzzy gp*-open set *B*, which is also fuzzy gp*-neighborhood of *y* (by Theorem 4.4) in *Y* such that $y \in B \leq 1-A$ I.e. $A \wedge B = \phi$.

4. FUZZY GENERALIZED PRE STAR INTERIOR (FUZZY gp*-INT).

Definition 5.1 Suppose (Y,τ) is a fuzzy topological space and α is a fuzzy subset in *Y*. Then a point $y \in \alpha$ is called *fuzzy gp*-interior point* of α, if α is fuzzy gp*-nbhd of *y*. The collection of all fuzzy gp*- interior points of α is said to be *fuzzy gp*-interior* of α, denoted by *fuzzy gp*-int* (α).

Theorem 5.2 *Suppose (Y,τ) is a fuzzy topological space and α, μ be two fuzzy subsets of Y. Then we have*

(a) *Fuzzy gp*-int*$(1_Y) = 1_Y$ *& fuzzy gp*-int*$(0_Y) = 0_Y$.

(b) *Fuzzy gp*-int*$(\alpha) \leq \alpha$.

(c) *Suppose α is any fuzzy gp*-open set contained in μ, Then $\alpha \leq$ fuzzy gp*-int(μ).*

(d) *Suppose $\alpha \leq \mu$, Then fuzzy gp*-int$(\alpha) \leq$ fuzzy gp*-int(μ).*

Proof: *(a)* We know that 1_Y and 0_Y are fuzzy gp*-open sets in *Y*, as every fuzzy open set is fuzzy gp*- open. So fuzzy gp*-int $(1_Y) = \vee$ {H: H is fuzzy gp*-open $H \leq 1_Y$ }

⇒ Fuzzy gp*-int $(1_Y) = 1_Y \vee$ {all fuzzy gp*-open sets}

⇒ Fuzzy gp*-int $(1_Y) = 1_Y$.

Now, as 0_Y is the only fuzzy gp*-open set contained in 0_Y. So fuzzy gp*-int $(0_Y) = 0_Y$

(b) Suppose y ∈ fuzzy gp*-int(α)

⇒ y is fuzzy interior point of α.

⇒ α is fuzzy neighborhood of y.

⇒ y ∈ α.

So, y ∈ fuzzy gp*-int (α) ⇒ y ∈ α, I.e. fuzzy gp*-int(α) ≤α.

(c) Suppose that α is any fuzzy gp*-open set contained in μ. Now, let $y \in \alpha$. So as α is fuzzy gp*-open set contained in μ, implies *y* is fuzzy gp*-interior point of μ. I.e. $y \in$ *gp*-int(μ)*. Hence $\alpha \leq$ *gp*-int(μ)*.

(d) Suppose α & μ are two fuzzy subsets of fuzzy

topological space (y,τ), such that $\alpha \le \mu$. Now we have to show that

$$\text{fuzzy } gp^*\text{-}int(\alpha) \le \text{fuzzy } gp^*\text{-}int(\mu).$$

Let $y \in$ fuzzy gp*-int(α), which implies y is fuzzy gp*-interior point of α I.e. α is fuzzy gp*-nbhd of y. Now as $\alpha \le \mu$, implies μ is also fuzzy gp*-nbhd of y I.e. $y \in$ fuzzy gp*-int(μ).

So $y \in$ fuzzy gp*-int(α) $\Rightarrow y \in$ fuzzy gp*-int(μ) I.e.

$$\text{fuzzy } gp^*\text{-}int(\alpha) \le \text{fuzzy } gp^*\text{-}int(\mu).$$

Theorem 5.3 *Suppose* (Y,τ) *is a fuzzy topological space & α is any fuzzy gp*-open subset in Y, then fuzzy gp*-int (α) = α.*

Proof: Suppose α is a fuzzy gp*-open subset in Y. Now from Theorem 5.2(b) we have fuzzy $gp^*\text{-}int(\alpha) \le \alpha$. Also α is fuzzy gp*-open subset contained in α. So by Theorem 5.2 (c), we have $\alpha \le$ *fuzzy gp*-int(α).* Hence fuzzy $gp^*\text{-}int(\alpha) = \alpha$.

Theorem 5.4 *Suppose* (Y,τ) *is a fuzzy topological space & α, μ are fuzzy subsets of Y. Then, fuzzy gp*-int (α) ∨ fuzzy gp*-int(μ) ≤ fuzzy gp*-int(α ∨ μ).*

Proof*: We have* $\alpha \le \alpha \vee \mu$ & $\mu \le \alpha \vee \mu$. Now by Theorem 5.2(d) we have,

$$\text{Fuzzy } gp^*\text{-}int\ (\alpha) \le \text{fuzzy } gp^*\text{-}int\ (\alpha \vee \mu)\ \&$$

$$\text{Fuzzy } gp^*\text{-}int\ (\mu) \le \text{fuzzy } gp^*\text{-}int\ (\alpha \vee \mu)$$

The above two inequalities implies that,

$$\text{Fuzzy } gp^*\text{-}int\ (\alpha) \vee \text{fuzzy } gp^*\text{-}int(\mu) \le \text{fuzzy } gp^*\text{-}int(\alpha \vee \mu).$$

Theorem 5.5 *Suppose* (Y,τ) *is a fuzzy topological space & α, μ are fuzzy subsets of Y. Then fuzzy gp*-int (α ∧ μ) = fuzzy gp*-int (α) ∧ fuzzy gp*-int (μ).*

Proof: Since we know that $\alpha \wedge \mu \le \alpha$ and $\alpha \wedge \mu \le \mu$, so by Theorem 5.2(d) we have

$$\text{Fuzzy } gp^*\text{-}int\ (\alpha \wedge \mu) \le \text{fuzzy } gp^*\text{-}int\ (\alpha) \quad \&$$
$$\text{Fuzzy } gp^*\text{-}int\ (\alpha \wedge \mu) \le \text{fuzzy } gp^*\text{-}int\ (\mu)$$

The above two inequalities imply that

$$\text{Fuzzy } gp^*\text{-}int\ (\alpha \wedge \mu) \le \text{fuzzy } gp^*\ int\ (\alpha) \wedge \text{fuzzy } gp^*\text{-}int\ (\mu) \rightarrow (1)$$

Now, Let y ∈ *fuzzy gp*-int (α)∧ fuzzy gp*-int (μ).*

$$\Rightarrow \quad y \in \text{fuzzy } gp^*\text{-}int(\alpha)\ \&\ y \in \text{fuzzy } gp^*\text{-}int(\mu).$$

So it follows that y is fuzzy gp*-interior point of both the sets α & μ. Which implies that both α & μ are fuzzy gp*-nbhds of y. Implying that $\alpha \wedge \mu$ is also fuzzy gp*-neighborhood of y.

$\Rightarrow y \in$ fuzzy gp*-int($\alpha \wedge \mu$). Therefore,

$$\text{Fuzzy } gp^*\text{-}int\ (\alpha) \wedge \text{fuzzy } gp^*\text{-}int\ (\mu) \le \text{fuzzy } gp^*\text{-}int\ (\alpha \wedge \mu) \rightarrow (2)$$

From (1) and (2), we have

$$\text{Fuzzy } gp^*\text{-}int\ (\alpha \wedge \mu) = \text{fuzzy } gp^*\text{-}int\ (\alpha) \wedge \text{fuzzy } gp^*\text{-}int\ (\mu).$$

Theorem 5.6 *Let* (Y,τ) *be a fuzzy topological space and α is a fuzzy subset of Y, Then fuzzy gp*-int (α) = ∨ {μ : μ is a fuzzy gp*-open set in Y and μ≤α}.*

Proof: Suppose that α is a fuzzy subset of Y. Now we know that,

$y \in$ fuzzy gp*-int(α) ⇔ y is a fuzzy gp*-interior point of α.

⇔ α is fuzzy gp*-nbhd of y.

⇔ ∃ a fuzzy gp*-open set μ such that $y \in \mu \le \alpha$ (by Definition 4.1)

⇔ y ∈ ∨ {μ : μ is a fuzzy gp*-open set in Y such that $y \in \mu \le \alpha$}.

⇒ fuzzy gp*-int(α) =. ∨ {μ : μ is a fuzzy gp*-open set in Y and $\mu \le \alpha$}.

Theorem 5.7 *Suppose* (Y,τ) *is a fuzzy topological space & α is any fuzzy subset in Y. Then, fuzzy int(α)≤ fuzzy gp*-int(α).*

Proof: Suppose that α is any fuzzy subset of Y. And let $y \in$ fuzzy int(α).

⇒ $y \in \vee \{\mu : \mu$ *is fuzzy open in Y,* $\mu \le \alpha\}$

⇒ ∃ A fuzzy open set μ such that $y \in \mu \le \alpha$

Now as every fuzzy open set is fuzzy gp*-open, Implies there exists a fuzzy gp*-open set μ such that $y \in \mu \le \alpha$.

⇒ $y \in \vee \{\mu : \mu$ *is fuzzy gp*-open in Y,* $\mu \le \alpha\}$

⇒ $y \in$ fuzzy gp*-int(α).

So $y \in$ fuzzy int (α) ⇒ $y \in$ fuzzy gp*-int (α) I.e. *fuzzy int (α)≤ fuzzy gp*-int(α).*

5. FUZZY GENERALIZED PRE STAR CLOSURE (FUZZY gp*-CLOSURE).

Definition 6.1 Suppose (Y,τ) is a fuzzy topological space and $\alpha \le Y$. Then *fuzzy gp*-closure of α* is defined as

$$\text{fuzzy } gp^*\text{-}cl(\alpha) = \wedge \{\mu : \alpha \le \mu,\ \mu \text{ is fuzzy } gp^*\text{-closed set in } Y\}$$

Theorem 6.2 *Suppose* (Y,τ) *is a fuzzy topological space and α, μ are fuzzy subsets of Y. Then*

(a) *fuzzy gp*-cl*(1_Y) = 1_Y *and fuzzy gp*-cl*(0_Y) = 0_Y .

(b) $\alpha \le$ *fuzzy gp*-cl(α).*

(c) *Suppose μ ≤ α, where α is fuzzy gp*-closed set. Then fuzzy gp*-cl(μ)≤α.*

(d) *If α ≤ μ, then fuzzy gp*-cl(α) ≤ fuzzy gp*-cl(μ).*

Proof: *(a)* We know that fuzzy gp*-cl(1_Y) is the intersection I.e. minimum of all fuzzy gp*-closed sets in Y containing 1_Y and since 1_Y is the minimum fuzzy gp*-closed set containing 1_Y. So *fuzzy gp*-cl*(1_Y)=1_Y. Now fuzzy gp*-cl(0_Y) is the intersection I.e. minimum of all fuzzy gp*-closed sets in Y containing 0_Y and we know that 0_Y is the minimum fuzzy gp*-closed set containing 0_Y. So *fuzzy gp*-cl(* 0_Y *)* = 0_Y.

Proof: *(b)* Since we know that fuzzy gp*-cl(α) is the intersection of all fuzzy gp*-closed sets containing α. So $\alpha \leq$ *fuzzy gp*-cl(α)* is obvious

Proof: *(c)* Suppose $\mu \leq \alpha$, where α is fuzzy gp*-closed set. Now,

fuzzy gp-cl(μ)=* $\wedge$ *{π:μ ≤ π, π is fuzzy gp*-closed set in Y}*

I.e. fuzzy gp*-cl(μ) is contained in all fuzzy gp*-closed sets, so in particular *fuzzy gp*-cl(μ)* $\leq \alpha$

Proof: (d) Suppose $\alpha \leq \mu$, and we know that

fuzzy gp-cl(μ)=*$\wedge$*{π:μ ≤ π, π is fuzzy gp*-closed set in Y}* $\rightarrow$*(d.1)*

Now if $\mu \leq \pi$, where π is fuzzy gp*-closed in *Y*, then by (c) of this theorem we have *fuzzy gp*-cl(μ)≤π.* Now by (b) of this theorem $\mu \leq$ *fuzzy gp*-cl(μ)* implies $\alpha \leq \mu \leq \pi$, where π is fuzzy gp*-closed. So we have *fuzzy gp*-cl(α)* $\leq \pi$ (by (c) of this theorem). Therefore

fuzzy gp-cl(α)≤* $\wedge$ *{π:μ ≤ π, π is fuzzy gp*-closed set in Y}*

$\Rightarrow$ *fuzzy gp*-cl(α) ≤ fuzzy gp*-cl(μ). (using (d.1))*

Theorem 6.3 *Suppose* (Y,τ) *is a fuzzy topological space and α is a fuzzy gp*-closed set in Y, then fuzzy gp*-cl(α)=α.*

Proof*:* Suppose α is fuzzy gp*-closed subset in *Y*. Now by Theorem 6.2 (b) $\alpha \leq$ *fuzzy gp*-cl(α).* Also *α≤α* & given that α is fuzzy gp*-closed set in *Y*, so by Theorem 6.2(c) *fuzzy gp*-cl(α)≤ α*. Therefore we have *fuzzy gp*-cl(α) = α.*

Theorem 6.4 *Suppose α & μ are fuzzy subsets in fuzzy topological space* (Y,τ). *Then we have fuzzy gp*-cl(*$\alpha \wedge \mu$*) ≤ fuzzy gp*-cl(α)* $\wedge$ *fuzzy gp*-cl(μ).*

Proof: Suppose α & μ are fuzzy subsets in *Y*. Then clearly $\alpha \wedge \mu \leq \alpha$ and $\alpha \wedge \mu \leq \mu$. Now by Theorem 6.2 (d) *fuzzy gp*-cl(*$\alpha \wedge \mu$*) ≤ fuzzy gp*-cl(α)* and *fuzzy gp*-cl(*$\alpha \wedge \mu$*) ≤ fuzzy gp*-cl(μ)*. Implying that

fuzzy gp-cl(*$\alpha \wedge \mu$*) ≤ fuzzy gp*-cl(α)* $\wedge$ *fuzzy gp*-cl(μ).*

Theorem 6.5 *Suppose* (Y,τ) *is a fuzzy topological space and α is a fuzzy set in Y, then fuzzy gp*-cl(α) ≤ cl(α).*

Proof**:** Suppose α is a fuzzy subset in *Y*. Now, we know that *cl(α) =* $\wedge$ *{π: α ≤ π,π is fuzzy closed}*. So if *{α ≤ π:π is fuzzy closed set in Y}* ⇒*{α ≤ π:π is fuzzy gp*-closed set in Y}*, as every fuzzy closed set is fuzzy gp*-closed. So by Theorem 6.2(c) *fuzzy gp*-cl(α) ≤ π.* Therefore *fuzzy gp*-cl(α) ≤* $\wedge$ *{π: α ≤ π,π is fuzzy closed} = cl(α)* I.e. *fuzzy gp*-cl(α) ≤ cl(α).*

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